# Lecture 10: Coordinate Algebra

*I am not come to destroy, but to fulfill.* Matthew 5:17

## 1. Rectangular Coordinates

To introduce coordinate computations for the algebra of points and vectors, we begin by fixing an origin and three orthogonal axes. Denote the origin by $O$, and the three unit vectors along the three orthogonal axes by $i,j,k$ (see Figure 1).

Any vector $v$ can be written in a unique way as a linear combination of the three basis vectors $i,j,k$. Thus for any vector $v$, there are three scalars $v_1,v_2,v_3$ such that

$$v = v_1 i + v_2 j + v_3 k .$$

The scalars $v_1,v_2,v_3$ are called the *rectangular coordinates* of the vector $v$, and the notation $v = (v_1,v_2,v_3)$ is simply shorthand for the equation $v = v_1 i + v_2 j + v_3 k$.

Similarly, any point $P$ can be written as $P = O + (P - O)$. Since $P - O$ is a vector, there are three scalars $p_1,p_2,p_3$ such that

$$P - O = p_1 i + p_2 j + p_3 k .$$

The scalars $p_1,p_2,p_3$ are called the *rectangular coordinates* of the point $P$, and the notation $P = (p_1,p_2,p_3)$ is once again simply shorthand for the equation $P = O + p_1 i + p_2 j + p_3 k$.

With these definitions and this notation in hand, we are now ready to introduce coordinate computations for the linear algebra of points and vectors.

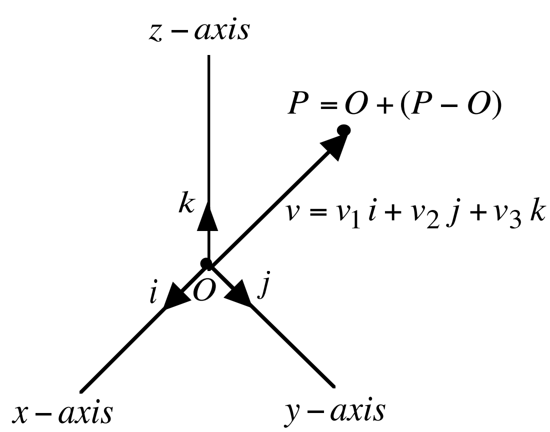


**Figure 1:** A rectangular coordinate system consists of an origin and three orthogonal axes. Every vector can be written as a unique linear combination of the three orthogonal unit vectors along the coordinate axes, and every point is the sum of the origin and the vector from the origin to the point.

## 2. Addition, Subtraction, and Scalar Multiplication

The rules for computing sums, differences, and scalar products in terms of coordinates follow directly from the associative, commutative, and distributive properties of addition, subtraction, and scalar multiplication. Thus

$$u \pm v = (u_1\, i + u_2\, j + u_3\, k) \pm (v_1\, i + v_2\, j + v_3\, k) = (u_1 \pm v_1) i + (u_2 \pm v_2) j + (u_3 \pm v_3) k$$

$$\therefore \quad u \pm v = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3)\,.$$

Similarly,

$$P \pm v = (O + p_1 i + p_2\, j + p_3\, k) \pm (v_1 i + v_2\, j + v_3\, k) = O + (p_1 \pm v_1) i + (p_2 \pm v_2) j + (p_3 \pm v_3) k$$

$$\therefore \quad P \pm v = (p_1 \pm v_1, p_2 \pm v_2, p_3 \pm v_3)\,,$$

and

$$Q - P = (O + q_1\, i + q_2\, j + q_3\, k) - (O + p_1 i + p_2\, j + p_3\, k) = (q_1 - p_1) i + (q_2 - p_2) j + (q_3 - p_3) k$$

$$\therefore \quad Q - P = (q_1 - p_1, q_2 - p_2, q_3 - p_3)\,.$$

Also, since scalar multiplication distributes through addition,

$$c\, u = c(u_1\, i + u_2\, j + u_3\, k) = c\, u_1\, i + c\, u_2\, j + c\, u_3\, k$$

$$\therefore \quad c\, u = (c\, u_1, c\, u_2, c\, u_3)\,.$$

## 3. Vector Products

The rules for computing vector products in terms of rectangular coordinates are derived in two stages:

i. Rules for products of the basis vectors are derived directly from the coordinate free definitions.
ii. Rules for the products of arbitrary vectors are derived from the rules for the basis vectors and the distributive property of multiplication.

