Recurrences

Applications

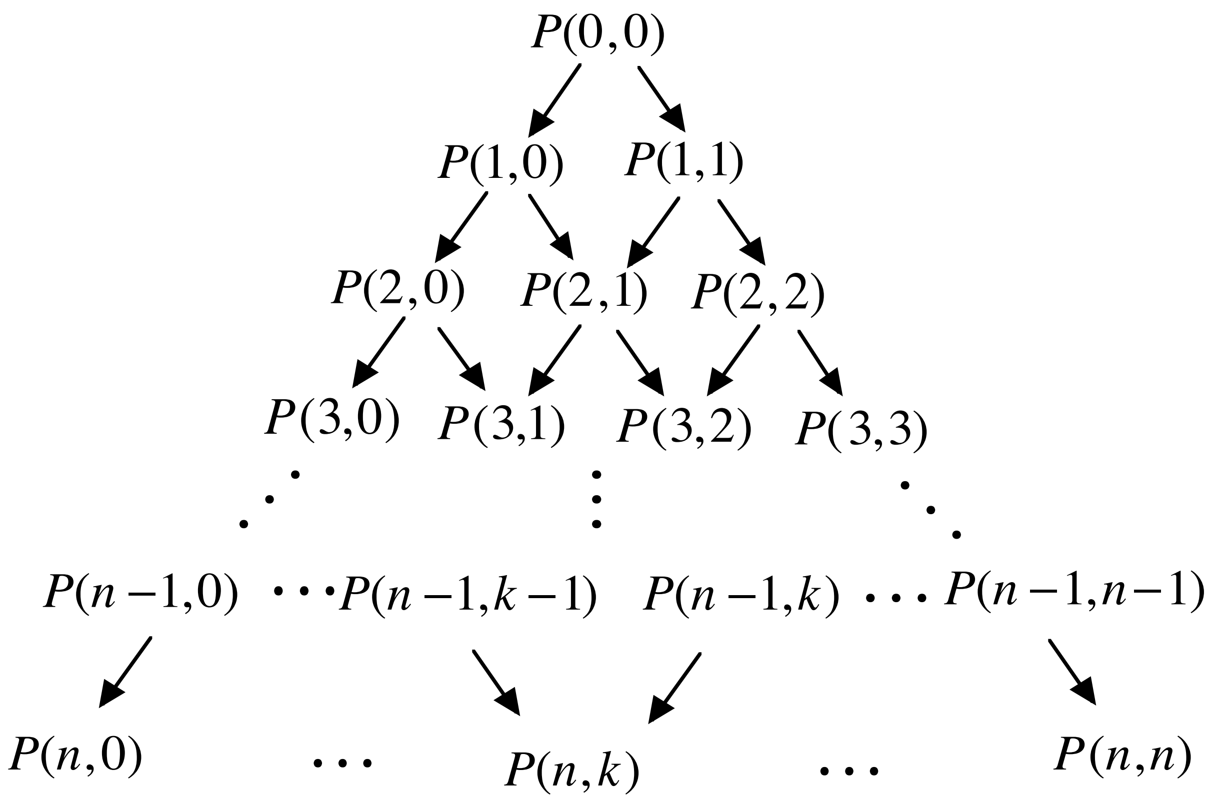
More Counting Problems

Complexity of Algorithms

Part I

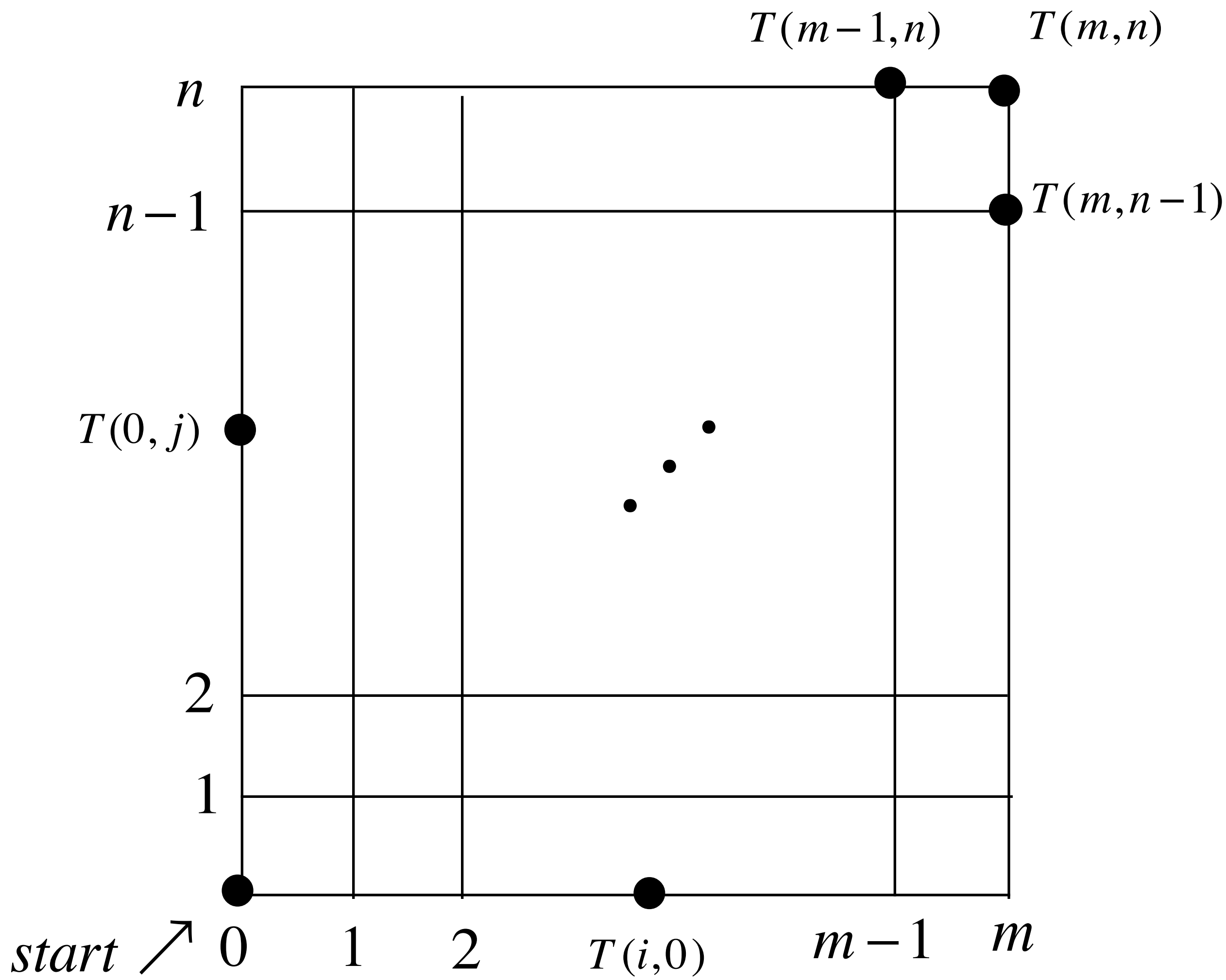
Recurrences and Binomial Coefficients

Paths in a Triangle



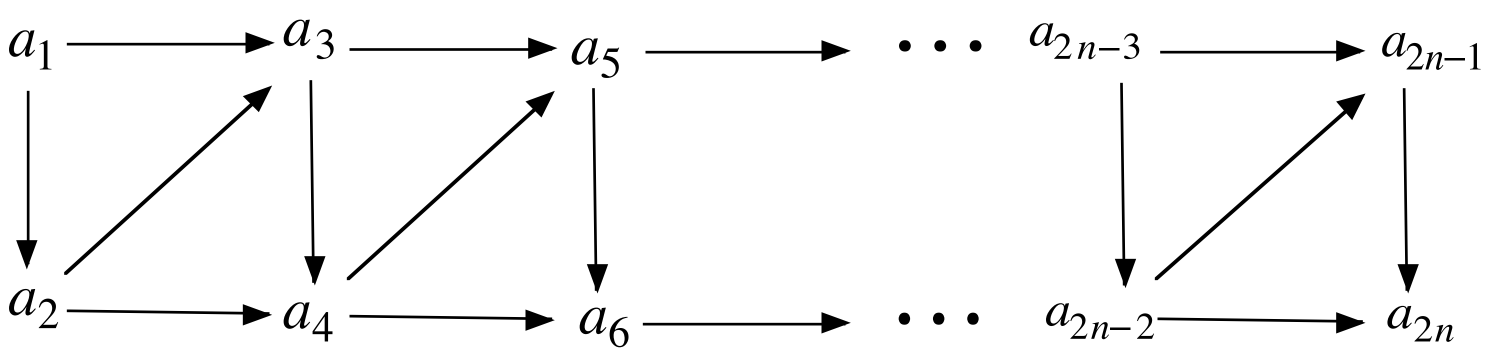
$$P(n,k) = P(n-1,k-1) + P(n-1,k)$$

Taxi Cab Paths in a Rectangle



$$T(m,n) = T(m,\, n-1) + T(m-1,\, n)$$

Paths in a Rectangle



$$a_{2n} = a_{2n-1} + a_{2n-2}$$

Path Problems

Triangle

- $P(n,k) = P(n-1,k-1) + P(n-1,k)$
- $C(n,k) = C(n-1,k-1) + C(n-1,k)$
- $P(n,k) = C(n,k)$

  -- $P(n,0) = 1 = C(n,0)$

Rectangle (Taxicab)

- $T(m,n) = T(m,n-1) + T(m-1,n)$
- $C(m+n,n) = C(m+n-1,n-1) + C(m+n-1,n)$
- $T(m,n) = C(m+n,n)$

  -- $T(m,0) = 1 = C(m,m)$

  -- $T(0,n) = 1 = C(n,n)$

Taxicab Problem

Solution

- $T(m,n) = C(m+n,n)$

Examples

- $T(8,8) = C(16,8) = 12{,}870 \approx 10^4$
- $T(16,16) = C(32,16) = 601{,}080{,}390 \approx 10^8$
- $T(32,32) = C(64,32) = 1{,}832{,}624{,}140{,}942{,}590{,}534 \approx 10^{18}$

Conclusion

- Searching even moderate size grids for optimal path is not feasible.

Pascal's Triangle and Fibonacci Numbers

Definitions

$$a_n = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n/2, n/2)$$

$$a_{n-1} = \qquad C(n-1,0) + C(n-2,1) + \cdots + C(n/2, n/2-1) + C\big((n-1)/2, (n-1)/2\big)$$

$$\text{Sum} = C(n+1,0) + C(n,1) + C(n-1,2) + \cdots + C(n/2+1, n/2) + C\big((n+1)/2, (n+1)/2\big)$$

Conclusion

$$a_{n-1} + a_n = a_{n+1}$$

Indistinguishable Objects in Distinguishable Boxes

Problem

- How many ways $W(n, m)$ can you put n similar objects into m different boxes placing at least $r_j \geq 0$ objects into box j?

