

## Forces

### Introduction

In this chapter, solutions are obtained by direct application of Newton's Laws. In a later chapter, an alternate solution approach based on "Energy Methods" will be introduced. Modern numerical methods for solving mechanics problems often rely primarily on energy methods, and use Newton's Laws only as necessary validation checks. However, for small or simple systems, like those illustrated in this chapter, direct application of Newton's Laws is usually the quickest way to reach a solution.

### Example 3.1

The system of plane, automobile, and cable represent a resultant system of three forces that are concurrent (pass through a common point). They are shown in Figure 1. The system is at rest, so the acceleration is zero, and so it is referred to as a statics problem. The plane actually applies normal (perpendicular) forces to the front and rear wheels. However, they combine to form a single normal force,  $N$ . (Later you will learn how to find both of the normal wheel forces.) The plane makes an angle of 20 degrees up from horizontal. Thus, the normal force is at an angle of 20 degrees to the left of vertical. This is a two-dimensional model with a single free body diagram (FBD), which is given in Figure 2. Therefore, you have only three laws (rules) available: gravity and equilibrium in the horizontal ( $x$ -) and vertical ( $y$ -) directions.

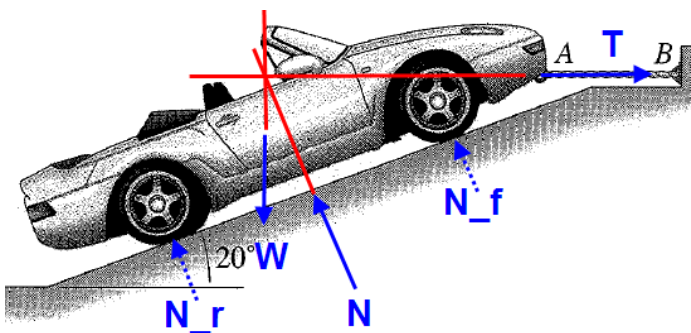


Figure 1 Inclined automobile sketch

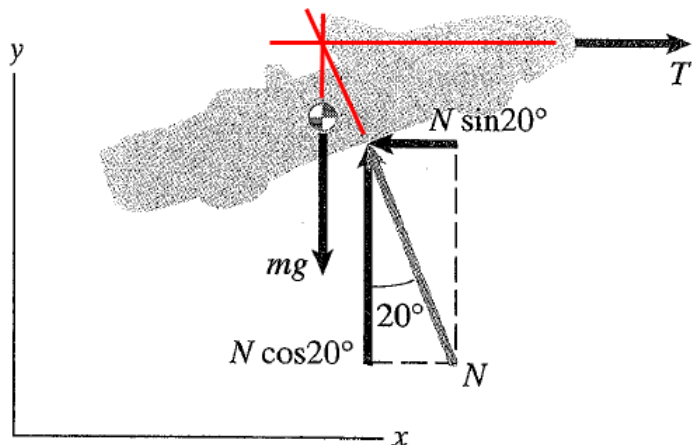


Figure 2 FBD of concurrent forces

Later you will have many FBD's to employ with several rules that apply to each. Therefore, it is desirable to have a computer tool for crunching numbers, while you concentrate on the mechanics principles that give you the rules that are necessary to solve a statics or dynamics problem.

TK Solver is just such a tool. It is a non-procedural "case solver", and not a programming language. Modern engineering programming languages like, Fortran 2003, and C++ (as well as the Matlab environment), require all their statements to be in a strict procedural (sequential) order. Any current statement can only refer to variables defined by previous statements. Furthermore, a programming statement can have only one variable to the left of the equals (=) assignment sign. None of those programming restrictions applies to TK rules. TK Solver rules can (and should) be written as you

think of them as governing principles. The equals (=) sign can be anywhere in the rule. So, place it where you are used to seeing it in a governing principle. Also, remember that any good engineering coding style should contain about 50% comments as self-documentation. In TK, a semi-colon (;) precedes a comment at the end of a rule.

TK Solver is called a ‘case-solver’ because it counts the number of unknowns and searches the rules to find enough equations to compute them. It also does that search in an efficient way. It first looks for a single valid equation to solve. Finding none it would search for two simultaneous equations to solve. If that fails it searches for three, and so on. When it solves a case, its successful solution reduces the number of unknowns, so it repeats its case search. Sometimes the system of simultaneous equations is solved by iteration and you start that process by assigning a Guess value to one or more selected variables.

You should always try to count the number of unknowns. You need to write as many equilibrium equations as there are unknowns. Here, there are two unknowns: the cable force and the resultant normal force from the plane. For the class of problems in this chapter, there are two equilibrium equations per FBD. Thus, you need (and have) only one FBD. The TK rules for this example are shown in Figure 3. As each rule is completed (by an ‘Enter’ key), the name of each new variable automatically begins the population of the variable sheet. The initial variable sheet appears in Figure 4. Applying the usual Windows ‘cut-and-paste’ procedures, you can move rows around later for a pretty report. The variable sheet shows that any variable could be selected as a known quantity to have an input numerical value assigned to it. If a variable is not input, then TK Solver will try to compute its output value.