### 3.1 Dot Product.

**3.1 Dot Product.** Recall that by definition,

$$u \bullet v = |u|\,|v|\cos(\theta)\,,$$

where $\theta$ is the angle between the vectors $u$ and $v$. Hence since the basis vectors $i, j, k$ are orthogonal unit vectors, we have the following rules:

$$i \bullet i = j \bullet j = k \bullet k = 1$$

$$i \bullet j = j \bullet i = j \bullet k = k \bullet j = k \bullet i = i \bullet k = 0\,.$$

Therefore by the distributive property of the dot product:

$$u \bullet v = (u_1 i + u_2\, j + u_3\, k) \bullet (v_1\, i + v_2\, j + v_3\, k) = u_1 v_1 + u_2 v_2 + u_3 v_3\,.$$

**3.2 Cross Product.** Recall that by definition,

$$|u \times v| = |u|\,|v|\,\sin(\theta)$$

$$u \times v \perp u,v$$

$$\mathrm{sgn}(u,v,u \times v) = 1$$

where $\theta$ is the angle between the vectors $u$ and $v$. Hence since the basis vectors $i,j,k$ are orthogonal unit vectors, we have the following rules:

$$i \times i = j \times j = k \times k = 0$$

$$i \times j = k, \quad j \times k = i \quad k \times i = j$$

$$j \times i = -k, \quad k \times j = -i \quad i \times k = -j\,.$$

Therefore by the distributive property of the cross product:

$$\begin{aligned} u \times v &= (u_1 i + u_2 j + u_3 k) \times (v_1 i + v_2 j + v_3 k) \\ &= (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k, \end{aligned}$$

so

$$u \times v = \left(u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1\right).$$

Equivalently, expanding by cofactors of the first row yields

$$u \times v = \det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

**3.3 Determinant.** We can derive an expression for computing the determinant of three vectors from the rules for computing the dot product and cross product. Recall that

$$\det(u,v,w) = (u \times v) \bullet w\,.$$

Therefore,

$$\det(u,v,w) = \left(u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1\right) \bullet (w_1, w_2, w_3),$$

so

$$\det(u,v,w) = (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3\,.$$

Equivalently, expanding by cofactors of the third column yields

$$\det(u,v,w) = \det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}.$$

## 4. Summary

Although we do not need coordinates to construct an algebra for points and vectors, we do need coordinates to communicate this algebra to a computer. For easy reference, we collect below the low level coordinate computations corresponding to the high level algebraic operations.

*Addition and Subtraction*

$$u \pm v = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3)$$

$$P \pm v = (p_1 \pm v_1, p_2 \pm v_2, p_3 \pm v_3)$$

$$Q - P = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$$

*Scalar Multiplication*

$$c\,u = (c\,u_1, c\,u_2, c\,u_3)$$

*Dot Product*

$$u \bullet v = u_1v_1 + u_2v_2 + u_3v_3$$

*Cross Product*

$$u \times v = \left(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\right) = \det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

*Determinant*

$$\det(u,v,w) == \det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

Notice that the cross product is much more expensive to compute than the dot product: cross product requires 6 multiplications and 3 subtractions, whereas dot product uses only 3 multiplications and 2 additions. Thus the cross product is about twice as expensive to compute as the dot product. For this reason given a choice, we usually prefer formulas that invoke dot product rather than cross product. The Lagrange identity from the previous lecture allows us to replace two cross products and a dot product by four dot products.

**Exercises:**

1. Explain the computational significance of Lecture 9, Exercise 1.

2. Using a computer algebra system such as *Mathematica* or *Maple* to simplify the coordinate computations, verify symbolically the following identities:

   a. $(\mathrm{v} \times \mathrm{w}) \times \mathrm{u} = (\mathrm{u} \bullet \mathrm{v})\mathrm{w} - (\mathrm{u} \bullet \mathrm{w})\mathrm{v}$

   b. $(\mathrm{u}_1 \times \mathrm{u}_2) \bullet (\mathrm{v}_1 \times \mathrm{v}_2) = (\mathrm{u}_1 \bullet \mathrm{u}_2)(\mathrm{v}_1 \bullet \mathrm{v}_2) - (\mathrm{u}_1 \bullet \mathrm{v}_2)(\mathrm{u}_2 \bullet \mathrm{v}_1)$

3. Show that if $v_1 = (r_1, \theta_1, z_1)$ and $v_2 = (r_2, \theta_2, z_2)$ are cylindrical coordinates for $v_1, v_2$, then

   $$v_1 \bullet v_2 = r_1r_2\cos(\theta_1 - \theta_2) + z_1z_2.$$

4. Using the formula $u \bullet v = u_1 v_1 + u_2 v_2 + u_3 v_3$, verify the commutative and distributive properties of the dot product.