Solution

- $W(n,m) = W(n-1,m) + W(n,m-1)$

  -- Either you put an object in the first box or you don't.

- $C(m+n-1,\ m) = C(m+n-1,\ m-1) + C(m+n-1,\ m)$

- $W(n,m) = C(m+n-1,m-1) = C(m+n-1,\ n)$

  -- $W(n,1) = 1 = C(n,0)$

Part II

Linear Recurrences with Constant Coefficients

Recurrences with Constant Coefficients

Linear Homogeneous Recurrence

- $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$

Linear Inhomogeneous Recurrence

- $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + f(n)$

Nonlinear Recurrence

- Some a_j not to first power
- Will not study
- See homework for simple examples

Problems

- Derive the recurrence
- Solve the recurrence

<u>Fibonacci Sequences (Homogeneous)</u>

Permutations where each element either stays in place or changes places with an adjacent neighbor

- Examples: 1234, 1243, 2143, 2134, 1324
- a_n = number of such permutations of n objects
- $a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 5, \ldots$
- $a_n = a_{n-1} + a_{n-2}$ (either n changes places or it doesn't)
- $1, \; 2, \; 3, \; 5, \; 8, \; 13, \ldots$

Rabbits breed one pair/month after the second month

- f_n = number of rabbit pairs after n months (no deaths)
- $f_0 = f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ (number last month + newborn)
- $1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \ldots$

More Fibonacci Sequences (Homogeneous)

Bitstrings of length n that do NOT have two consecutive zeros

- Examples: 010, 011, 101, 110, 111
- $a_1 = 2 \quad a_2 = 3$
- $a_n = a_{n-1} + a_{n-2}$ (sequence either ends in 1 or ends in 0)
- $2, 3, 5, 8, 13, 21, \ldots$

Pascal's Triangle

- Arrange binomial coefficients in a right triangle
- Sum along diagonal slant
- $f_{n+1} = C(n,0) + C(n-1,1) + \cdots + C(n/2, n/2)$
- $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Exponential Sequences (Homogeneous)

Compound Interest

- r = interest rate
- A_n = amount of money after n years

-- A_0 = Initial deposit

-- $A_n = (1+r)A_{n-1} \Leftrightarrow A_n - A_{n-1} = rA_{n-1}$

- $A_n = (1+r)^n A_0$

Biological Growth

- r = yearly (hourly) rate of growth or decay
- A_n = population after n years (hours)

-- A_0 = Initial population

-- $A_n = (1 \pm r)A_{n-1} \Leftrightarrow A_n - A_{n-1} = \pm rA_{n-1}$

-- $A_n = (1 \pm r)^n A_0$

Analogy to Differential Equations

Differential Equations

- $\dfrac{dA}{dt} = rA$
- $A(t) = A_0 e^{rt}$

Difference Equations

- $\nabla A_n = A_n - A_{n-1} = rA_{n-1}$
- $A_n = (1 \pm r)^n A_0$

Inhomogeneous Equations

Tower of Hanoi

- h_n = number of moves needed to solve Tower of Hanoi
  - -- $h_1 = 1$
  - -- $h_n = 2h_{n-1} + 1$
  - -- $h_n = 2^n - 1$ (by induction on *n*)

Regions of Space

- r_n = number of planar regions generated by *n* nonconcurrent lines
  - -- $r_1 = 2$
  - -- $r_n = r_{n-1} + n$ (every crossing creates a new region)
  - -- $r_n = \frac{n(n+1)}{2} + 1$ (by induction on *n* or by inspection)

Backwards Difference

Differences

- $\nabla a_n = a_n - a_{n-1}$
- $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$

Examples

- $\nabla a_n = a_n - a_{n-1}$
- $\nabla^2 a_n = \nabla a_n - \nabla a_{n-1} = (a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) = a_n - 2a_{n-1} + a_{n-2}$
- $\nabla^3 a_n = \nabla^2 a_n - \nabla^2 a_{n-1} = (a_n - 2a_{n-1} + a_{n-2}) - (a_{n-1} - 2a_{n-2} + a_{n-3})$
  $= a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3}$

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j}$

Proof: By induction on k.

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j}$

Proof: By induction on k.

Base cases already verified for $k = 1,2,3$

Inductive Step:

Assume:

$$\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j}$$

Must Show:

$$\nabla^{k+1} a_n = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} a_{n-j}$$

Use the Inductive Definition:

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$$

Must Show: $\nabla^{k+1} a_n = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} a_{n-j}$

But by definition: $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$, so

$$\nabla^{k+1} a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j} - \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-1-j}$$

$$\nabla^{k+1} a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j} - \sum_{j=1}^{k+1} (-1)^{j-1} \binom{k}{j-1} a_{n-j} \qquad \{reindex\ j \to j-1\}$$

$$\nabla^{k+1} a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j} + \sum_{j=1}^{k+1} (-1)^{j} \binom{k}{j-1} a_{n-j} \qquad \{adjust\ signs\}$$

$$\nabla^{k+1} a_n = \sum_{j=0}^{k+1} (-1)^j \left\{ \binom{k}{j} + \binom{k}{j-1} \right\} a_{n-j}$$

$$\nabla^{k+1} a_n = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} a_{n-j}$$

<u>Backwards Differences and Recurrences</u>

Recurrence

- $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$

Differences

- $\nabla a_n = a_n - a_{n-1}$
- $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j}$

Corollary 1: a_{n-k} *can be written in terms of* $a_n, \nabla^n a, \ldots, \nabla^k a_n$.

