Weak Induction

Principle of Weak Induction

Let $P(n)$ be a statement about the *nth* integer.

If the following hypotheses hold:

   i. $P(1)$ is True.

   ii. The statement $P(n) \to P(n+1)$ is True for all $n \geq 1$.

Then we can conclude that:

   iii. $P(n)$ is True for Every Integer $n \geq 1$.

Template for Proofs by Weak Induction

Prove

   i. $P(1)$ {Base Case}

  ii. $P(n) \rightarrow P(n+1)$ {Inductive Step}

Conclude

  iii. $P(n)$ is True for Every Integer $n \geq 1$ {Conclusion}

Remark

- Do not always need to start at 1.

Examples -- Weak Induction

1. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{4} \right)^2$$

4. $$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Proposition: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof:

Base Case: $n = 1$.

$$\sum_{k=1}^{1} k = 1 \quad and \quad \frac{1(1+1)}{2} = \frac{1*2}{2} = 1.$$

Inductive Step:

Assume: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Must Show: $\displaystyle\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

Inductive Proof:

$$\sum_{k=1}^{n+1} k = (n+1) + \sum_{k=1}^{n} k \qquad (recursion)$$

$$= (n+1) + \frac{n(n+1)}{2} \qquad (inductive\ hypothesis)$$

$$= \frac{2(n+1)}{2} + \frac{n(n+1)}{2} \qquad (common\ denominator)$$

$$= \frac{(n+1)(2+n)}{2} \qquad (factoring)$$

Good Semantics

Proposition: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Good Proof:

Base Case: $n = 1$.

$\displaystyle\sum_{k=1}^{1} k^2 = 1$ *and* $\dfrac{1(1+1)(2*1+1)}{6} = \dfrac{1*2*3}{6} = 1$.

Inductive Step:

Assume: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Must Show: $\displaystyle\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

Inductive Proof:

$$\sum_{k=1}^{n+1} k^2 = (n+1)^2 + \sum_{k=1}^{n} k^2 \qquad (recursion)$$

$$= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \qquad (inductive\ hypothesis)$$

$$= (n+1)\frac{\left(6(n+1) + n(2n+1)\right)}{6} \qquad (factoring)$$

$$= (n+1)\frac{\left(6n + 6 + 2n^2 + n\right)}{6} \qquad (expanding)$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6} \qquad (adding)$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} \qquad (factoring)$$

Bad Semantics

Prop: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Bad Proof:

Base Case: $n = 1$.

$$\sum_{k=1}^{1} k^2 = \frac{1(1+1)(2*1+1)}{6} = \frac{1*2*3}{6} = 1.$$

Inductive Step:

Assume: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Must Show: $\displaystyle\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)^2 + \sum_{k=1}^{n} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)^2 + \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)\frac{\bigl(6(n+1)+n(2n+1)\bigr)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)\frac{\left(6n+6+2n^2+n\right)}{6} = \frac{(n+1)(n^2+7n+6)}{6}$$

$$\frac{(n+1)(n^2+7n+6)}{6} = \frac{(n+1)(n^2+7n+6)}{6} \qquad \text{BAD}$$

Bad Semantics (continued)

1. The implications go the wrong way
   a. From q and $p \Rightarrow q$, we cannot conclude p.   BAD

2. Good style for discovery; bad style for proof.

3. Start at bottom; work to the top.

4. Start with what you know; prove what you do not know.

5. Do not assume what you are trying to prove.

6. Sometimes this proof style actually is correct because
   the implications are if and only if.
   But this style is highly dangerous because often the
   implications are only one way.

Good Style for Inductive Proofs

A. First verify the base case, using the following template:

      LHS(1)=something

      RHS(1)=something

and say "base case holds".

DO NOT WRITE:

      LHS(1)=RHS(1)

BEFORE showing they are both equal to some common result.

B. Clearly specify your induction hypothesis.

Omitting this step can cause serious confusion.

