

COMP 360 -- Midterm Exam (200 Points)

Below are the rules for this examination:

- a four hour time limit to finish the entire exam;
- you need not do the entire exam in one session, but once you read a problem you must finish the problem without taking time out (except for obvious emergencies);
- please either **TYPE** your answers or write all your answers in **DARK INK**. Do **NOT** write in pencil! Pencil is difficult to read. If you want to change an answer, just cross out your old answer and write your new answer. Do not try to erase.
- closed textbook, closed notes; some helpful identities appear on pages 2,3;
- all work must be entirely your own;
- you may not confer on any problems with any person except the professor;
- the exam must be handed in to no later than the start of class on Thursday, October 20.

There are 5 problems on this exam. Each problem is worth 40 points. For full credit you must do **ALL** 5 problems.

Good luck.

Dot Product

$$u \bullet v = |u| |v| \cos(\theta) \quad (\text{definition})$$

$$u \bullet v = v \bullet u \quad (\text{commutative})$$

$$u \bullet (v + w) = u \bullet v + u \bullet w \quad (\text{distributive})$$

$$(v + w) \bullet u = v \bullet u + w \bullet u$$

$$|v|^2 = v \bullet v \quad (\text{length})$$

$$\cos(\theta) = \frac{u \bullet v}{|u| |v|} \quad (\text{angle})$$

$$u_{\parallel} = \left(\frac{u \bullet v}{v \bullet v} \right) v \quad (\text{parallel projection})$$

$$u_{\parallel} = (u \bullet v)v \quad \text{if } |v| = 1$$

$$u_{\perp} = u - u_{\parallel} = u - \left(\frac{u \bullet v}{v \bullet v} \right) v \quad (\text{perpendicular projection})$$

$$u_{\perp} = u - (u \bullet v)v \quad \text{if } |v| = 1$$

$$u \bullet v = 0 \Leftrightarrow u \perp v \quad (\text{orthogonality})$$

Cross Product

$$|u \times v| = |u| |v| \sin(\theta) \quad (\text{definition})$$

$$u \times v \perp u, v$$

$$\text{sgn}(u, v, u \times v) > 0$$

$$\text{area}(u, v) = |u \times v| \quad (\text{area})$$

$$(u \times v) \bullet u = 0 \quad (\text{orthogonality})$$

$$(u \times v) \bullet v = 0$$

$$u \times u = 0 \quad (\text{parallelism})$$

$$u \times v = 0 \Leftrightarrow v \parallel \pm u$$

$$u \times (v + w) = u \times v + u \times w \quad (\text{distributive})$$

$$(v + w) \times u = v \times u + w \times u$$

$$u \times v = -v \times u \quad (\text{anti-commutative})$$

$$u \times (v \times w) = (u \bullet w)v - (u \bullet v)w \quad (\text{non-associative})$$

$$(v \times w) \times u = (u \bullet v)w - (u \bullet w)v$$

$$|u \times v|^2 = |u|^2 |v|^2 - (u \bullet v)^2 \quad (\text{length})$$

$$(u_1 \times u_2) \bullet (v_1 \times v_2) = (u_1 \bullet v_1)(u_2 \bullet v_2) - (u_1 \bullet v_2)(u_2 \bullet v_1) \quad (\text{Lagrange Identity})$$

Determinant

$\det(u,v,w) = (u \times v) \bullet w$	(definition)
$\text{vol}(u,v,w) = \det(u,v,w) $	(volume)
$\det(u,v,w) > 0 \Leftrightarrow \text{sgn}(u,v,w) > 0$	(orientation)
$\det(u,v,w) \neq 0 \Leftrightarrow u,v,w \text{ are linearly independent}$	(linear independence)
$\det(u,u,w) = \det(u,v,u) = \det(u,v,v) = 0$	
$\det(u,v,w) = \det(v,w,u) = \det(w,u,v)$	(skew symmetry)
$\det(v,u,w) = -\det(u,v,w)$	
$\det(u_1 + c u_2, v, w) = \det(u_1, v, w) + c \det(u_2, v, w)$	
$\det(u, v_1 + c v_2, w) = \det(u, v_1, w) + c \det(u, v_2, w)$	(multi-linearity)
$\det(u, v, w_1 + c w_2) = \det(u, v, w_1) + c \det(u, v, w_2)$	

1. (40 points)

Consider the three fractal curves depicted in Figure 1.

- a. Write a turtle program to generate each of these fractal curves.
- b. Describe an IFS to generate each of these fractal curves.
- c. Compute the fractal dimension for each of these fractal curves.