Corollary 2: $a_n = b_1 \nabla a_n + \cdots + b_k \nabla^k a_n$.

Characteristic Equation

Explicit Solution for Linear Homogeneous Recurrence

- $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$

  -- $a_n = r^n$

  -- $r^n = c_1 r^{n-1} + \cdots + c_k r^{n-k}$

Characteristic Equation

- $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

Analogy to Linear Homogeneous Differential Equations

Difference Equation

- $a_n = b_1 \nabla a_n + \cdots + b_k \nabla^k a_n$

  -- $a_n = r^n$ (Exponential in n)

  -- r = solution of characteristic equation

Differential Equation

- $A(t) = b_1 \frac{dA}{dt} + \cdots + b_k \frac{d^k A}{dt^k}$

  -- $A(t) = e^{rt}$ (Exponential in t)

  -- r = solution of characteristic equation

Solving Linear Homogeneous Recurrence Relations

Hypotheses

- Recurrence Relation: $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$
- Characteristic Poly: $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

Theorem

- If $r_1, \ldots, r_k$ are k *distinct* roots of the characteristic polynomial,

  then the only solutions of the recurrence are of the form:

  $$a_n = b_1 r_1^n + \cdots + b_k r_k^n$$

Proof

- $a_n = r_j^n$ is a solution since r_j^n satisfies the characteristic polynomial.
- if $a_n = r(n)$ and $a_n = s(n)$ are two solutions of the recurrence,

  then *by linearity* so are

  -- $a_n = r(n) + s(n)$

  -- $a_n = b\,r(n)$
- Given k initial conditions, we can solve for the constants $b_1, \ldots, b_k$,

  since the coefficient matrix is a Vandermonde in $r_1, \ldots, r_k$.
- Therefore $a_n = b_1 r_1^n + \cdots + b_k r_k^n$ is a solution that

  satisfies the initial conditions.
- Since the initial conditions specify a unique solution,

  all solutions must have this form.

Linearity

Lemma: *If* $a_n = r(n)$ *and* $a_n = s(n)$ *are two solutions of the recurrence,*

- $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$

then so are

i. $a_n = r(n) + s(n)$

ii. $a_n = b\,r(n)$

Proof: If $a_n = r(n)$ and $a_n = s(n)$ are two solutions of the recurrence, then

- $r(n) = c_1 r(n-1) + \cdots + c_k r(n-k)$
- $s(n) = c_1 s(n-1) + \cdots + c_k s(n-k)$

Therefore

i. $r(n) + s(n) = c_1 \big(r(n-1) + s(n-1) \big) + \cdots + c_k \big(r(n-k) + s(n-k) \big)$

ii. $b\,r(n) = c_1 \big(b\,r(n-1) \big) + \cdots + c_k \big(b\,r(n-k) \big)$

Initial Conditions and Vandermonde Matrices

Initial Conditions

$$\begin{aligned} a_0 &= b_1 r_1^0 + \cdots + b_k\, r^0 \\ a_1 &= b_1 r_1^1 + \cdots + b_k\, r_k^1 \\ &\vdots \\ a_{k-1} &= b_1 r_1^{k-1} + \cdots + b_k\, r_k^{k-1} \end{aligned} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}$$

Vandermonde Determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{pmatrix} \neq 0 \qquad \textit{provided } r_1 \neq r_1 \neq \cdots \neq r_k$$

Vandermonde Determinants

2×2 *Vandermonde Determinant*

- $$\det\begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} = r_2 - r_1 \neq 0 \quad \textit{if} \;\; r_2 \neq r_1$$

$k \times k$ *Vandermonde Determinant*

- $$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{pmatrix} \neq 0 \qquad \textit{provided } r_1 \neq r_1 \neq \cdots \neq r_k$$

since a polynomial of degree $k-1$ *has at most k-1 roots*.

What if

Question

- What happens if the roots $r_1, \ldots, r_k$ are not unique?

Answer

- Given: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and $r_1 = r_2$.

- Solutions: $a_n = b_1 r_1^n + b_2 n\, r_k^n$ (Homework).

Solving Recurrences

- Recurrence Relation: $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} \qquad n \geq k$
- Initial Conditions: $a_0, \ldots, a_{k-1}$
- Characteristic Poly: $r^k - c_1 r^{k-1} - \cdots - c_k = 0$
- Distinct Roots: $r_1, \ldots, r_k$
- General Solution: $a_n = b_1 r_1^n + \cdots + b_k r_k^n$
- Constants from Initial Conditions:

$$a_0 = b_1 + \cdots + b_k$$

$$a_1 = b_1 r_1 + \cdots + b_k r_k$$

$$\vdots$$

$$a_{k-1} = b_1 r_1^{k-1} + \cdots + b_k r_k^{k-1}$$

Fibonacci -- Revisited

Fibonacci Sequence

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$

Characteristic Polynomial

- $r^2 - r - 1 = 0$
- $r = (1 \pm \sqrt{5})/2$ (Golden Ratio)

Fibonacci -- Revisited (continued)

Solution

- $$f_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Initial Conditions

- $b_1 + b_2 = 0 \qquad (n = 0)$

- $$b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad (n = 1)$$

-- $b_1 = \frac{1}{\sqrt{5}} \qquad b_2 = -\frac{1}{\sqrt{5}}$

Solution

- $$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Polynomial Recurrences

Motivation

Question

- What happens when $r_1 = r_2 = \cdots = r_k = 1$ are the roots of the characteristic polynomial?