5. Using the formulas

$$u \bullet v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$u \times v = \det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix},$$

verify the following identities:

a. $u \times (v + w) = u \times v + u \times w$

b. $(v + w) \times u = v \times u + w \times u$

c. $u \times u = 0$

d. $u \times v = -v \times u$

e. $(u \times v) \bullet u = 0$

f. $(u \times v) \bullet v = 0$

g. $|u \times v|^2 = |u|^2 |v|^2 - (u \bullet v)^2$

6. Using the formula

$$\det(u,v,w) = \det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix},$$

verify the following identities:

a. $\det(u,u,w) = \det(u,v,u) = \det(u,v,v) = 0$

b. $\det(u,v,w) = \det(v,w,u) = \det(w,u,v)$

c. $\det(v,u,w) = -\det(u,v,w)$

d. $\det(u_1 + c\,u_2, v, w) = \det(u_1, v, w) + c\det(u_2, v, w)$

e. $\det(u, v_1 + c\,v_2, w) = \det(u, v_1, w) + c\det(u, v_2, w)$

f. $\det(u, v, w_1 + c\,w_2) = \det(u, v, w_1) + c\det(u, v, w_2)$

In the following exercises we will develop an alternative approach to deriving expressions for the dot and cross products in terms of rectangular coordinates by a method that does not invoke the distributive law (Lecture 9, Exercise 6).

7. Recall from Lecture 4 that in 2-dimensions the matrix representing rotation of vectors around the origin by the angle $\theta$ is given by

$$Rot(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

a. Show that in 3-dimensions, the matrices representing rotation of vectors around the coordinate axes are given by

i. $Rot(k,\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ -- rotation around the *z*-axis

ii. $Rot(j,\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$ -- rotation around the *y*-axis

iii. $Rot(i,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$ -- rotation around the *x*-axis

b. Show that

i. $Rot(k,\theta)^T = Rot(k,\theta)^{-1}$

ii. $Rot(j,\theta)^T = Rot(j,\theta)^{-1}$

iii. $Rot(i,\theta)^T = Rot(i,\theta)^{-1}$

8. Let *R* be a matrix representing a rotation about an arbitrary vector in 3-dimensions.
   a. Show that *R* is equivalent to a product of rotations around the coordinate axes.
   b. Using part *a* and Exercise 7, conclude that $R^T = R^{-1}$.

9. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, and let $\theta$ be the angle between *u* and *v*. Show that:
   a. $u * v^T = u_1 v_1 + u_2 v_2 + u_3 v_3$.
   b. For any rotation matrix *R*
      $(u * R) * (v * R)^T = u * v^T$.
   c. If *u* lies along the *x*-axis and $v$ lies in the *xy*-plane, then
      $u * v^T = |u|\,|v| \cos(\theta)$.
   d. There exists a rotation matrix *R* such that $u * R$ lies along the *x*-axis and $v * R$ lies in the *xy*-plane.
   e. Conclude from part *a-d* that:
      $u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\,|v| \cos(\theta) = u \bullet v$.

10. Let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$, and let $\theta$ be the angle between $u$ and $v$. Set

$$u \bullet v = u_1v_1 + u_2v_2 + u_3v_3$$

$$u \times v = \left(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\right)$$

a. Using these definitions, show that:

i. $(u \times v) \bullet (u \times v) = (u \bullet u)(v \bullet v) - (u \bullet v)^2$

ii. $(u \times v) \bullet u = (u \times v) \bullet v = 0$

iii. $\det(u,v,u \times v) = (u \times v) \bullet (u \times v)$

b. Using Exercise 9 and part *a,* show that:

iv. $|u \times v| = |u||v|\sin(\theta)$

v. $u \times v \perp u,v$

vi. $\operatorname{sgn}(u,v,u \times v) > 0$

c. Conclude from part *b* that $\left(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\right)$ represents the cross product in terms of rectangular coordinates.