Status	Rule
Comment	; Ex 3.1 Equilibrium by J.E. Akin, Rice University
Comment	; The plane angle is 20 degrees up from horizontal,
Comment	; thus, the normal force is 20 degrees left of vertical
Comment	; Assume the cable is in tension and normal force is compression
Comment	;NOTE: TK is non-procedural. Put rules in any order.
Comment	; Put '=' anywhere in each rule
Comment	; From Free Body Diagram (FBD) #1
Satisfied	$W = m * g$ ; Gravity
Satisfied	$T - N * \sin(\text{angle}) = 0$ ; $\Sigma F_x = m * a_x == 0$ Horizontal
Satisfied	$N * \cos(\text{angle}) - W = 0$ ; $\Sigma F_y = m * a_y == 0$ Vertical


Figure 3 Rules for the inclined automobile

There is one important aspect of assigning units to a variable. TK Solver allows you to compute using one set of consistent units, and display the (input or output) in any other units. You can use any units (within reason as described below), BUT the first time you put a unit (symbol) in the Unit column of the Variable Sheet, that unit is the one used for calculation unit. Here, that is important because you are using trigonometric functions that expect radians as arguments. Therefore, you must enter ‘radians’ or ‘rad’ as the computation (first) unit for the variable *angle*.

Variables					
Status	Input	Name	Output	Unit	Comment
		W			
		m			
		g			
		T			
		N			
		angle			

Figure 4 The initial variables sheet

A completed Variables Sheet is in Figure 5. After first entering the calculation units, you can re-enter display units for any variable. Select enough input items to make the problem solvable, and assign them numerical input values.

Attempt to find the outputs by executing a TK ‘direct solve’, by clicking on the light bulb icon, . Then, all the output variables found will appear. It is important to note here that the output forces had positive signs. That means that the corresponding *assumption* of tension or compression was correct. When in doubt, it is a good practice to assume any unknown force is in tension. Then, a negative output value would show that it is actually in compression. Remember, that the usable number of significant figures in the output cannot exceed the number available in the inputs. The bottom three comment lines in Figure 5 illustrate the power of TK to let you play the necessary engineering game of “what-if”. What-if the cable has a safe load limit of 5,000 N.? What is the maximum angle for building the ramp?

Variables					
Status	Input	Name	Output	Unit	Comment
		W	14126.4	N	Car weight (in negative y-direction)
	1440	m		kg	Car mass
	9.81	g		m/s^2	Gravitational constant
		T	5141.59	N	Cable force (tension)
		N	15033	N	Plane normal force (compression)
	20	angle		degrees	Angle of plane up from horizontal
					; set cable tension tension to 5000 N
					; and solve for required angle
					; with a starting Guess angle

Figure 5 Variables with input choices and display unit choices

The result of such an engineering study is given in Figure 6. Likewise, the TK model can be used to check your common sense expectations for the mechanics principles. What would you expect the cable and plane forces to become if you set the angle to zero? (Try it.) Likewise, your boss may ask, what is the weight in pounds? Figure 7 shows that you just have to change the output unit to *lb* to see that the answer is about 3,176 lb. Or, you can specify the weight in pounds and solve for its mass. You can download this completed example workbook from file Bedford\_Chap\_3\_1.tkw. Better still, begin with a blank workbook containing just the most common units symbols sub-sheet, and add your own rules like shown above. That file name is Units\_only.tkw. Just save it with a new name after you have begun your application.

Now, some more words to the wise about units in TK Solver. Units are just character strings or symbols. Therefore, if you make a typo in entering units you have defined a new engineering unit. If TK cannot find a conversion in the Units Sheet you will see a question mark (?) before a numerical value showing that the value is probably wrong. The units

	5000	T		N	Cable force (tension)
		N		N	Plane normal force (compression)
Guess	20	angle		degrees	Angle of plane up from horizontal

Input	Name	Output	Unit	Comment
	W	14126.4	N	Car weight (in negative y-direction)
1440	m		kg	Car mass
9.81	g		m/s <sup>2</sup>	Gravitational constant
5000	T		N	Cable force (tension)
	N	14985.2	N	Plane normal force (compression)
	angle	19.4913	degrees	Angle of plane up from horizontal

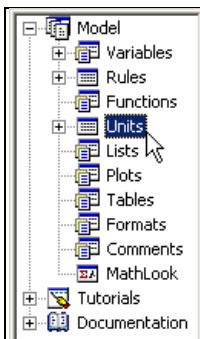
Figure 6 Guess, and find the angle for a limited cable force

Input	Name	Output	Unit	Comment
	W	3175.76	lb	Car weight (in negative y-direction)
1440	m		kg	Car mass

Input	Name	Output	Unit	Comment
3500	W		lb	Car weight (in negative y-direction)
	m	1587.02	kg	Car mass

Figure 7 Changing the weight display unit and Input/Output status

sheet contains many of the standard symbols for units along with the equations to convert from one symbol (unit) to another. To see what is built into the supplied Units Sheet, go to the Navigation Menu and double-click on Units. The first four conversion rules (of many) are shown in Figure 8. You can insert new rows for any special units you need to use. Just right click in a From column and select "Insert Row" from the displayed options.