Answer

- Recurrence is no longer an exponential.
- Recurrence is a polynomial.

Characteristic Polynomial

- $(1-r)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} r^{k-j} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} r^{k-j}$

Recurrence

- $a_n - \binom{k}{1} a_{n-1} + \binom{k}{2} a_{n-2} - \cdots + (-1)^{k+1} a_{n-k} = 0$

Backwards Difference

Differences

- $\nabla a_n = a_n - a_{n-1}$
- $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$

Examples

- $\nabla a_n = a_n - a_{n-1}$
- $\nabla^2 a_n = \nabla a_n - \nabla a_{n-1} = (a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) = a_n - 2a_{n-1} + a_{n-2}$
- $\nabla^3 a_n = \nabla^2 a_n - \nabla^2 a_{n-1} = (a_n - 2a_{n-1} + a_{n-2}) - (a_{n-1} - 2a_{n-2} + a_{n-3})$
  $= a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3}$

Linearity

- $\nabla^k (a_n + b_n) = \nabla^k a_n + \nabla^k b_n$
- $\nabla^k (c a_n) = c \nabla^k a_n$

Backwards Differences and Polynomial Recurrences

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n-j}$

Proof: By induction on k.

Polynomial Recurrences

- $a_n - \binom{k}{1} a_{n-1} + \binom{k}{2} a_{n-2} - \cdots + (-1)^{k+1} a_{n-k} = 0$

- $\nabla^k a_n = 0$

Examples

- See IQ-tests

Theorem: $\nabla^{k+1} n^k = 0$ *{k constant, n varies}*

Proof: By induction on k.

Base Case: $k = 0, \quad n^0 = 1 \;\Rightarrow\; \nabla n^0 = 1 - 1 = 0$.

Inductive Step. Assume $\nabla^k n^j = 0, \quad j < k$.

Must Show: $\nabla^{k+1} n^k = 0$.

Apply the binomial theorem.

$$\nabla^{k+1} n^k = \nabla^k n^k - \nabla^k (n-1)^k$$

$$= \nabla^k \left(n^k - (n-1)^k \right)$$

$$= \nabla^k \left(k n^{k-1} + \textit{lower order terms} \right)$$

$$= 0 \quad \{\textit{by the inductive hypothesis}\}$$

Consequences

Theorem: $\nabla^{k+1} n^k = 0$ *{k constant, n varies}*

Corollary 1: $\nabla^{k+1} p = 0$ if p is any polynomial of degree $\leq k$.

Corollary 2: $\nabla^k p$ = constant if p is any polynomial of degree $\leq k$.

Analogy: $\frac{d^k p}{dt} = constant$ if p is any polynomial of degree $\leq k$.

Theorem: *The only solutions of the recurrence* $\nabla^{k+1} a_n = 0$ *are of the form:*

$$a_n = b_k n^k + b_{k-1} n^{k-1} + \cdots + b_0$$

Proof: By the Corollary 1, $a_n = b_k n^k + b_{k-1} n^{k-1} + \cdots + b_0$ is a solution.

Given $k+1$ initial conditions, we can solve for the constants $b_0, \ldots, b_k$ since the coefficient matrix is Vandermonde in 1, $2, \ldots, k$.

Therefore $a_n = b_k n^k + b_{k-1} n^{k-1} + \cdots + b_0$ is a solution that satisfies the initial conditions.

Since the initial conditions specify a unique solution, all solutions must have this form.

Forward Differences

Differences

- $\nabla a_n = a_{n+1} - a_n$
- $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$

Examples

- $\nabla a_n = a_{n+1} - a_n$
- $\nabla^2 a_n = \nabla a_{n+1} - \nabla a_n = (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+2} - 2a_{n+1} + a_n$
- $\nabla^3 a_n = \nabla^2 a_n - \nabla^2 a_{n-1} = (a_{n+3} - 2a_{n+2} + a_{n+1}) - (a_{n+2} - 2a_{n+1} + a_n)$

  $= a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n$

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_{n+j}$

Proof: By induction on k.

Forward Differences

Differences

- $\nabla a_n = a_{n+1} - a_n$
- $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$

Example

- $$\begin{aligned}\nabla\big((t-1)\cdots(t-k)\big) &= t(t-1)\cdots(t-k-1) - (t-1)\cdots(t-k) \\ &= (t-1)\cdots(t-k-1)\big(t-(t-k)\big) \\ &= k(t-1)\cdots(t-k-1)\end{aligned}$$
- Similar to Differentiation

Linearity

- $\nabla^k (a_n + b_n) = \nabla^k a_n + \nabla^k b_n$
- $\nabla^k (c\, a_n) = c\nabla^k a_n$

Forward Differences and Polynomial Recurrences

Lemma: $\nabla^k a_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_{n+j}$

Proof: By induction on k.

