One-dimensional Analytic (Finite Element) Spring Models (draft 4 Feb 20)

Introduction

In basic physics, we learn to utilize springs connected in series and/or in parallel. Elastic bodies behave as springs. For 1-D bars, shafts, and beams we can utilize their analytic stiffness relations to derive approximate (and sometimes exact) analytic solutions at discrete points on the body. That allows us to treat axial forces, axial torsion, and transverse forces in simple models.

The equilibrium relation for a simple spring between its stiffness, *k*, displacement, *u*, and axial force, *f*, is usually seen as: $k\,u = f$, or $u = f/k$. This simple form arises only because we assumed that one end of the spring was always restrained from displacement (fixed), and the opposite end was subjected to a constant force. If the spring model is generalized to allow either end to be restrained or loaded then the equilibrium equation takes a matrix form:

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}, \tag{1}$$

where the subscripts 1 and 2 refer to the left and right ends of the spring, respectively. Note that the determinant of the matrix is zero. That is because were must later add restraint information about at least one end to obtain a unique physical solution.

For example, assume that the left node has a known displacement, $u_1 = u_{given}$, (which is usually zero) and the right end has a known force, $f_2 = F$. Then, the unknowns are the right displacement, u_2, and the left end reaction force, say $f_1 = R$. The revised equilibrium relation is

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{given} \\ u_2 \end{Bmatrix} = \begin{Bmatrix} R \\ F \end{Bmatrix},$$

and the independent displacement is found from the second row:

$$k[-u_{given} \quad u_2] = F$$

or

$$u_2 = u_{given} + F/k.$$

This is the same as the common form when u_{given} is zero, namely $u_2 = F/k$. Now the reaction force necessary to maintain u_{given} is found from the first row of the matrix system:

$$k[u_{given} \quad -(u_{given} + F/k)] = R$$

or simply $R = -F$, as expected.

Single bar spring

An elastic bar acts like a simple spring. However, in addition to end point loads it can have distributed mechanical loads per unit length, and/or thermal loading due to a temperature change, say ΔT, over its length. The resultants of such effects are lumped at the ends as additional point loads. For a bar with a cross-sectional area, *A*, length, *L*, a distributed load per unit length of w_1 on the left and w_2 on the right, and material with an elastic modulus of *E* and a coefficient of thermal expansion of *α*, the corresponding matrices are

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \alpha \Delta T E A \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{2}$$

where, again, the f_k terms represent external point loads or reactions. The ratio *k* = *EA/L* is called the axial stiffness of a bar. If the line load is constant then the second load vector (transposed) reduces to $F_w^T = \frac{wL}{2}[1 \quad 1]$ which places half the total appled line load at each end of the bar. The displacement between the two ends is assumed linear. That causes the strain to be constant $\epsilon = \Delta L/L = (u_2 - u_1)/L$, which is only correct at the mid-length for a line load.

Multiply spring connections

When a system of springs is combined in series or parallel, they share (sum) square matrix diagonal terms and source vector row terms at any connecting points (or rows) in the system matrix equilibrium equation, see Figure 1. The size of the overlapping summed regions depends on how many degrees of freedom (dof) are at a connection point. Usually it is only a single shared term, but for a beam there is usually a deflection and slope at the connection and two rows are summed. The (generalized) forces from the individual spring internal forces (or sources) at a node connection equals the resultant external point force there. Often the resultant is zero (the external force is absent). If the node point is restrained then the external resultant is an unknown point reaction.

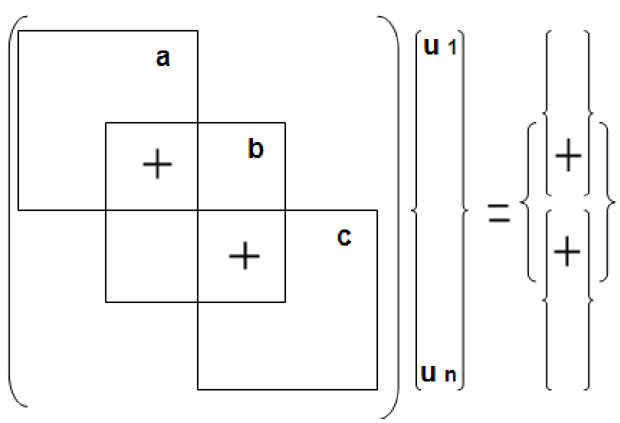

Figure 1 Overlapping element sums at connection points

Consider two springs, say *a* and *b*, connected in series. Let the first (left) point be restrained while the third (right) point has a known force, *F*. The second (middle) has no resultant external force applied. Then the assembled system equilibrium matrices become:

$$\begin{bmatrix} k^a & -k^a & 0 \\ -k^a & (k^a + k^b) & -k^b \\ 0 & -k^b & k^b \end{bmatrix} \begin{Bmatrix} u_1 = u_{given} \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^a \\ f_2^a + f_1^b \\ f_2^b \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}.$$

This is a set of three equations for three unknowns: u_2, u_3 and R. The reaction cannot be found until the displacements are known, so we use the last two rows to do that (with u_{given} zero):

$$\begin{bmatrix} (k^a + k^b) & -k^b \\ -k^b & k^b \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix} - u_{given} \begin{Bmatrix} -k^a \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

so inverting the 2 by 2 square matrix gives the displacements

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{d} \begin{bmatrix} k^b & k^b \\ k^b & (k^a + k^b) \end{bmatrix} \begin{Bmatrix} 0 \\ F \end{Bmatrix} = \frac{F}{d} \begin{Bmatrix} k^b \\ k^a + k^b \end{Bmatrix}$$

where the determinant is $d = k^a k^b$. Returning to the top row gives the necessary reaction

$$\frac{F}{k^a k^b} [0 \quad -k^a k^b \quad 0] = -F = R$$

which gives the reaction as equal and opposite to the applied force, as expected.