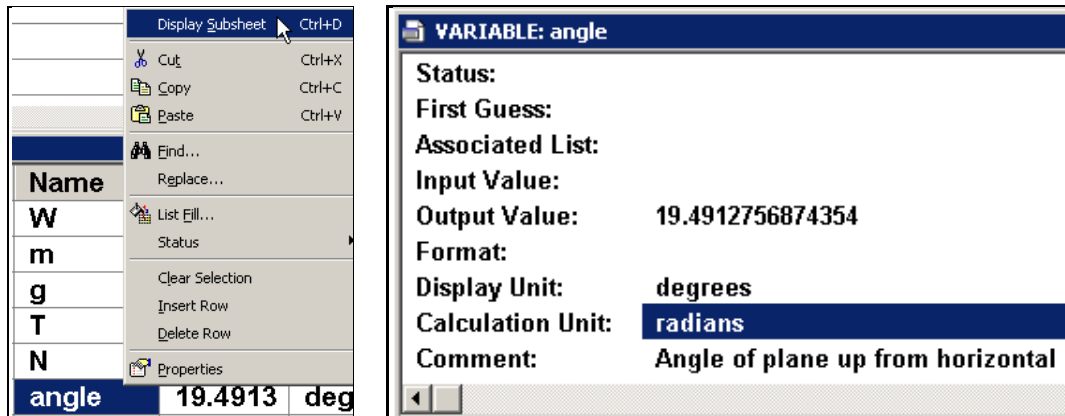


From	To	Multiply By	Add Offset	Comment
				Unit Master Sheet by R.L. Norton, appended by J.E. Akin
				length
cm	mm	10		
m	cm	100		
mm	um	1000		

From	To	Multiply By	Add Offset	Comment
				angles
radians	degrees	57.29577951		
rad	deg	57.29577951		
rev	deg	360		
rad	rad			

Figure 8 Checking for available calculation or display units

Sometime you will forget to enter the calculation unit first into the Variable Sheet. Then, you will need to change it. To do so, right-click on the variable name and select its Sub sheet. There, you can assign any symbol in the Units Sheet to be the calculation and/or display unit. They are shown in Figure 9.



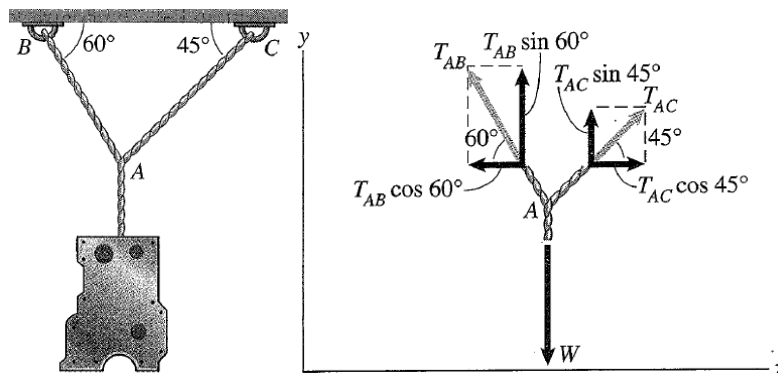
**Figure 9 Define or change a variable calculation or display unit**

TK Solver includes a “Report Wizard” (File → Create Report) to provide neat written reports. See the help file for the options. A simple default report for this example is given in the Appendix.

**Example 3.2**

Again, the system in this example is three concurrent forces in a vertical plane. The same three principles apply, just the two equilibrium rules change. The engine block support ropes and their FBD are given in Figure 10. Here, there are two unknowns: the forces in the inclined ropes. (By inspection, you should see that the vertical force carries the full weight. If you do not see that, draw a second FBD of just the weight and the vertical cable.) Like the prior example, you need only one FBD to define the equilibrium rules. The corresponding Rule Sheet is in Figure 11 and the assigned units are in

Figure 12. Executing by clicking solve, , gives the results in Figure 13.



**Figure 10 Cable supported engine block**

Again, it is always wise to check a model with common sense tests. For example, if the left cable was vertical you should expect it to support the entire weight. Then the right cable should carry no portion of the weight (unless the wind blows and we modify the FBD). Check that case by setting the left angle to 90 degrees, and solve again (as in Figure 14). The

completed worksheet for this example has file name Bedford\_Chap3\_2.tkw. While you can use it to duplicate these figures, you will learn more by building your own worksheet by starting with the default work sheet (Units\_only.tkw).

Rules	
Status	Rule
Comment	; Ex 3.2 Equilibrium by J.E. Akin, Rice University
Comment	; Assume all cables are in tension
Comment	; NOTE: TK is non-procedural. Put rules in any order.
Comment	; Put '=' anywhere in each rule
Comment	; From Free Body Diagram (FBD) #1
Satisfied	$W = m * g$ ; Gravity
Satisfied	$T_{AC} * \cos(\text{angle\_CA}) - T_{AB} * \cos(\text{angle\_BA}) = 0$ ; $\Sigma F_x = m * a_x == 0$ Horizontal
Satisfied	$T_{AC} * \sin(\text{angle\_CA}) + T_{AB} * \sin(\text{angle\_BA}) - W = 0$ ; $\Sigma F_y = m * a_y == 0$ Vertical