Polynomial Recurrences

- $a_{n+k} - \binom{k}{1} a_{n+k-1} + \binom{k}{2} a_{n+k-2} - \cdots + (-1)^{k+1} a_n = 0$

- $\nabla^k a_n = 0$

Examples

- See IQ-Tests

IQ Tests

Quadratic

$\nabla^2 a_n$:			6		6		6		**?**		
∇a_n :		9		15		21		27		**?**	
a_n :	4		13		28		49		76		**?**

$$p(t) = ?$$

Cubic

$\nabla^3 a_n$:				6		6		**?**			
$\nabla^2 a_n$:			12		18		24		**?**		
∇a_n :		11		23		41		65		**?**	
a_n :	5		16		39		80		145		**?**

$$p(t) = ?$$

<u>IQ Tests -- continued</u>

Quadratic

$\nabla^2 a_n$:			6	6	6	**6**	
∇a_n :		9	15	21	27	**33**	
a_n :	4	13	28	49	76	**109**	

$$p(t) = \frac{6(t-1)(t-2)}{2} + 9(t-1) + 4$$

Cubic

$\nabla^3 a_n$:				6	6	**6**
$\nabla^2 a_n$:			12	18	24	**30**
∇a_n :		11	23	41	65	**95**
a_n :	5	16	39	80	145	**240**

$$p(t) = \frac{6(t-1)(t-2)(t-3)}{3!} + \frac{12(t-1)(t-2)}{2} + 11(t-1) + 5$$

IQ Tests -- Revisited

Exponential

$\nabla^2 a_n :$ 1 2 4 8

$\nabla a_n :$ 1 2 4 8 16

$a_n :$ 1 2 4 8 16 32

$$\frac{de^t}{dt} = e^t \quad \textit{analogous to} \quad \nabla 2^n = 2^n$$

Fibonacci

$\nabla^2 a_n :$ 1 1 2 3 5 8

$\nabla a_n :$ 1 1 2 3 5 8 13

$a_n :$ 1 1 2 3 5 8 13 21

$\nabla a_n = a_n$ *is the signature of an exponential*

Polynomial Evaluation

Evenly Spaced Points

- $a_{n+1} = a_n + h$

Differencing

- Calculate n evenly spaced values by Horner's method
- Build difference triangle
- Propagate the difference triangle
- Evaluate new points using only addition, no multiplication

Generating Functions

Applications

Counting Solutions to Diophantine Equations

- Placing Indistinguishable objects into Indistinguishable boxes
- Making Change

Combinatorial Identities

- Expectation of Binomial Distribution
- Vandermonde Identity

Solving (In)Homogeneous Linear Equations

- Analogous to series solutions to differential equations

Methods

Coding Information in Exponents

- Counting Solutions to Diophantine Equations

Coding Information in Coefficients

- Combinatorial Identities
- Solving Linear Recurrences

<u>**Linear Equations with Unit Coefficients**</u>

Unconstrained

- Solve $x_1 + \cdots + x_k = n$ -- order matters
- # Solutions in non-negative integers = $C(n+k-1, n)$

Constrained

- Solve $x_1 + \cdots + x_k = n$ -- order matters
- Constraints: $a_1 \le x_1 \le b_1 \ \cdots \ a_k \le x_k \le b_k$
- # Solutions
  - -- Apply Inclusion/Exclusion (Old Method)
  - -- Take Coeff of x^n in $(x^{a_1} + \cdots + x^{b_1}) \cdots (x^{a_k} + \cdots + x^{b_k})$
- Equations $\Leftrightarrow$ n Indistinguishable objects into k Distinguishable boxes

Linear Equations with Unit Coefficients -- Constrained

Example

- Solve $x_1 + x_2 + x_3 = 10$ -- order matters
- Constraints: $1 \le x_1 \le 2, \quad 2 \le x_2 \le 5, \quad 4 \le x_3 \le 7$
- 10 Indistinguishable Objects into 3 Distinguishable Boxes
- # Solution

  -- Coeff of x^{10} in $(x + x^2)\,(x^2 + x^3 + x^4 + x^5)\,(x^4 + x^5 + x^6 + x^7)$

  -- $4 + 3 = 7$

Partitioning the Integers

Partitions

- Find all partitions of n into non-negative integers -- ignore order
- Solve $x_1 + \cdots + x_k = n$ -- ignore order
- # Solutions in non-negative integers is coeff of x^n in a polynomial product
- Partitions with distinct summands

  -- $(1+x)(1+x^2)\cdots(1+x^n)$
- Partitions with repeated summands

  -- $(1+x+x^2 \cdots + x^n)(1+x^2+x^4+\cdots)\cdots(1+x^n)$
- Partitions $\Leftrightarrow$ n Indistinguishable objects into k Indistinguishable boxes

<u>Examples</u>

Partitions of 6 with Distinct Summands

- Coeff of x^6 in

  -- $(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4)\,(1+x^5)\,(1+x^6)$

- 4 Solutions: $(6,0),\ (5,1),\ (4,2),\ (3,2,1)$

Partitions of 6 with Repeated Summands

- Coeff of x^6 in

  -- $(1+x+\cdots+x^6)(1+x^2+x^4+x^6)\,(1+x^3+x^6)\,(1+x^4)\,(1+x^5)\,(1+x^6)$

- 11 Solutions:

  $(6,0),\ (5,1),\ (4,2),\ (4,1,1),\ (3,3),\ (3,2,1),\ (3,1,1,1),$

  $(2,2,2),\ (2,2,1,1),\ (2,1,1,1,1),\ (1,1,1,1,1,1)$

Solutions to Diophantine Equations

Diophantine Equation -- Solutions in Non-Negative Integers

- $a_1 x_1 + \cdots + a_k x_k = n$
- $a_1, \ldots, a_k$ non-negative integers
- Coeff x^n in $(1 + x^{a_1} + x^{2a_1} \cdots)(1 + x^{a_2} + x^{2a_2} + \cdots)\cdots(1 + x^{a_k} + x^{2a_k} + \cdots)$