As a second example, let the above springs represent a bar with the first half having an area of *2A* while the last half has an area of *A*. The two axial stiffness's *k* = *EA* / *L* are $k^a = E(2A)/(L/2) = 4EA/L$ and $k^b = EA/(L/2) = 2EA/L$. Then the deflections are

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{FL}{4EA} \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

and the narrow right end's relative deflection is only half as much as the thicker left half.

Individual bar end forces

Because there was no distributed load in the above example, it is simple enough to see that the force in each segment of the bar is clearly just *F*. However, there is a standard process for finding the end reactions on each spring once the displacements of the system are found. Simply substitute the known displacements for each individual bar into Equation 2 (including it's line load vector). To illustrate all these features, and to explain the stress recovery, a numerical example will be given for a compound axial bar.

Compound bar

A large compound hanging bar made of an upper steel section and lower brass section carries it's weight and a point end load of 10,000 pounds (see Figure 2). There is no thermal loading. The properties of the system are tabulated as:

Element	Length	Area	Modulus	Specific Weight	Connections	
1	420 "	10 sq.in.	30e6 psi	0.283 lb/cubic in.	1	2
2	240 "	8 sq.in.	13e6 psi	0.300 lb/cubic in.	2	3

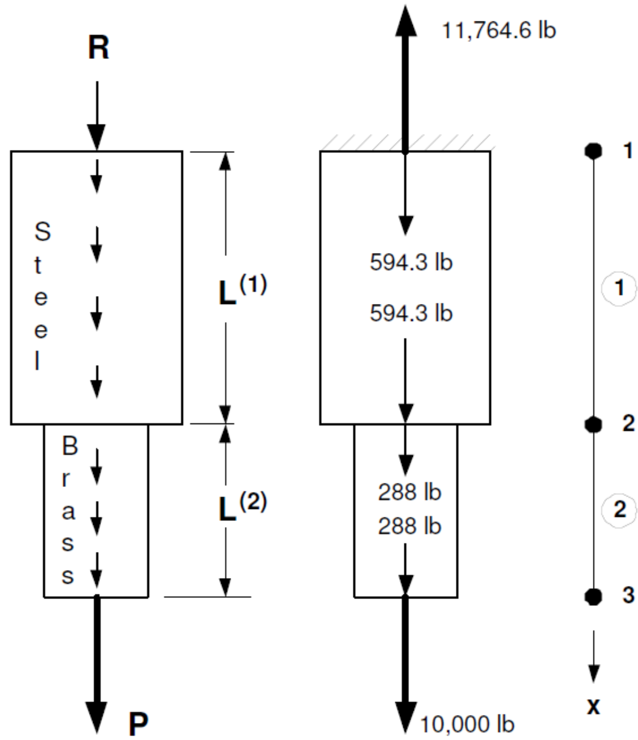

Figure 2 Steel-Brass compound hanging structure

Those data show that the axial stiffness of the steel bar is $k^a = 7.143e5\ lb/in$, its total weight is $W^a = 1{,}188.6$ lb corresponding to a distributed load of $w^a = 2.83\ lb/in$. For the lower brass section the data are: $k^b = 4.333e5\ lb/in, W^b = 576\ lb, w^b = 2.4\ lb/in$. Assembling the two axial members gives the system equilibrium equation of:

$$10^5 \begin{bmatrix} 7.143 & -7.143 & 0 \\ -7.143 & (7.143 + 4.333) & -4.333 \\ 0 & -4.333 & 4.333 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 10{,}000 \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} 1{,}188.6 \\ 1{,}188.6 + 576 \\ 576 \end{Bmatrix}$$

where R is the unknown top reaction force. From the bottom two rows

$$10^5 \begin{bmatrix} 11.476 & -4.333 \\ -4.333 & 4.333 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 882.3 \\ 10{,}288 \end{Bmatrix}$$

The displacements are determined to be $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 10^{-2} \begin{Bmatrix} 1.5638 \\ 3.9381 \end{Bmatrix} inches$. From the top row the reaction is recovered as $10^5[0 - 7.143(0.015638) + 0] = \{R\} + \{594.3\}$, or $R = -11{,}764.6\ lb$ which is equal and opposite to the combined weights and the bottom point load, as expected.

As discussed above, the two end reactions are found by inserting the system displacements into each bar's equation of equilibrium. For the top steel bar:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^a \\ u_2^a \end{Bmatrix} = \begin{Bmatrix} f_1^a \\ f_2^a \end{Bmatrix} + \frac{wL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \alpha\Delta TEA\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \text{ or}$$

$$7.143e5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 10^{-2}\begin{Bmatrix} 0 \\ 1.5638 \end{Bmatrix} = \begin{Bmatrix} f_1^a \\ f_2^a \end{Bmatrix} + \frac{1{,}188.6}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

so $\begin{Bmatrix} f_1^a \\ f_2^a \end{Bmatrix} = \begin{Bmatrix} -11{,}764.4 \\ 10{,}576 \end{Bmatrix} lb$. Likewise, for the lower bar $\begin{Bmatrix} f_1^b \\ f_2^b \end{Bmatrix} = \begin{Bmatrix} -10{,}576 \\ 10{,}000 \end{Bmatrix} lb$. These member and system reactions are sketched in Figure 3 to illustrate that they are in equilibrium with each other, and the external loads.