Figure 11 Gravity and equilibrium rules

Variables					
Status	Input	Name	Output	Unit	Comment
		W		N	Engine weight (in negative y-direction)
	200	m		kg	Engine mass
	9.81	g		m/s^2	Gravitational constant
Guess	100	T_AC		N	Right cable (tension) Enter a Guess
		T_AB		N	Left cable force (tension)
	45	angle_BA		degrees	Angle from B to A
	60	angle_CA		degrees	Angle from C to A

Figure 12 Initially populated Variable Sheet with an initial guess

Input	Name	Output	Unit	Comment
	W	1962	N	Engine weight (in negative y-direction)
200	m		kg	Engine mass
9.81	g		m/s^2	Gravitational constant
	T_AC	1436.2837	N	Right cable (tension) Enter a Guess
	T_AB	1015.6059	N	Left cable force (tension)
45	angle_BA		degrees	Angle from B to A
60	angle_CA		degrees	Angle from C to A

Figure 13 A solution for a pair of active ropes

Input	Name	Output	Unit	Comment
	W	1962	N	Engine weight (in negative y-direction)
200	m		kg	Engine mass
9.81	g		m/s^2	Gravitational constant
	T_AC	2.403E-13	N	Right cable (tension) Enter a Guess
	T_AB	1962	N	Left cable force (tension)
90	angle_BA		degrees	Angle from B to A
60	angle_CA		degrees	Angle from C to A

Figure 14 The solution for the left rope vertical

**Hint for Debugging Rules**

Everyone makes a mistake sometime, so consider a debugging trick. If an expression should equal zero, equate it equal to an error variable name instead. You can always input the error variable as zero. When your rules are debugged, you can equate the expression exactly to zero, in the Rules Sheet, and delete the rows with the error variables, in the Variable Sheet. To illustrate such an approach let the last rule have a sign error in the second term, as in Figure 15 (top). The horizontal equilibrium equation is the simplest. Try setting its error to zero and guess a value of 100 N for T<sub>AC</sub>. While a solution is obtained, there is a large negative (downward) error in the y-direction, as seen at the bottom of Figure 15. That suggests that one (or both) of the vertical cable forces may be in error. Correcting the AB tension vertical sign and setting both error variables to zero yields the results in the text, and in Figure 13 above.

Status	Rule
* Unsatisfied	$T_{AC} * \cos(\text{angle}_{CA}) - T_{AB} * \cos(\text{angle}_{BA}) = \text{Error}_x$ ; $\Sigma F_x = 0$ Horizontal
* Unsatisfied	$T_{AC} * \sin(\text{angle}_{CA}) - T_{AB} * \sin(\text{angle}_{BA}) - W = \text{Error}_y$ ; $\Sigma F_y = 0$ Vertical

Status	Input	Name	Output	Unit	Comment
		W	1962	N	Engine weight (in negative y-direction)
	200	m		kg	Engine mass
	9.81	g		m/s^2	Gravitational constant
		T <sub>AC</sub>	100	N	Right cable (tension) Enter a Guess
		T <sub>AB</sub>	70.7107	N	Left cable force (tension)
	45	angle <sub>BA</sub>		degrees	Angle from B to A
	60	angle <sub>CA</sub>		degrees	Angle from C to A
	0	Error <sub>x</sub>		N	Error in x-equilibrium
		Error <sub>y</sub>	-1925.4	N	Error in y-equilibrium

Status	Rule
Satisfied	$T_{AC} * \cos(\text{angle}_{CA}) - T_{AB} * \cos(\text{angle}_{BA}) = \text{Error}_x$ ; $\Sigma F_x = 0$ Horizontal
Satisfied	$T_{AC} * \sin(\text{angle}_{CA}) + T_{AB} * \sin(\text{angle}_{BA}) - W = \text{Error}_y$ ; $\Sigma F_y = 0$ Vertical

Input	Name	Output	Unit	Comment
	T <sub>AC</sub>	1436.28	N	Right cable (tension) Enter a Guess
	T <sub>AB</sub>	1015.61	N	Left cable force (tension)
45	angle <sub>BA</sub>		degrees	Angle from B to A
60	angle <sub>CA</sub>		degrees	Angle from C to A
0	Error <sub>x</sub>		N	Error in x-equilibrium
0	Error <sub>y</sub>		N	Error in y-equilibrium

**Figure 15** Correcting the sign error and setting both error values to zero yields a solution

**List Solves and Graphs**

You may be interested in looking at the results for a range of angles for the rope AB. You may also wish to graph those results. To illustrate that process assume you wish to vary the AB angle from 45 to 90 degrees while keeping the weight and AC angle constant (by changing the length of AC). Clearly, forces T<sub>AB</sub> and T<sub>AC</sub> will vary as angle BA varies. In other words, there will be one input list (for angle<sub>BA</sub>) and two output lists (for T<sub>AB</sub> and T<sub>AC</sub>). To indicate that task,

you have to put an “L” (for List) in the status column for each of those three variable rows, and build the specific list of desired input angles. Since the angle is currently selected for the unit of degrees, we need to fill its list with equal increments from 45 to 90 degrees, as illustrated in Figure 16.