Example: Change for a Dollar

- $x_1 + 5x_2 + 10x_3 + 25x_4 + 50x_5 = 100$
- Coeff x^{100} in $(1 + x + x^2 + \cdots + x^{100})(1 + x^5 + x^{10} + \cdots + x^{100})\cdots(1 + x^{50} + x^{100})$

<u>**Generating Functions**</u>

Definition

- $A = a_0, a_1, a_2, \ldots =$ sequence finite or infinite
- $G(x) = \sum_k a_k x^k$ = Generating Function of A

Examples

- $a_k = 1 \qquad 0 \le k \le n$

$$G(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

- $a_k = 1 \qquad 0 \le k < \infty$

$$G(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

Generating Functions

More Examples

- $a_k = r^k \qquad 0 \le k < \infty$

$$G(x) = \sum_{k=0}^{\infty} r^k x^k = \frac{1}{1 - rx}$$

- $a_k = B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad 0 \le k \le n$

$$G(x) = \sum_{k=0}^{n} B_k^n(t) x^k = \sum_{k=0}^{n} \binom{n}{k} t^k x^k (1-t)^{n-k} = \left((1-t) + tx\right)^n$$

Binomial Distribution

Generating Function

- $B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad 0 \le k \le n$

- $G(x) = \sum_{k=0}^{n} B_k^n(t) x^k = \sum_{k=0}^{n} \binom{n}{k} t^k x^k (1-t)^{n-k} = \big((1-t) + tx\big)^n$

- $G'(x) = \sum_{k=0}^{n} k B_k^n(t) x^{k-1} = nt\big((1-t) + tx\big)^{n-1}$

Expectation

- $G'(1) = \sum_{k=0}^{n} k B_k^n(t) = nt$

- $\dfrac{G^{(j)}(1)}{j!} = \sum_{k=j}^{n} C(k,j) B_k^n(t) = C(n,j) t^j$

Vandermonde's Identity

Generating Function

- $(1+t)^{m+n} = (1+t)^m (1+t)^n$

Binomial Theorem

- $\sum_{k=0}^{m+n} C(n+m,k)\, t^k = \left(\sum_{i=0}^{m} C(m,i)\, t^i \right) \left(\sum_{j=0}^{n} C(n,j) t^j \right)$

Identity

- $C(m+n,k) = \sum_{i+j=k} C(m,i) C(n,j) \qquad k \le \min(m,n)$

<u>Algebra of Generating Functions</u>

Setup

- $F(x) = \sum_k a_k x^k$
- $G(x) = \sum_k b_k x^k$

Addition

- $F(x) + G(x) = \sum_k (a_k + b_k) x^k$

Multiplication {Discrete Convolution}

- $F(x)\, G(x) = \sum_k \left(\sum_j a_j b_{k-j} \right) x^k$

Geometric Series

Sums

- $S = c + cr + cr^2 + \cdots$
- $rS \;=\; cr + cr^2 + cr^3 + \cdots$
- $S - rS = c$
- $S = \dfrac{c}{1-r}$

Examples

- $x + x^2 + x^3 \cdots = \dfrac{x}{1-x}$
- $1 + ax + a^2x^2 + \cdots = \dfrac{1}{1-ax}$

Solving Recurrence Relations -- Tower of Hanoi (Inhomogeneous)

Tower of Hanoi (Inhomogeneous)

- $h_0 = 0 \qquad h_1 = 1$
- $h_n = 2h_{n-1} + 1 \qquad n \geq 1$

Generating Function

- $h_n x^n = 2h_{n-1}x^n + x^n$
- $H(x) = \sum_{n=1}^{\infty} h_n x^n = 2x \sum_{n=1}^{\infty} h_{n-1} x^{n-1} + \sum_{n=1}^{\infty} x^n$
- $H(x) = 2xH(x) + x/(1-x)$ *(Functional Equation)*

Solving Recurrence Relations -- Tower of Hanoi (continued)

Generating Function (continued)

- $H(x) = \sum_{n=1}^{\infty} h_n x^n$

- $H(x) = 2xH(x) + x/(1-x)$

  -- $(1-2x)H(x) = x/(1-x)$

  -- $H(x) = x/(1-x)(1-2x) = 1/(1-2x) - 1/(1-x)$

- $H(x) = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2^n - 1)x^n$

Conclusion

- $h_n = 2^n - 1$

Solving Recurrence Relations -- Fibonacci Sequence (Homogeneous)

Fibonacci Sequence

- $f_0 = 0 \qquad f_1 = 1$
- $f_n = f_{n-1} + f_{n-2} \qquad n \geq 2$

Generating Function

- $F(x) = \sum_{n=1}^{\infty} f_n x^n$
- $f_n x^n = f_{n-1} x^n + f_{n-2} x^n$
- $\sum_{n=2}^{\infty} f_n x^n = x \sum_{n=2}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} f_{n-2} x^{n-2}$
- $F(x) - x = xF(x) + x^2 F(x)$ *(Functional Equation)*

Solving Recurrence Relations -- Fibonacci Sequence (continued)

Generating Function (continued)

