Convex Hulls

Convex Hull

Convex Hull (2 Points)

$$Convex\ \ Hull(P_0, P_1) = \{(1-c)P_0 + cP_1 \mid c \geq 0\ \ and\ \ c \leq 1\}$$
$$= \{c_0 P_0 + c_1 P_1 \mid c_0 + c_1 = 1\ \ and\ \ c_0, c_1 \geq 0\}$$

Convex Hull ($n+1$ Points)

$$Convex\ \ Hull(P_0, \ldots, P_n) = \left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1\ \ and\ \ c_k \geq 0 \right\}$$

Lemma 1: The set $S = \left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ and \ \ c_k \geq 0 \right\}$ *is convex.*

Proof: Let $Q, R \in S$:

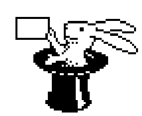
$$Q = \sum_{k=0}^{n} a_k P_k \qquad \sum_{k=0}^{n} a_k = 1 \ \ and \ \ a_k \geq 0$$

$$R = \sum_{k=0}^{n} b_k P_k \qquad \sum_{k=0}^{n} b_k = 1 \ \ and \ \ b_k \geq 0$$

Must show: $(1-t)Q + tR \in S \quad 0 \leq t \leq 1$.

$$(1-t)Q + tR = (1-t) \sum_{k=0}^{n} a_k P_k + t \sum_{k=0}^{n} b_k P_k = \sum_{k=0}^{n} \left\{ (1-t)a_k + tb_k \right\} P_k$$

- $\sum_{k=0}^{n} (1-t)a_k + tb_k = (1-t) \sum_{k=0}^{n} a_k + t \sum_{k=0}^{n} b_k = (1-t) + t = 1$
- $(1-t)a_k + tb_k \geq 0$



Lemma 2: $\left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ \text{and} \ \ c_k \geq 0 \right\} \subseteq \text{Convex Hull}(P_0, \ldots, P_n)$

Proof: By induction on n. For $n = 1$, we have equality, since both sets are equal to the line segment joining P_0 and P_1. Suppose that

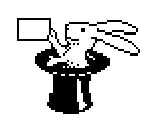
$$S_n = \left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ \text{and} \ \ c_k \geq 0 \right\} \subseteq \text{Convex Hull}(P_0, \ldots, P_n).$$

Let $\sum_{k=0}^{n+1} c_k P_k \in S_{n+1}$. Must show that $\sum_{k=0}^{n+1} c_k P_k \in \text{Convex Hull}(P_0, \ldots, P_{n+1})$.

If $c_{n+1} = 1$, then $\sum_{k=0}^{n+1} c_k P_k = P_{n+1} \in \text{Convex Hull}(P_0, \ldots, P_{n+1})$. If $c_{n+1} \neq 1$, then

$$\sum_{k=0}^{n+1} c_k P_k = (1 - c_{n+1}) \sum_{k=0}^{n} \frac{c_k}{1 - c_{n+1}} P_k + c_{n+1} P_{n+1} \in \text{Convex Hull}(P_0, \ldots, P_{n+1})$$

because by the inductive hypothesis $\sum_{k=0}^{n} \frac{c_k}{1 - c_{n+1}} P_k \in \text{Convex Hull}(P_0, \ldots, P_n)$.



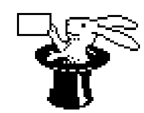
Theorem: $Convex\ \ Hull(P_0,\ldots,P_n) = \left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ and \ \ c_k \geq 0 \right\}$

Proof: By definition, $Convex\ \ Hull(P_0,\ldots,P_n)$ is the smallest convex set containing the points $P_0,\ldots,P_n$. Hence by the Lemma 1

$$Convex\ \ Hull(P_0,\ldots,P_n) \subseteq \left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ and \ \ c_k \geq 0 \right\}.$$

But by Lemma 2

$$\left\{ \sum_{k=0}^{n} c_k P_k \mid \sum_{k=0}^{n} c_k = 1 \ \ and \ \ c_k \geq 0 \right\} \subseteq Convex\ \ Hull(P_0,\ldots,P_n)\ .$$



Convex Hull Property

Curve Schemes

$$D(t) = \sum_{k=0}^{n} D_k^n(t) P_k$$

$$\sum_{k=0}^{n} D_k^n(t) \equiv 1 \quad and \quad D_k^n(t) \geq 0 \Rightarrow D(t) \in Convex \quad Hull(P_0, \ldots, P_n)$$

Bernstein Polynomials

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

$$\sum_{k=0}^{n} B_k^n(t) \equiv 1 \quad and \quad B_k^n(t) \geq 0 \quad 0 \leq t \leq 1 \qquad \text{(Inspection)}$$

Lagrange Polynomials

$$L_k^n(t \mid t_0, \ldots, t_n) = \frac{\prod_{j \neq k}(t - t_j)}{\prod_{j \neq k}(t_k - t_j)}$$

$$\sum_{k=0}^{n} L_k^n(t) \equiv 1 \quad but \quad L_k^n(t) < 0 \quad \text{for some values of } t \qquad \text{(Inspection)}$$