Often you need to start a Solve (or List Solve) by assigning a first guess value to begin a Newton-Raphson iterative solution for the outputs. Do that for cable force T\_AB and execute a “List Solve” by clicking on the multiple light bulb icon, as shown in Figure 17. Then the Navigation menu shows three lists available for review and the end of the force lists are given in Figure 18. As you might expect, variables for which lists are available can be graphed. Right click “Plots” in the Navigation Menu and define a line chart as shown in Figure 19. As you might expect, the two cables carry equal portions of the weight when both angles are 60 degrees, but thereafter all the weight is gradually shifted to the left cable, as seen in the list of Figure 18, and the plot in Figure 20.

Status	Input	Name
		W
	200	m
	9.81	g
L		T_AC
L		T_AB
L	45	angle_BA
	60	angle_CA

**List Fill**

List Name:  Fill List

Fill Type:

Linear       Fill By Step  
 Logarithmic       Fill By Range  
 Distribution     

First Element:       Last Element:

First Value:       Last Value:

Cancel      Help

Linear Scaling

Figure 16 Filling the input angle list from 45 to 90 degrees

**VARIABLE: T\_AB**

Status: LF

First Guess: 10

Associated List: T\_AB

Input Value:

Output Value:

Format:

Display Unit: N

Calculation Unit: N

Comment: Left cable force (tension)

Status	Input	Name
	1962	W
	200	m
	9.81	g
L		T_AC
LF		T_AB
L	45	angle_BA
	60	angle_CA

Figure 17 Assign a first guess to T\_AB and execute a list solve

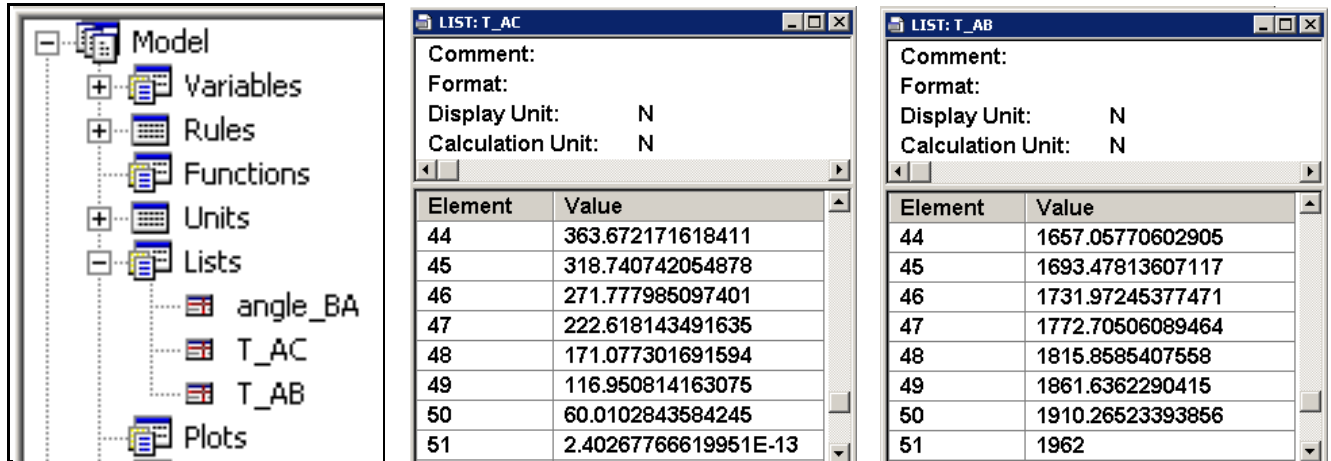


Figure 18 The end of the output cable force lists (to be plotted)