- $F(x) - x = xF(x) + x^2F(x)$

  -- $(1 - x - x^2)F(x) = x$

  -- $F(x) = x/(1 - x - x^2) = x/(1 - ax)(1 - bx)$

- $$F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - ax} - \frac{1}{1 - bx}\right)$$

  -- $a = \dfrac{1 + \sqrt{5}}{2}$ $\qquad$ $b = \dfrac{1 - \sqrt{5}}{2}$

- $$F(x) = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(a^n - b^n)x^n \Rightarrow f_n = \frac{a^n - b^n}{\sqrt{5}}$$

- *Analogous to series solutions for ordinary differential equations.*

<u>Algorithm for Solving Linear Recurrences</u>

Algorithm

1. Multiply both sides of the recurrence by x^n.

2. Sum over n.

3. Replace the power series by generating functions
   a. Find the functional equation

4. Solve the functional equation for the generating function -- explicit formula.

5. Convert the explicit formula back to a power series.

6. Read off the coefficients of the power series.

Fun Exercises

1. Prove that the generating function of a linear homogeneous recurrence is a rational function.

2. Prove that the roots of the denominator of the generating function are the reciprocals of the roots of the characteristic polynomial of the recurrence.

3. Prove that the numerator of the generating function for the recurrence

   $$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

   is given by taking the product

   $$(a_0 + a_1 x + \cdots a_{k-1} x^{k-1})\,(c_1 x + \cdots + c_{k-1} x^{k-1})$$

   and truncating to the first $k-1$ powers of x.

<u>**Placing Objects into Boxes**</u>

Objects	*Boxes*	*Constraints*	*Solution*
n–type I	k–type D	$x_j \geq 1$	$C(n-1, k-1)$
n–type I	k–type D	$x_j \geq 0$	$C(n+k-1, n)$
n–type D	k–type D (Onto Functions)	$x_j \geq 1$	$\sum_{k=0}^{n} (-1)^k C(n,k)(n-k)^m$
m–type D	n–type I	$x_j \geq 1$	$\sum_{k=0}^{n} \frac{(-1)^k}{n!} C(n,k)(n-k)^m$
n–type I	k–type I	$x_j \geq 0$	Coeff x^n $(1+x+\cdots+x^n)(1+x^2+\cdots)\cdots(1+x^n)$

I = indistinguishable *D = distinguishable*

The Master Theorem
and
Algorithmic Complexity

The Master Theorem -- Divide and Conquer

Binary Search

- $f(n) = f(n/2) + 2$

Maxima and Minima

- $f(n) = 2f(n/2) + 2$

Divide and Conquer

- $f(n) = af(n/b) + c$

  -- Divide problem into $\underline{a}$ subproblems, each of size n/b, using c additional operations.

  -- Conquer $f(n)$ by solving the smaller problems

Master Theorem

Theorem: *Let* $f(n) = a f(n/b) + c$

Then

$i. \quad f(n) = O\left(\log_b(a)\right) \qquad a > 1$

$ii. \quad f(n) = O\left(\log(n)\right) \qquad a = 1$

Proof: $f(n) = a f(n/b) + c$ $\qquad \{ f(n/b) = af(n/b^2) + c \}$

$$= a^2 f(n/b^2) + ac + c \qquad \{ f(n/b^2) = af(n/b^3) + c \}$$

$$= a^3 f(n/b^3) + a^2 c + ac + c$$

•

•

•

$$= a^k f(n/b^k) + a^{k-1} c + \cdots + ac + c \qquad (*)$$

Case 1. $a > 1$:

a. Suppose $n = b^k$.

- $k = \log_b(n) \Rightarrow a^k = a^{\log_b(n)} = n^{\log_b(a)}$ (take $\log_b$ of both sides)

Then by (*)

- $f(n) = a^k f(1) + c\dfrac{a^k - 1}{a-1} = \left(f(1) + \dfrac{c}{a-1}\right)a^k - \dfrac{c}{a-1} = O\left(n^{\log_b(a)}\right)$

b. Suppose $b^k < n < b^{k+1}$. Then since *f* is increasing

- $f(n) < f(b^{k+1}) = C_1 a^{k+1} + C_2$

$$= (C_1 a)a^k + C_2$$

$$< (C_1 a)a^{\log_b(n)} + C_2$$

$$= (C_1 a)n^{\log_b(a)} + C_2$$

$$= O\left(n^{\log_b(a)}\right)$$

Case 2. $a = 1$:

a. Suppose $n = b^k$. Then by (*)

- $f(n) = f(1) + kc = f(1) + c\log_b(n) = O\left(\log_b(n)\right)$

b. Suppose $b^k < n < b^{k+1}$. Then since *f* is increasing

- $f(n) < f(b^{k+1}) = C_1(k+1) + C_2$

$$= (C_1 + C_2) + C_1 k$$

$$= (C_1 + C_2) + C_1 \log_b(n)$$

$$= O\left(\log_b(n)\right)$$

QED

Examples

Binary Search

- $f(n) = f(n/2) + 2$
- $f(n) = O\left(\log(n)\right)$ since $a = 1$

Maxima and Minima

- $f(n) = 2f(n/2) + 2$
- $f(n) = O(n)$ since $a = b = 2$

Theorem: *Let* $f(n) = af(n/b) + cn^d$, *where* $a \geq 1$

Then

i. $f(n) = O(n^d)$ $a < b^d$

ii. $f(n) = O\left(n^d \log(n)\right)$ $a = b^d$

iii. $f(n) = O\left(n^{\log_b(a)}\right)$ $a > b^d$

Proof: Homework