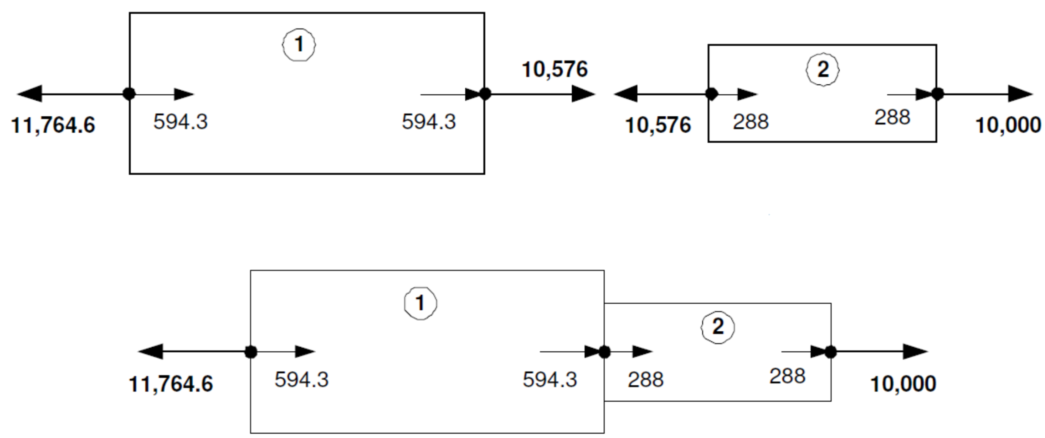

Figure 3 Member and combined system reaction equilibrium

Stress recovery

There are consistent methods for estimating stresses in two node analytic springs. However, they are not generally accurate when line loads are present. Instead, the use of basic mechanics and the above member reactions will yield exact stresses at both ends of the member. In the above axial bar example remember that the axial stress, σ, is simply $\sigma = F/A$ where *F* is the axial force acting over area *A*. For the steel bar the two end stresses are simply

$$\begin{Bmatrix} \sigma_1^a \\ \sigma_2^a \end{Bmatrix} = \begin{Bmatrix} 11{,}764.4 \\ 10{,}576 \end{Bmatrix} lb/(10\ \text{sq. in}) = \begin{Bmatrix} 1.18 \\ 1.06 \end{Bmatrix} \text{ksi, tension.}$$

The stress is tension because the *external* end forces are acting away from the member. For the brass member,

$$\begin{Bmatrix} \sigma_1^b \\ \sigma_2^b \end{Bmatrix} = \begin{Bmatrix} 10{,}576 \\ 10{,}000 \end{Bmatrix} lb/(8\ \mathrm{sq.in}) = \begin{Bmatrix} 1.32 \\ 1.25 \end{Bmatrix} \mathrm{ksi, tension.}$$

Assuming a linear axial stress distribution between the two ends results in the sketch of stresses shown in Figure 4. Note that the stress is discontinuous due to the change in the two cross-sectional areas (even though the axial force is continuous). In this example, the linear stress change in both bars is exact.

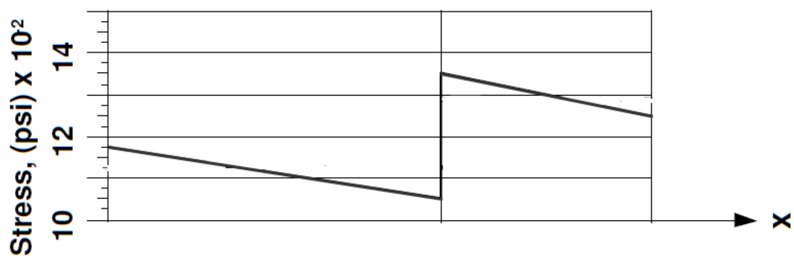

Figure 4 Stress in the steel (left) and brass bars

The finite element theory for the stress in such a bar is only accurate at the middle of the bar. It is given by

$$\sigma = \frac{E}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{E}{L}(u_2 - u_1) = \frac{E\Delta L}{L} = E\varepsilon.$$

Here, ε is the strain and its value (change in length over original length) is exact only for a bar with a constant point load at one end. Thus, in this example a combination of basis solid mechanics and the element stiffness matrix and load vectors is more accurate than the complete finite element theory. That is only because we are employing only one or two analytic elements whereas fully numerical solutions can use hundreds of small elements to give accurate numerical results everywhere in the system.

Single bars with more than two nodes each are available and will be discussed in a later section. Additional bar examples to illustrate these techniques will be given in the Appendix. Next, the analytic spring example will be extended to include the transverse bending of beams.

Analytic beam members

An elastic beam acts like a spring with two nodal degrees of freedom: the transverse displacement, *v*, and the slope, θ.. However, in addition to end point transverse shear load, *V*, and moment, *M*, it can have distributed transverse loads per unit length, *q(x),* and/or thermal moment due to a temperature change, say ΔT, through its thickness, *h*, from bottom to the top. The resultants of such effects are lumped at the ends as additional point forces and/or moments. For a beam with a cross-sectional moment of inertia, *I*, length, *L*, and material with an elastic modulus of *E* and a coefficient of thermal expansion of *α*, the corresponding matrices are

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} + \frac{L}{60}\begin{bmatrix} 21 & 9 \\ 3L & 2L \\ 9 & 21 \\ -2L & -3L \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \frac{\alpha \Delta T EI}{h}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}, \qquad (3)$$

where q_1 and q_2 are the load per unit length at the first (left) and second end, respectively. This member is call the cubic beam because the deflection, $v(x)$, varies cubically with *x*.