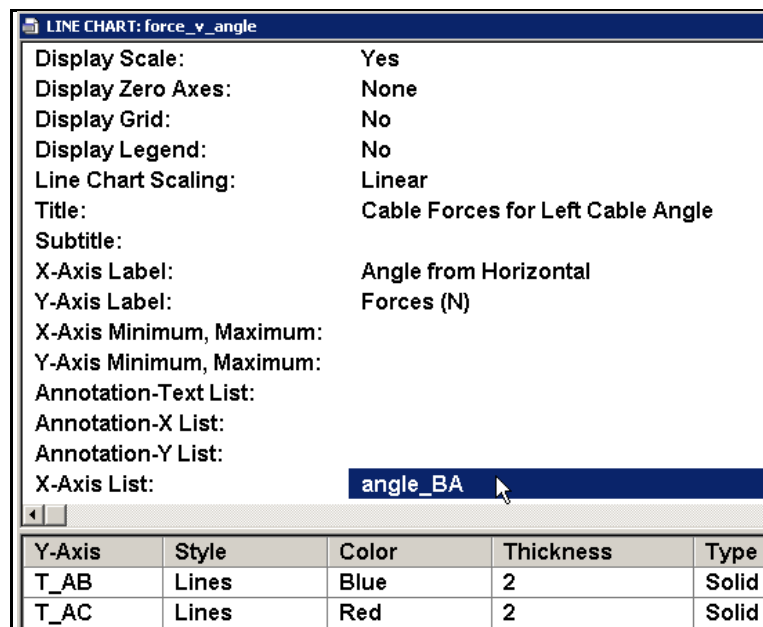
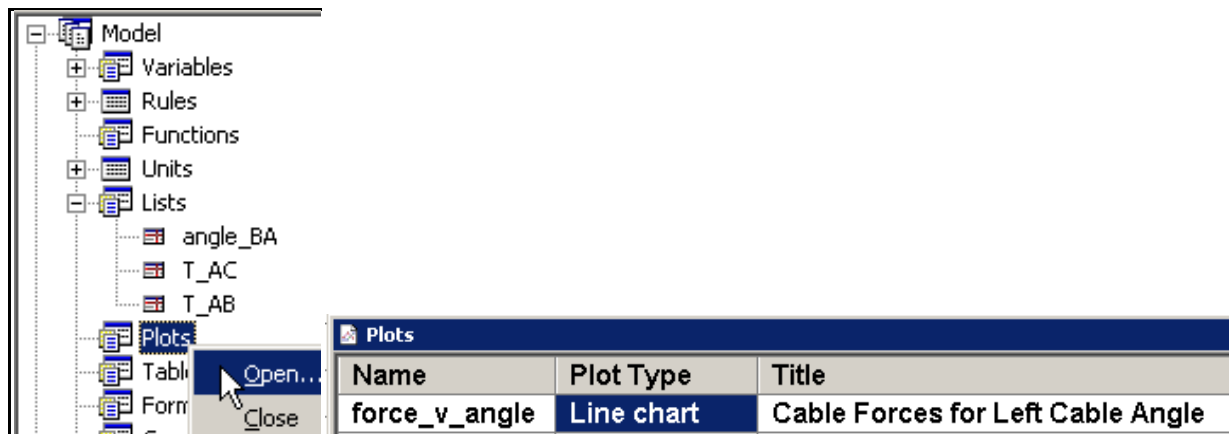


Figure 19 Defining the forces line charts

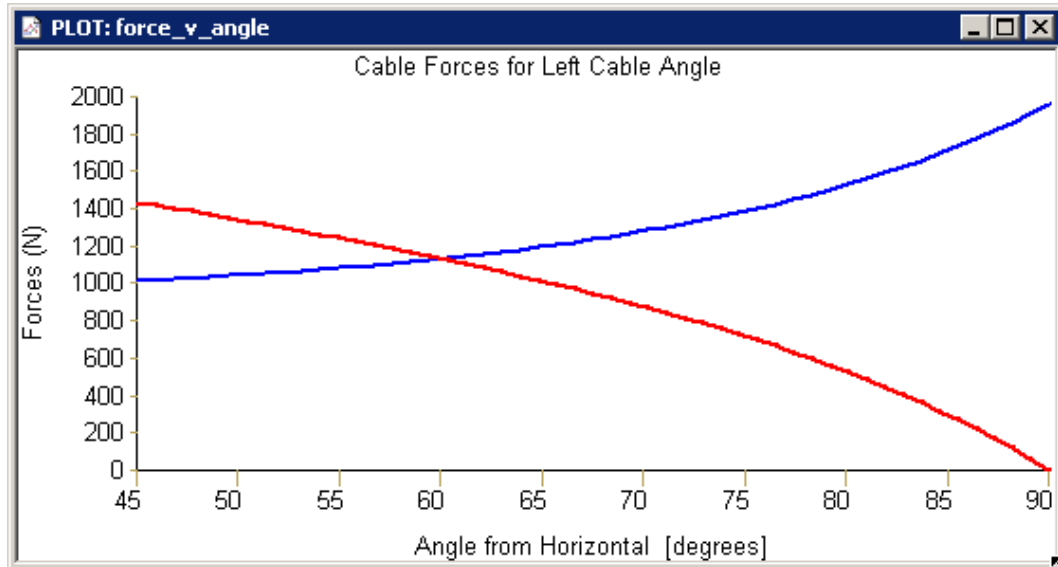


Figure 20 Graphing the cable forces versus left cable angle

### Example 3.3

In this problem neither the value of the hand tension,  $T$ , nor the weight,  $W$ , of the mass of body A is given. Thus, one will have to be given an arbitrary value, such as unity, so that the other can be computed. Usually you would assign a unit value to the hand tension and solve for the weight it would hold in equilibrium. The ratio of those two values is defined as the “mechanical advantage” of the pulley system. Here, the pulleys are assumed to be frictionless, and have weights that can be neglected.

You always need to count the number of unknowns to determine the minimum number of FBD's you will need.

Assuming the hand tension,  $T$ , is known frictionless pulleys B and C just change its direction until that rope ends at the top left connector. In addition to the unknown weight,  $W$ , there is an additional internal cable force from the support of pulley C, over pulley D, and down to support pulley B. Call that force  $T_D$ . Then, the last unknown is the top right force, say  $T_{top}$ , supporting pulley D. That means this system has a total of three unknowns (relative to the magnitude of  $T$ ), namely  $W$ ,  $T_D$ , and  $T_{top}$ . Figure 21 shows the system, and at least four FBD's that you could construct.