C. For the inductive step, NEVER start from the conclusion

      LHS(n+1)=RHS(n+1)

and perform arithmetic operations on both sides to derive a well-known fact such as

      0=0.

This style is VERY BAD and the semantics are incorrect.

      q and p$\Rightarrow$q, do not imply p.

You MAY verify the given statement in this way.

You MAY NOT prove the result in this way.

D. I strongly recommend one of the following three correct approaches:

i. Start from LHS(n+1), show

$$LHS(n+1) \quad = \quad ....$$
$$= \quad ....$$
$$= \quad ....$$
$$= \quad RHS(n+1)$$

ii. Start from RHS(n+1), show

$$RHS(n+1) \quad = \quad ....$$
$$= \quad ....$$
$$= \quad ....$$
$$= \quad LHS(n+1)$$

iii. Start from LHS(n+1), show

$$LHS(n+1) \quad = \quad ....$$
$$= \quad ....$$
$$= \quad ....$$

$$= \text{some intermediate result;}$$

and do the same thing on RHS(n+1)

$$\text{RHS}(n+1) \quad = \quad \ldots\ldots$$
$$= \quad \ldots\ldots$$
$$= \quad \ldots\ldots$$
$$= \text{same intermediate result;}$$

E. After you finish the inductive step, you should state a conclusion, summarizing what your proof has shown.

Questions

1. Is induction circular?
   - Aren't we assuming what we are trying to prove?
   - If we assume the result, can't we prove anything at all?

2. Does induction ever lead to false results?

3. Can we change the base case?

4. Why do we need induction?

5. Is proof by induction finite?
   - Don't we need infinitely many steps to establish $P(n)$ for every n?

6. Proof of Induction?
   - Well Ordering Principle

Failed Induction

Proposition: $\displaystyle\sum_{k=1}^{n} k = n^2$

Proof:

    Base Case: $n = 1$.

$$\sum_{k=1}^{1} k = 1 \quad and \quad 1^2 = 1.$$

    Inductive Step:

        Assume: $\displaystyle\sum_{k=1}^{n} k = n^2$

        Must Show: $\displaystyle\sum_{k=1}^{n+1} k = (n+1)^2$

$$\sum_{k=1}^{n+1} k = (n+1) + \sum_{k=1}^{n} k \qquad (recursion)$$

$$= (n+1) + n^2 \qquad (inductive\ hypothesis)$$

$$= (n^2 + 2n + 1) - n \qquad (add\ and\ subtract\ n)$$

$$= (n+1)^2 - n \qquad (factoring)$$

$$\neq (n+1)^2 \qquad (when\ \ n \neq 0)$$

PROOF FAILS

Theorem: All horses have the same color.

Theorem: All horses have the same color.

Theorem: For all integers $n \geq 1$, in every collection of n horses, all the horses have the same color.

The Island of Perfect Logicians -- Extra Credit

There once was an island where all the inhabitants were *Perfect Logicians* and also *Perfectly Law Abiding*. The islanders had the following three laws:

1. Anyone who finds out that they have blue eyes must commit suicide that same day before sundown by jumping off a cliff.
2. No one is allowed to tell another person that the other person has blue eyes.
3. No mirrors are allowed on the island.

One day a man from the *Island of Perfect Truth Tellers* visits the *Island of Perfect Logicians*. After the dinner that night the visitor from the *Island of Perfect Truth Tellers* says to all the inhabitants of the *Island of Perfect Logicians* that he is happy to have dinner on an island where at least one person has blue eyes. (All the residents of the *Island of Perfect Truth Tellers* have brown eyes.)

There are 100 inhabitants on the *Island of Perfect Logician*, and all 100 inhabitants have blue eyes. One hundred days after the visit of the person from the *Island of Perfect Truth Tellers* all the inhabitant of the *Island of Perfect Logicians* jump off a cliff. Why?