Triangular line load

Consider a cantilever beam with a triangular line load shown in Figures 5 and 6. The resultant end forces and moments from the line load become:

$$F_w = \frac{L}{60}\begin{bmatrix} 21 & 9 \\ 3L & 2L \\ 9 & 21 \\ -2L & -3L \end{bmatrix}\begin{Bmatrix} 0 \\ w \end{Bmatrix} = \frac{wL}{60}\begin{Bmatrix} 9 \\ 2L \\ 21 \\ -3L \end{Bmatrix}.$$

The first (left) end is free, so $V_1 = 0,\ M_1 = 0$, and the wall reaction shear and moment acts on the second node. The resultant load $R = wL/2$ acts at a third of the distance from the wall. From the bottom of Figure 5 you see that analytic model puts 30% at the left end and 70% at the right end.

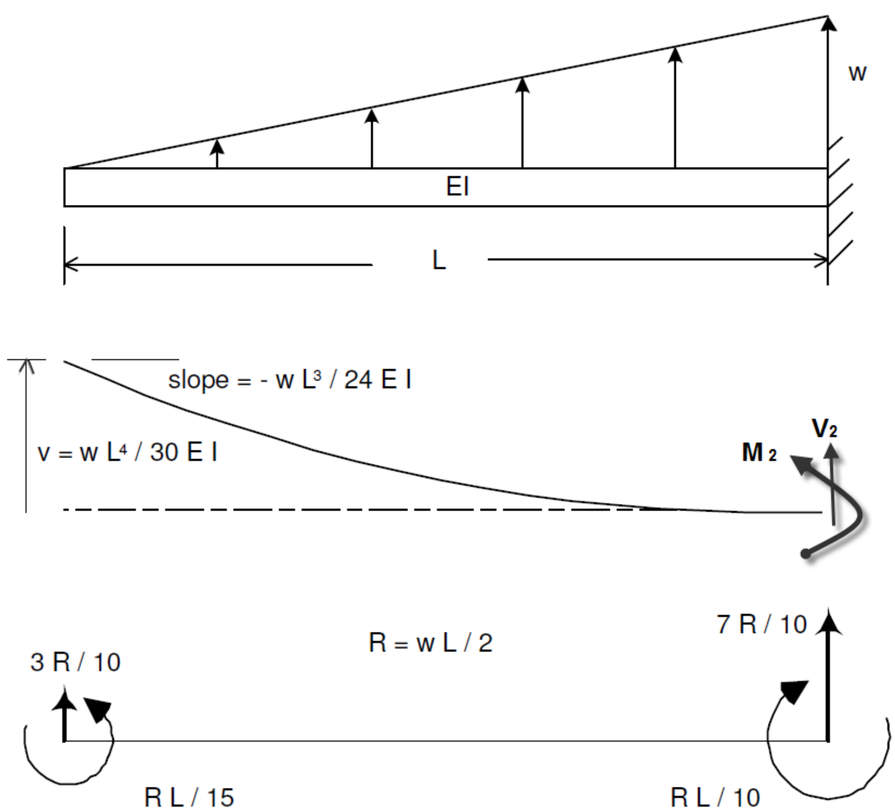

Figure 5 Line load deflections and resultant member loadings

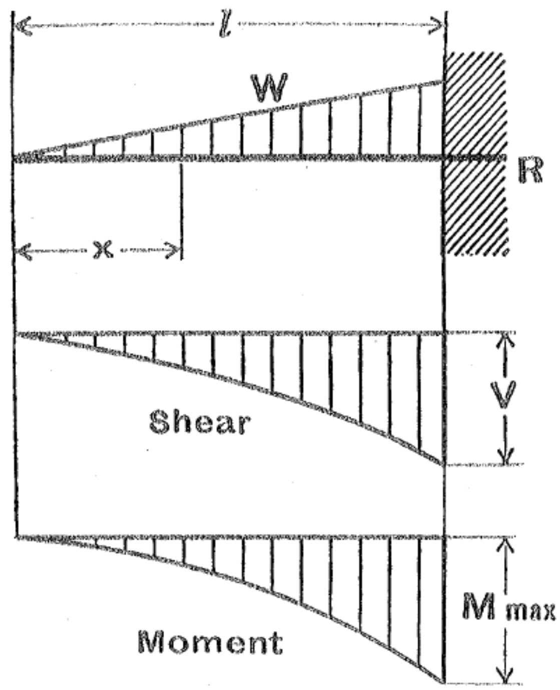

Figure 6 Exact free-fixed moment and shear diagrams

The analytic deflection solution (for no temperature change) comes from the top two rows of Eq. 3, since the right end deflection and slope are zero:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \frac{wL}{60}\begin{Bmatrix} 9 \\ 2L \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Inverting the square matrix gives the free end deflections as:

$$\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \frac{L^3}{12EIL^2}\begin{bmatrix} 4L^2 & -6L \\ -6L & 12 \end{bmatrix}\frac{wL}{60}\begin{Bmatrix} 9 \\ 2L \end{Bmatrix} = \frac{wL^3}{120}\begin{Bmatrix} 4L \\ -5 \end{Bmatrix}.$$

These are analytically exact. The known solution is a fifth degree polynomial, namely

$$120\, EI\, v(x) = wL^4[4 - 5\, x/L + (x/L)^5].$$

That means the exact moment in the member ($EI\, v''$) will be a cubic polynomial, while the transverse shear force ($EI\, v'''$) will be a quadratic polynomial. Nevertheless, we can get accurate end moments and shears from the system and member reactions. The system reactions comes from the last two rows of Eq. 4:

$$\frac{EI}{L^3}\begin{bmatrix} -12 & -6L \\ 6L & 4L^2 \end{bmatrix}\frac{wL^3}{120}\begin{Bmatrix} 4L \\ -5 \end{Bmatrix} = \begin{Bmatrix} V_2 \\ M_2 \end{Bmatrix} + \frac{wL}{60}\begin{Bmatrix} 21 \\ -3L \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 \\ M_2 \end{Bmatrix} = \frac{wL}{6} \begin{Bmatrix} -3 \\ L \end{Bmatrix}$$

which are again exact.