Since all the forces are vertical, you can only get one independent equation (vertical equilibrium) per FBD. Four diagrams are shown, but you only need three. That is okay, since it is useful in TK Solver to have additional valid equations. Historically, statics classes required students to solve a huge number of assignments so they would learn how to spot the ones that reduce the number of equations to be solved by hand. Here, the emphasis is for you to demonstrate correct application of the principles, and let TK find and solve the cases of interest. To demonstrate the correct application of the principles, you must ALWAYS first build correct free body diagrams. Applying the equilibrium equations (and later the equations of motion) becomes repetitious.

The four vertical equilibrium equations and the gravity law are shown as rules in Figure 22. Input a value of unity for  $T$  and a direct solve yields the output of Figure 23. That figure shows that the system will support a weight four times larger than the applied hand tension. It also gives the other cable and support forces. Remember, since all the outputs had a positive sign that confirms the (tension) directions assumed in constructing the FBDs.

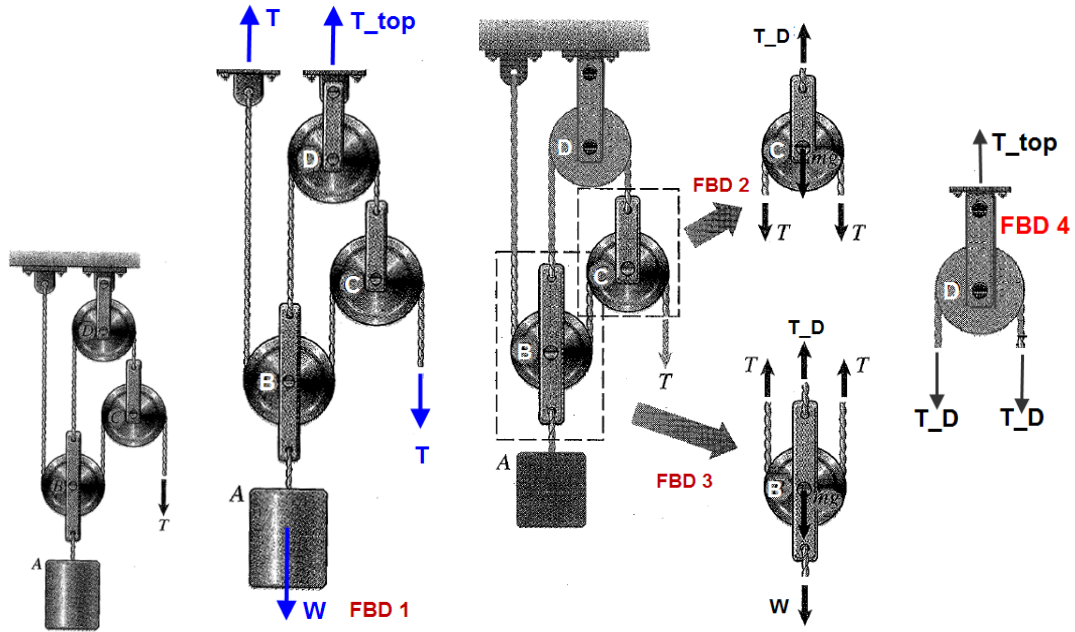


Figure 21 A vertical pulley system and some FBD's

Rules	
Status	Rule
Comment	; Ex 3.3 Equilibrium by J.E. Akin, Rice University
Comment	; Assume all cables are in tension. NEGLECT pulley weights.
Comment	; NOTE: TK is non-procedural. Put rules in any order.
Comment	; Put '=' anywhere in each rule
Satisfied	$W = m * g$ ; Gravity
Comment	; Vertical force equilibrium, $\Sigma F_y = m * a_y == 0$
Satisfied	$T + T_{top} - T - W = 0$ ; System Free Body Diagram (FBD) #1
Satisfied	$T_D - T - T = 0$ ; Pulley C, FBD # 2
Satisfied	$T + T_D + T - W = 0$ ; Pulley B, FBD # 3
Satisfied	$-T_D + T_{top} - T_D = 0$ ; Pulley D, FBD # 4

Figure 22 Rules for the three-pulley system

Variables					
Status	Input	Name	Output	Unit	Comment
		W	4	N	Hanging weight (in negative y-direction)
		m	.4077472	kg	Hanging mass
	9.81	g		m/s^2	Gravitational constant
	1	T		N	Long cable force (tension)
		T_D	2	N	Short cable force (tension)
		T_top	4	N	Pulley D support force (tension)

Figure 23 The system has a mechanical advantage of four (W/T)

If you had only cared about the value of the supported weight, then an experienced analyst would have seen that you only needed two FBD's (2 and 3) and two rules. That experience will come in time, but now just concentrate on the principles and let TK solve the systems. If you fail to provide enough rules, then TK will not provide outputs. That will be

your hint to look for more rules and/or more FBD's. Now that you have built rules for the complete system you can solve related problems. For example, you could specify the mass value and find all the required forces. That change in inputs is shown in Figure 24. Note that one of the unknowns is given an initial guess value. Sometimes that is necessary to prevent very small round off errors from causing the TK logic to assume that the solution has failed.

