Part IV: Quantified Logic Homework Problems

1 First-order logic

Exercise 1:

For the sentence $\forall x.\ \forall y.((\mathrm{A}(x) \wedge \mathrm{A}(y)) \rightarrow \mathrm{B}(x,y))$ state whether it is true or false, relative to the following interpretations:

1. The domain of the natural numbers, where A is interpreted as "even?" and B is interpreted as "equals"
2. The domain of the Booleans (just {true, false}), where A is interpreted as "false?" and B is interpreted as "equals"
3. The domain of WaterWorld locations in the particular board where locations Y and Z contain pirates, but all other locations are safe, and the relation symbol A is interpreted as "safe?" and B is interpreted as "neighbors"
4. All WaterWorld boards, where A is interpreted as "safe?" and B is interpreted as "neighbors". (That is, is the formula valid for WaterWorld?)

Solution:

(solution set will be posted later)

Exercise 2:

Translate the following conversational English statements into first-order logic, using the suggested predicates, or inventing appropriately-named ones if none provided. (You may also freely use =, which (for us) is always interpreted as equality.) For the WaterWorld problems, use the set of locations as the domain, and use predicates nhbr (binary), and unary predicates hasPirate, shows3, and shows2.

- Raspberry sherbet with hot fudge ("rshf") is the tastiest dessert. Use tastier as your only relation. (What is the intended domain for your formula? What is a relation which makes this statement true? One which makes it false?)
- The conjunction of
  - All for one, and one for all!
  - If you're not for us, you're against us.
  - The enemy of your enemy is your friend.
  - Somebody has an enemy.

  Use just the relations isFor, and isAgainst, both to be interpreted on pairs of people. Clarifying:

  - We'll take "one" to mean "one particular person". (The original musketeers presumably meant something different – that *each* one of them was for all; but this makes the sentence more boring if the domain includes everybody, not just musketeers.) The English is meant to convey group unity – so, the first part is trying to say "All for [the same] one particular person", rather than "each for their own one favorite." (Whew, natural language sure can be ambiguous! Actually, that's one of its strengths, in non-engineering areas.)

  - For "us", we'll take it to mean "any one particular individual". You can think of it as each person saying to each other, "if you're not for me, you're against me". Similarly for "you". This captures the intent of the English – these sayings are meant to apply to everybody, not just one particular person.
  - "your enemy" will mean "somebody you are against", and "your friend" will mean "somebody you are for". (Careful – this may be different than "somebody who is against/for you").

  Find two (fundamentally different) relations which each satisfy this formula's isFor on a domain of three people. (Depict your relation as a graph – three vertices, with directed edges.)
- In WaterWorld, location λ has exactly two neighbors. (Your answer will be a formula with λ free.)
- If a WaterWorld location shows a 3, then any neighboring location is a pirate.
- If a WaterWorld location shows a 2, and has exactly two neighbors, then both those neighbors have a pirate.

Solution:
(solution set will be posted later)

Exercise 3:

Translate the following statements into predicate logic. The domain is to be numbers, and the binary relation kth(n,k) indicates whether or not n is the kth number, in some sequence-of-numbers. That is, for the sequence <4,7,4>, the relation kth is {(4,1), (7,2), (4,3)}. You can also use the binary relations =, <, and/or <= but no others.

1. The sequence is finite.
2. The sequence contains three unique numbers.
3. The sequence is sorted. (What, exactly, does your formula "sorted" capture – ascending, descending; allowing duplicates (i.e. non-decreasing or non-increasing)? Any of those four are fine, but say which yours is.)
4. The sequence is sorted except for (up to) one location.

Solution:
(solution set will be posted later)

Exercise 4:

Some relations can be treated as functions. In fact, we can specify this relationship within our logic. Given some arbitrary binary relation R, write a first-order formula that says "R is a (unary) function."

Solution:
(solution set will be posted later)

Exercise 5:

Alternation of quantifiers: Determine the truth of each of the following sentences in each of the indicated domains.

HINT: To help yourself, you might want to develop an English version of what the logic sentences say.

- S1: $\forall x.\ \exists y.\ \forall z.(\text{likes}(x,y) \wedge (\neg (z = y) \rightarrow \neg\text{likes}(y,z)))$
- S2: $\forall x.\ \forall y.\ \exists z.(\text{likes}(x,y) \wedge (\neg (z = y) \rightarrow \neg\text{likes}(y,z)))$
- S3: $\exists x.\ \forall y.\ \forall z.(\text{likes}(x,y) \wedge (\neg (z = y) \rightarrow \neg\text{likes}(y,z)))$
- S4: $\exists x.\ \exists y.\ \forall z.(\text{likes}(x,y) \wedge (\neg (z = y) \rightarrow \neg\text{likes}(y,z)))$

- D0: The empty domain.
- D1: A world with one person, who doesn't like herself.
- D2: A world with Yorick and Zelda, where Yorick likes Zelda, Zelda likes herself, and that's all.
- D3: A world with many people, including CJ (Catherine Zeta-Jones), JC (John Cusack), and JR (Julia Roberts). Everybody likes themselves; everybody likes JC; everybody likes CJ except JR; everybody likes JR except CJ and IB. Any others may or may not like each other, as you choose, subject to the preceding. (You may wish to sketch the graph of this likes relation.)

Note how we don't just specify the domain which the logic variables range over, but also how to interpret the logic's relation-symbol likes.
Determine the truth of these combinations.

\\	D0	D1	D2	D3
S1				
S2				
S3				
S4				

Solution:
(solution set will be posted later)

2 Reasoning with Equivalences

Exercise 6:

In class, we characterized a prime number as a number z satisfying $\neg\, \exists q.\ \exists r.(\,(qr = z) \wedge \neg\,(q = 1) \wedge \neg\,(r = 1)\,)$. Using the equivalences for first-order logic, show step-by-step that this is equivalent to the formula $\forall q.\ \forall r.(\,(qr = z) \rightarrow (\,(q = 1) \vee (r = 1)\,))$

Solution:
(solution set will be posted later)

Exercise 7:

Simplify the following formula, so that the body of each quantifier contains only a single atomic formula involving that quantified variable. Provide reasoning for each step of your simplification.

$\forall x.\ \forall y.\ \exists z.((\text{A}(x) \wedge \text{B}(y)) \rightarrow \text{C}(z))$

Solution:
(solution set will be posted later)

3 Reasoning with Inference rules

Exercise 8:

What is wrong with the following "proof"?

1	subproof:$\exists x.\mathrm{E}(x) \vdash \mathrm{E}(c)$		
1.a		$\exists x.\mathrm{E}(x)$	[premise for subproof]
1.b		$\mathrm{E}(c)$	[$\exists$Elim, by line 1.1]
2	$(\exists x.\mathrm{E}(x) \rightarrow \mathrm{E}(c))$		[$\rightarrow$Intro, by line 1]

Solution:
(solution set will be posted later)

Exercise 9:

1. Prove the following: $\exists x.\mathrm{P}(x), \forall y.(\mathrm{P}(y) \rightarrow \mathrm{Q}(y)) \vdash \exists z.\mathrm{Q}(z)$
2. Give an interpretation in which these premises are true.
3. Why *can't* and *shouldn't* we conclude $\forall z.\mathrm{Q}(z)$?

Solution:
(solution set will be posted later)