Since each beam has four constants, the deflection of any point along the length is approximated by a cubic polynomial (for $r = x/L$):

$$v(x) = v_1(1 - 3r^2 + 2r^3) + \theta_1(r - 2r^2 + r^3)L + v_2(3r^2 - 2r^3) + \theta_2(r^3 - r^2)L$$

The flexural and shear stresses at any point depend on the moment and transverse shear force at the section, respectively. Knowing the analytic moments and shear forces at both ends of the beam we could estimate the internal moments and shear as straight lines between the two values. The exact moment and shear diagrams have cubic and quadratic variations, respectively. They are shown in non-dimensional form in Figure 7. The analytic estimate has a large error, but is conservative in this example. Other examples (such as this beam with both ends fixed) show the analytic moment and shear estimates to be highly non-conservative.

The consistent finite element theory for the cubic beam moment and shear are linear and constant along the beam length, respectively. Specifically, they are:

$$M(x) = [v_1(12r - 6) + \theta_1(6r - 4)L + v_2(6 - 12r) + \theta_2(6r - 2)L]/L^2$$

$$V(x) = [v_1(12) + \theta_1(6)L + v_2(-12) + \theta_2(6)L]/L^3.$$

For the current example, substituting the tip deflections into these expressions gives

$$M(x) = \frac{wL^2}{120}[4(12r - 6) - 5(6r - 4)] = \frac{wL^2}{120}[18r - 4]$$

$$V(x) = \frac{3wL}{20}$$

Those crude estimates are shown in Figure 8.

Of course numerical solutions with many elements give very good results. Here we are emphasizing quick analytic estimates based on a mixture of solid mechanics theory and finite element theory. If you used two analytic elements you would get much better moment and shear estimates (try it).

Thermal load through the depth

Now consider the cantilever with only a temperature change (cooler on top) that is constant along its full length. For a thermal loading only the right hand side changes to:

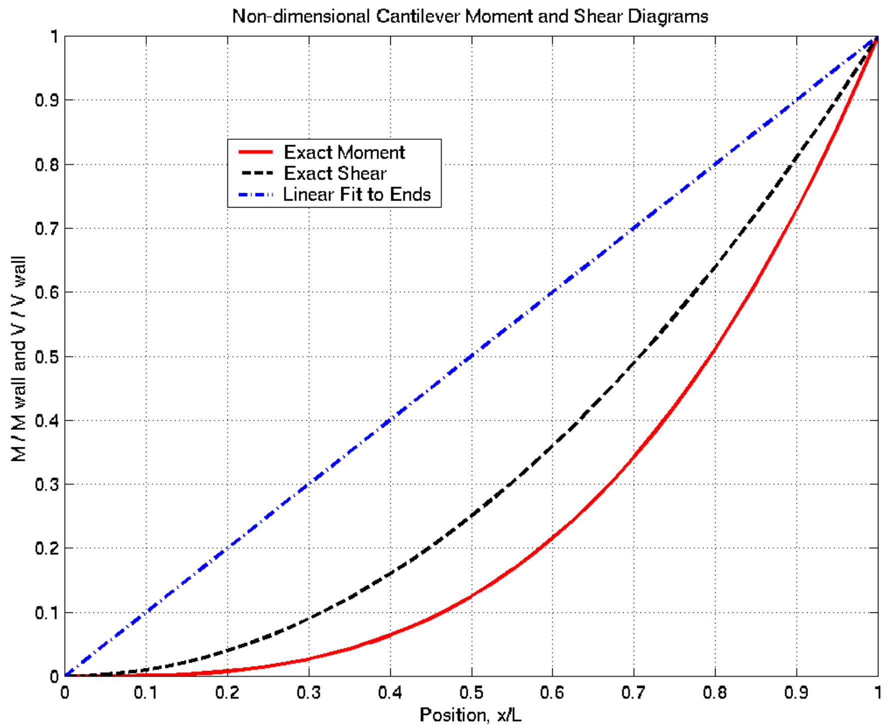

Figure 7 Analytic versus exact moment and shear distribution

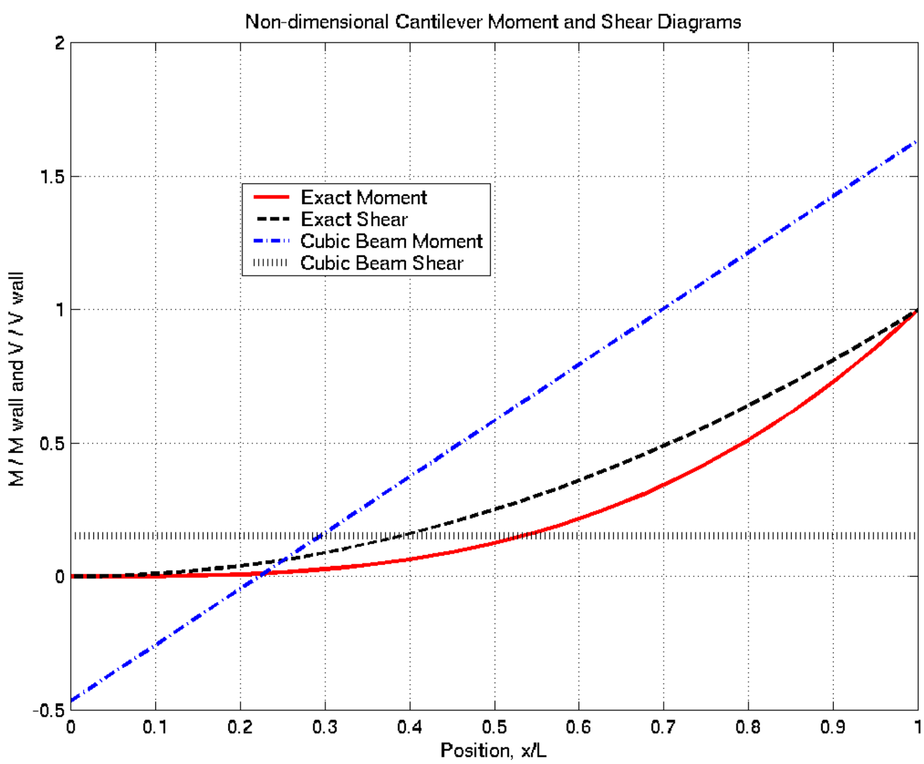

Figure 8 Cubic beam versus exact moment and shear

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & 4L^2 & 6L \\ 6L & 4L^2 & -6L & 12 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ V_2 \\ M_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{\alpha\Delta TEI}{h}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}.$$

Since the right end deflection and slope are zero the independent displacement relations (for a negative temperature change) are obtained from the top two rows of the equilibrium equations:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \frac{-\alpha\Delta TEI}{h}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

and the free tip deflections (similar to the top of Figure 5) are

$$\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \frac{\alpha\Delta TL}{12h}\begin{Bmatrix} 6L \\ -12 \end{Bmatrix}$$

which results in wall reactions of

$$\begin{Bmatrix} V_2 \\ M_2 \end{Bmatrix} = \frac{\alpha\Delta TEI}{h}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

This means that since the beam was free to expand, there is no thermally introduced wall reaction shear or moment. That will not be true for a statically indeterminate beam. For example, had the beam been fixed-fixed the two end reaction moments would have been $\mp \alpha\Delta TEI/h$. That is, the beam would have a constant moment and corresponding flexural stresses, but no transverse shear force.

Beam-column member

When you have a member that carries both axial loads, like the bar, and transverse loads, like the beam, the member is known as a column or beam-column. For the small deflection analytic models given above the axial and transverse effects are assumed uncoupled and you just solve both models independently. However, for an existing axial load, large deflections or buckling that is not true and an additional coupling matrix (the geometric stiffness) must be added to the matrix equations.

Members with three nodes

For problems with discontinuous materials, cross-sections, and or line loading the analytic models are usually simpler when various two-noded elements are assembled with nodes at the location of any such discontinuity. Usually accurate or exact displacements are obtained at those nodes. However, internal displacements are typically less accurate when line loads are present. For continuous data, more accurate solutions are obtained by using three-noded elements. That comes at the expense of often solving larger algebraic problems. Here the nodes are numbered sequentially, with the second node being at the mid-point of the total length, *L*.

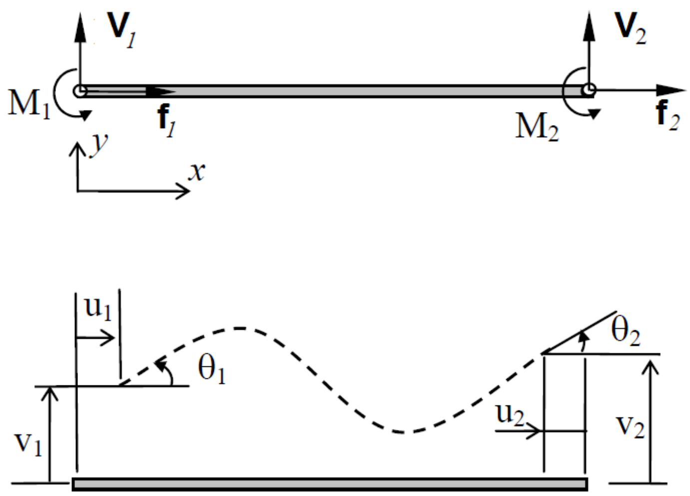

Figure 9 Combined bar and beam response

Three-noded bar

For this quadratic displacement bar Eq. 2 expands to:

$$\frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} + \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} + \alpha\Delta TEA\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad (4)$$

for a quadratic (three point) variation in the line load. A constant line load reduces the resultant load vector to

$$F_w^T = \frac{wL}{6}[1 \quad 4 \quad 1], \ (for\ w_1 = w_2 = w_3 = w)$$

while an increasing triangular load yields

$$F_w^T = \frac{wL}{3}[0 \quad 2 \quad 1], \ (for\ w_1 = 0,\ w_2 = w/2, w_3 = w).$$

