

## COMP 280, Spring 2009, Exam 1

You have **5 contiguous hours** of your choice to look at and work on the exam. Hand it in during class or at my office, DH 3118. Remember that evening and weekend access to my office is limited.

You may use any material provided as part of this course, your textbook, online notes, your personal notes, your graded assignments, and provided solutions, but no other materials. You may use a computer only to access the course's online material, including sample assignment solutions, or type your answers. In particular, you will likely want access to the logical equivalences and inference rules used in the course.

Since this is a take-home exam, it is not practical to ask for clarifications during the exam. If you feel that a problem is not sufficiently specified, clearly state the assumptions that you are making, and continue with the exam. Of course, your assumptions should be reasonable ones.

Please include your name, honor code pledge, starting and ending times. Write legibly, clearly label each problem, and show your work. You need to provide your own paper. Staple all your pages together.

You may choose which proof techniques to use, unless specifically instructed as part of the problem.

The 6 problems follow on separate pages. The problems total to 75 points.

1. (10 points total)

Let  $\phi = \neg(\neg P \vee Q) \vee (P \wedge \neg R)$ .

- (a) (5 points) Give the truth table for  $\phi$ , listing rows in the order FFF, FFT, FTF, ..., TTT. Showing the columns for subformulas is not necessary, but may be helpful for partial credit.
- (b) (5 points) Give an equivalent formula in Conjunctive Normal Form.

2. (15 points total)

Let  $T(x, y)$  mean “ $x$  is a teacher of  $y$ ”. Let  $\neq$  be the standard inequality relation on people.

Let

$$\phi = \exists x. \exists y. [T(x, y) \wedge \neg \exists u. \exists v. (T(u, v) \wedge (u \neq x \vee v \neq y))].$$

- (a) (5 points) Give an example satisfying interpretation.
- (b) (10 points) A logical statement is in *prenex normal form* iff it is of the form

$$Q_1 x_1. Q_2 x_2. \dots Q_n x_n. \psi,$$

where each  $Q_i$  is either  $\forall$  or  $\exists$ , and  $\psi$  is a formula without any quantifiers.

Use the first-order logical equivalences to find a statement in prenex normal form that is equivalent to  $\phi$ . Give a justification for each step of your proof. Use only the equivalences introduced in the course notes. Do not use any theorems previously shown in this course – you shouldn't need them.

3. (10 points)

Prove that for every integer  $x$ ,  $x^2 + x$  is even.

4. (10 points)

Consider the following claim:

**Theorem?:** For any sets  $A$ ,  $B$ ,  $C$ , and  $D$ , if  $A \times B \subseteq C \times D$ , then  $A \subseteq C$  and  $B \subseteq D$ .

**Proof?:** Suppose  $A \times B \subseteq C \times D$ . Let  $a$  be an arbitrary element of  $A$ , and let  $b$  be an arbitrary element of  $B$ . Then  $(a, b) \in A \times B$ , so since  $A \times B \subseteq C \times D$ , then  $(a, b) \in C \times D$ . Therefore,  $a \in C$  and  $b \in D$ . Since  $a$  and  $b$  were arbitrary elements of  $A$  and  $B$ , respectively, this shows that  $A \subseteq C$  and  $B \subseteq D$ .

Is the proof correct? If not, why not, and can it be fixed?

Is the theorem correct? If not, fix it and the proof.

5. (20 points total)

Suppose  $f : A \rightarrow B$  and  $C \subseteq A$ . The set  $f \cap (C \times B)$ , which is a function from  $C$  to  $B$ , is called the *restriction* of  $f$  to  $C$ , and is sometimes denoted  $f \upharpoonright C$ .

Prove or disprove each of the following, for arbitrary  $f$  and  $C$ :

- (a) (10 points) If  $f \upharpoonright C$  is onto, then so is  $f$ .

(b) (10 points) If  $f$  is onto, then so is  $f \upharpoonright C$ .

6. (10 points)

Show that the set of real numbers in the range  $[0, 1]$  has the same cardinality as the set of real numbers in  $[0, 2]$ .