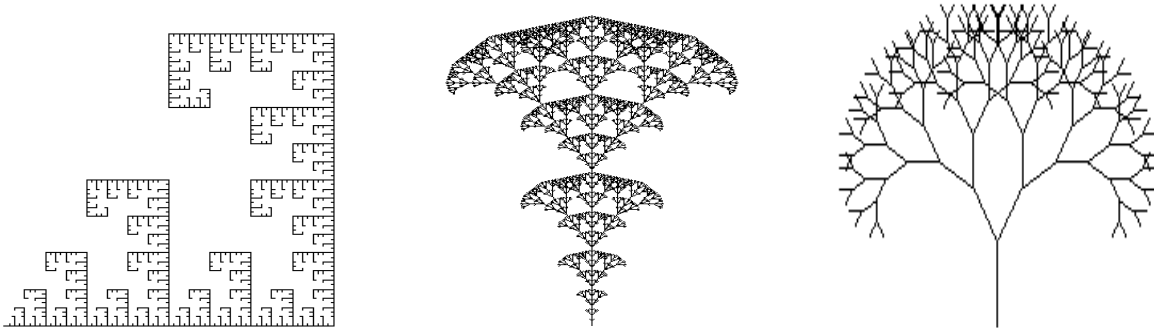


Figure 1: Three fractal curves.

2. (40 points)

Consider the three bump curves depicted in Figure 2.

- a. Write a turtle program to generate each of these bump curves.
- b. Write a recursive turtle program to generate the bump fractals corresponding to each of these bump curves.
- c. Describe the transformations in the conformal iterated function systems that generate each of these bump fractals.

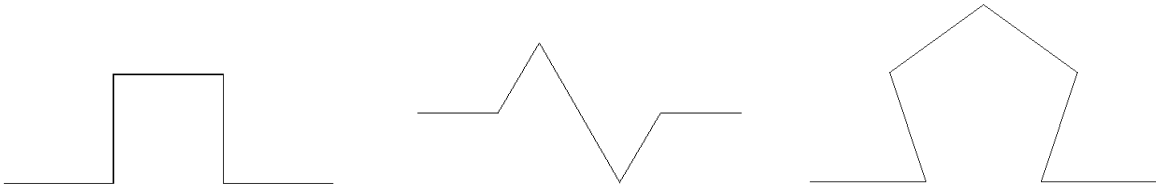


Figure 2: Three bump curves.

3. (40 points)

a. Let F be the fractal generated by the iterated function system

$W = \{w_1, \dots, w_l\}$, and let T be a nonsingular affine transformation.

i. Show that $T(F)$ is the fractal generated by the set of transformations

$$T^{-1} * W * T = \{T^{-1} * w_1 * T, \dots, T^{-1} * w_l * T\}.$$

ii. Conclude from part a that an affine transformation applied to a fractal generates another fractal.

b. Let F_i be the fractal generated by the iterated function system W_i , $i = 1, 2$.

Show that if $W_1 \subset W_2$, then $F_1 \subset F_2$.

c. Consider two iterated functions systems $F = \{f_1, \dots, f_m\}$

and $G = \{g_1, \dots, g_m\}$. Show that

$$H(t) = (1-t)F + tG = \{(1-t)f_1 + t g_1, \dots, (1-t)f_m + t g_m\}$$

is an iterated function system for all $0 \leq t \leq 1$.

4. (40 points)

Prove the following identities:

a. $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$

b. $(u_1 \times u_2) \times (v_1 \times v_2) = \det(u_1, u_2, v_2)v_1 - \det(u_1, u_2, v_1)v_2$

c. $(u_1 \times u_2) \times (v_1 \times v_2) = \det(u_1, v_1, v_2)u_2 - \det(u_2, v_1, v_2)u_1$

d. $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$

5. (40 points)

Recall that the image of three non-collinear points determines a unique affine transformation in the plane.

- a. Show that the image of four non-coplanar points determines a unique affine transformation in 3-space.
- b. Fix 2 sets of 4 points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 , and define

$$A(R) = \frac{\text{Det}(P_2 - R \ P_3 - R \ P_4 - R)Q_1 - \text{Det}(P_1 - R \ P_3 - R \ P_4 - R)Q_2}{\text{Det}(P_2 - P_1 \ P_3 - P_1 \ P_4 - P_1)} + \frac{\text{Det}(P_1 - R \ P_2 - R \ P_4 - R)Q_3 - \text{Det}(P_1 - R \ P_2 - R \ P_3 - R)Q_4}{\text{Det}(P_2 - P_1 \ P_3 - P_1 \ P_4 - P_1)}.$$

Show that

$$A(P_i) = Q_i \quad i = 1, 2, 3, 4.$$