Generally, you want to substitute the three nodal line load values, w_k, to obtain the F_w resultant vector before assembling with other elements. If you want to interpolate the solution or strain (gradient) between the nodes, the interpolation functions are (for $r = x/L$):

$$u(x) = u_1(1 - 3r + 2r^2) + u_2(4r - 4r^2) + u_3(-r + 2r^2)$$

$$du/dx = [u_1(-3 + 4r) + u_2(4 - 8r) + u_3(-1 + 4r)]/L.$$

Three-noded beam

For this quintic beam Eq 3 expands to:

$$\frac{EI}{35L^3}\begin{bmatrix} 5,092 & 1,138L & -3,584 & 1,920L & -1,508 & 242L \\ 1,138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3,584 & -896L & 7,168 & 0 & -3,584 & 896L \\ 1,920L & 320L^2 & 0 & 1,280L^2 & -1,920L & 320L^2 \\ -1,508 & -242L & -3,584 & -1,920L & 5,092 & -1,138L \\ 242L & 38L^2 & 896L & 320L^2 & -1,138L & 332L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} =$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} + \frac{L}{14,700}\begin{bmatrix} 1,995 & 1,540 & -105 \\ 105L & 140L & 0 \\ 560 & 6,720 & 560 \\ -280L & 0 & 280L \\ -105 & 1,540 & 1,995 \\ 0 & -140L & -105L \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} + \frac{\alpha\Delta TEI}{h}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix} \quad (5)$$

for a quadratic (three point) variation in the transverse line load. A constant transverse load reduces the resultant load and moment vector to

$$F_w^T = \frac{wL}{14,700}[3,430 \quad 245L \quad 7,840 \quad 0 \quad 3,430 \quad -245L], \ (for\ w_1 = w_2 = w_3 = w)$$

with a total load of wL, while an increasing triangular load yields

$$F_w^T = \frac{wL}{14,700}[665 \quad 70L \quad 3,920 \quad 280L \quad 2,765 \quad -175L], \ (for\ w_1 = 0,\ w_2 = w/2, w_3 = w)$$

with a total load of $wL/2$, and a symmetric quadratic hump yields

$$F_w^T = \frac{wL}{14,700}[1,540 \quad 140L \quad 6,7200 \quad 0 \quad 1,540 \quad -140L], \ (for\ w_1 = 0,\ w_2 = w, w_3 = 0)$$

with a total load of $2wL/3$. Generally, you want to substitute the three nodal transverse load values, w_k, to obtain the six F_w resultant generalized load entries before assembling with other elements.

As an example of the three noded beam, consider a fixed-fixed beam with a constant line load. The essential boundary conditions are $v_1 = 0,\ \theta_1 = 0, v_3 = 0,\ \theta_3 = 0$. The external point force and moment at center node 2 are zero ($V_2 = 0, M_2 = 0$). The middle two rows define the remaining unknown center point displacements:

$$\frac{EI}{35L^3}\begin{bmatrix} 7,168 & 0 \\ 0 & 1,280L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \frac{wL}{14,700}\begin{Bmatrix} 7,840 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Multiplying by the inverse of the square matrix gives

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{35wL^4}{14{,}700EI}\begin{bmatrix} 1/7{,}168 & 0 \\ 0 & 1/1{,}280L^2 \end{bmatrix}\begin{Bmatrix} 7{,}840 \\ 0 \end{Bmatrix} = \frac{wL^4}{384EI}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

which are exact. The slope was expected to be zero due to symmetry. It could have been used as a boundary condition to compute v_2 from one equation. Then $M_2 = 0$ would have been found in a reaction recovery. Here, the reactions on the left are found from the first two rows (since all the displacements are now known):

$$\frac{EI}{35L^3}\begin{bmatrix} -3{,}584 & 1{,}920L \\ -896L & 320L^2 \end{bmatrix}\frac{wL^4}{384EI}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \end{Bmatrix} + \frac{wL}{14{,}700}\begin{Bmatrix} 3{,}430 \\ 245L \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \end{Bmatrix} = \frac{-wL}{2}\begin{Bmatrix} 1 \\ L/6 \end{Bmatrix}.$$

Likewise, at the right end, utilizing the last two rows of the equilibrium equations

$$\begin{Bmatrix} V_3 \\ M_3 \end{Bmatrix} = \frac{-wL}{2}\begin{Bmatrix} 1 \\ -L/6 \end{Bmatrix}.$$

Since a 5-th degree element was used this problem is exact everywhere. That includes the moment and shear diagrams. They are shown in Figure 10. This analytic model would also give the exact results everywhere for the beam in Figure 11.

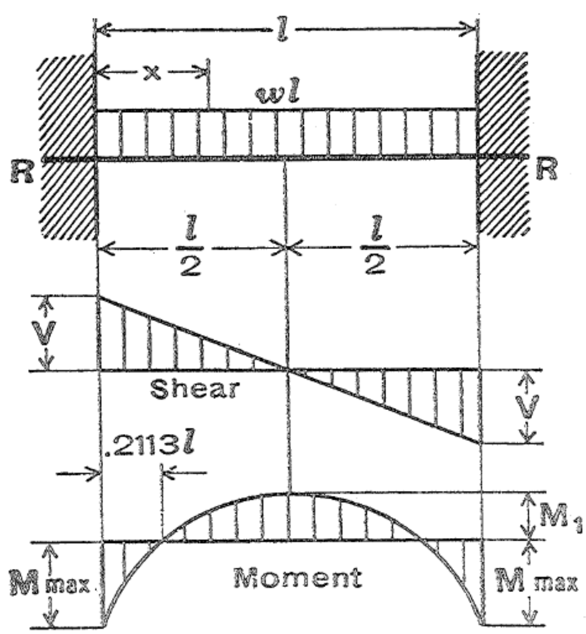

Figure 10 Fixed-fixed moment and shear diagrams

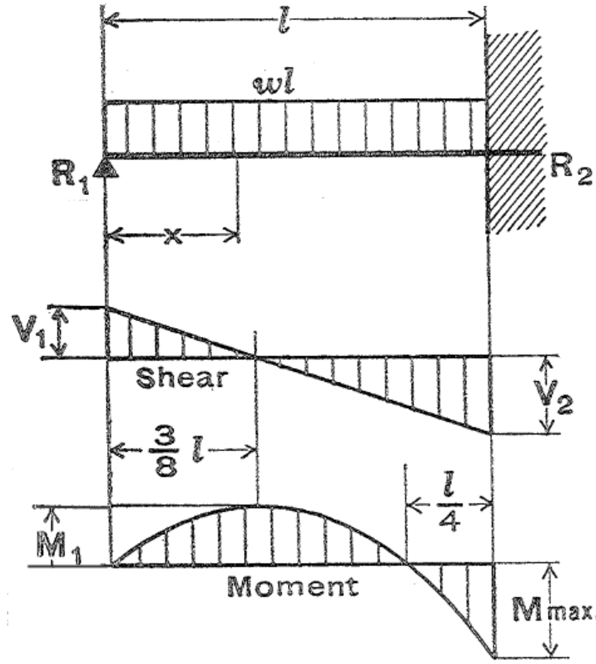

Figure 11 Pinned-fixed moment and shear diagrams

In terms of the 6 dof for this member, the finite element theory gives the following values along the beam (for $r = x/L$) for the deflection, slope, moment and shear are:

$$\begin{aligned} v(r) = [&v_1(1 - 23r^2 + 66r^3 - 68r^4 + 24r^5) + \theta_1(r - 6r^2 + 13r^3 - 12r^4 + 4r^5)L \\ &+ v_2(16r^2 - 32r^3 + 16r^4) + \theta_2(-8r^2 + 32r^3 - 40r^4 + 16r^5)L \\ &+ v_3(7r^2 - 34r^3 + 52r^4 - 24r^5) + \theta_3(-r^2 + 5r^3 - 8r^4 + 4r^5)L] \end{aligned}$$

$$\begin{aligned} \theta(r) &= v'(r) \\ &= [v_1(-46r + 198r^2 - 272r^3 + 120r^4)/L + \theta_1(1 - 12r + 39r^2 - 48r^3 + 20r^4) \\ &+ v_2(32r - 96r^2 + 64r^3)/L + \theta_2(-16r + 96r^2 - 160r^3 + 80r^4) \\ &+ v_3(14r - 102r^2 + 208r^3 - 120r^4)/L + \theta_3(-2r + 15r^2 - 32r^3 + 20r^4)] \end{aligned}$$

$$\begin{aligned} M(r) &= EIv''(r) \\ &= [v_1(-46 + 396r - 816r^2 + 480r^3)/L + \theta_1(12 + 78r - 144r^2 + 80r^3) \\ &+ v_2(32 - 192r + 192r^2)/L + \theta_2(-16 + 192r - 480r^2 + 320r^3) \\ &+ v_3(14 - 204r + 624r^2 - 480r^3)/L + \theta_3(-2 + 30r - 96r^2 + 80r^3)]/L \end{aligned}$$

$$\begin{aligned} V(r) &= EIv'''(r) \\ &= [v_1(396 - 1{,}632r + 1{,}440r^2)/L + \theta_1(78 - 288r + 240r^2) + v_2(-192 + 384r)/L \\ &+ \theta_2(192 - 960r + 960r^2) + v_3(-204 + 1{,}248r - 1{,}350r^2)/L \\ &+ \theta_3(30 - 192r + 240r^2)]/L^2 \end{aligned}$$

Since these are continuous functions they do not give the exact results when there are point forces or moments, or discontinuous line loads. This is illustrated in Figure 12, where the shear force must be discontinuous. However, the results would be exact everywhere if two quintic elements were connected at the point of the concentrated point load of Figure 10.

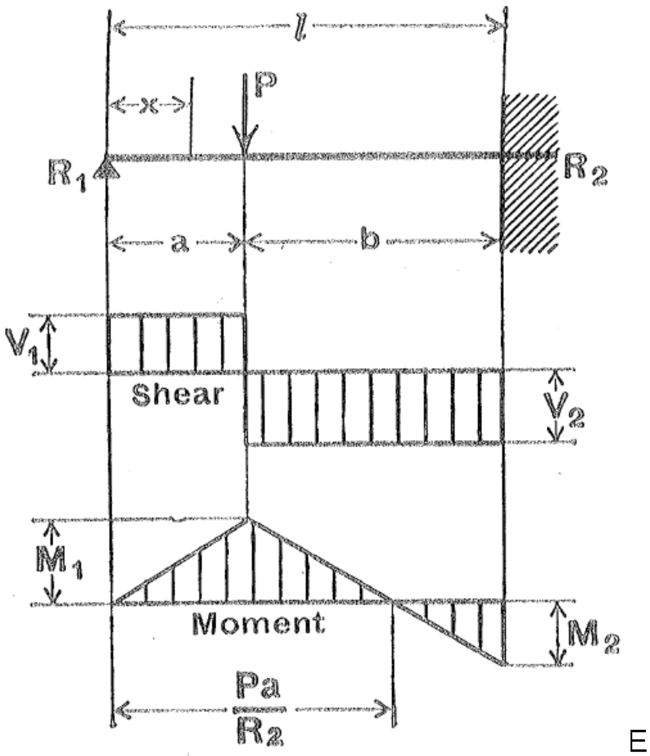

Figure 12 Point loads require two elements

Appendix: Example Solutions