Variables					
Status	Input	Name	Output	Unit	Comment
		W		N	Hanging weight (in negative y-direction)
	100	m		kg	Hanging mass
	9.81	g		m/s^2	Gravitational constant
Guess	1	T		N	Long cable force (tension)
		T_D		N	Short cable force (tension)
		T_top		N	Pulley D support force (tension)

Figure 24 Assigning the mass as an alternate set of input

Even if the guess is not close to the final output, the system usually converges. All iterative equation solvers are sensitive to their starting values. Thus, if one guess fails to converge try another one before searching your rules for possible errors. Figure 25 shows the answers for the assumed mass input.

Input	Name	Output	Unit	Comment
	W	981	N	Hanging weight (in negative y-direction)
100	m		kg	Hanging mass
9.81	g		m/s^2	Gravitational constant
	T	245.25	N	Long cable force (tension)
	T_D	490.5	N	Short cable force (tension)
	T_top	981	N	Pulley D support force (tension)

Figure 25 Cable tensions and weight for a given supported mass

### Example 3.5

In this system four three-dimensional forces act through a common point (they are concurrent). The mass, and thus the weight is given. By inspection, the vertical cable force is equal to the weight. That leaves three unknowns. Each FBD will yield three equilibrium equations, so we need (and have) only one FBD. By definition, straight cable and rope forces act in the direction of the cable or rope. A cable force vector is simply the product of the scalar magnitude of the force times a (non-dimensional) unit vector that defines its direction, such as  $\mathbf{T}_{AB} = T_{AB} \cdot \mathbf{e}_{AB}$ . The necessary unit vectors are obtained by dividing the position vector between the cable end points by the length of the cable, such as  $\mathbf{e}_{AB} = \mathbf{r}_{AB} / L_{AB}$ . The components of any unit vector are also known as the direction cosines. They are the cosines of the angles that the line makes with the x-, y-, and z-axes, respectively. For this problem, there are more analytical geometry rules than equilibrium rules. The system, a typical position vector, and the FBD are given in Figure 26. The analytic geometry portion of the rules for this system are in Figure 27, while the mechanics portion is given in Figure 28.

Note that the first direct solve satisfied the geometry and gravity rules, but not the equilibrium. That was probably due to round off error, so assign the cable forces three starting guess values and execute another direct solve. As seen in Figure 29 that yields the final solution. Note that secondary items, like the position vectors, were placed at the bottom of the variable sheet (by cut and paste). They can be hidden.



Status	Rule
Satisfied	$W = m * g$ ; Gravity
Comment	; x-direction equilibrium, $\Sigma F_x = 0$
* Unsatisfied	$T_{AB} * e_{ABx} + T_{AC} * e_{ACx} + T_{AD} * e_{ADx} = 0$
Comment	; Vertical force equilibrium, $\Sigma F_y = 0$
* Unsatisfied	$T_{AB} * e_{ABy} + T_{AC} * e_{ACy} + T_{AD} * e_{ADy} - W = 0$
Comment	; z-direction equilibrium, $\Sigma F_z = 0$
* Unsatisfied	$T_{AB} * e_{ABz} + T_{AC} * e_{ACz} + T_{AD} * e_{ADz} = 0$

Figure 28 Mechanics rules for the hanging mass

Status	Input	Name	Output	Unit	Comment
Guess	100	T_AB		N	Cable force AB (tension)
Guess	100	T_AC		N	Cable force AC (tension)
Guess	100	T_AD		N	Cable force AD (tension)

Status	Input	Name	Output	Unit	Comment
		T_AB	519.3529	N	Cable force AB (tension)
		T_AC	636.0749	N	Cable force AC (tension)
		T_AD	168.2401	N	Cable force AD (tension)

Figure 29 Final iterative three-dimensional concurrent system cable results

This example is available as file name Bedford\_Chap\_3\_5.tkw. Having verified a valid model it can be used for other purposes. Assume that you measured the force in cable AC to be 700 N, and you need to know the weight (and/or other two cable forces). The resulting four outputs are given in Figure 30, for that case.

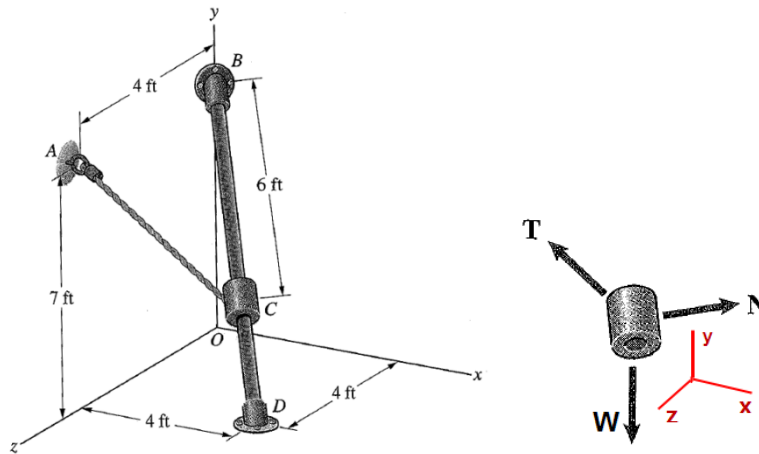
Status	Input	Name	Output	Unit	Comment
Guess	100	W		N	Hanging weight (negative y-direction)
		m		kg	Hanging mass
	9.81	g		m/s^2	Gravitational constant
Guess	100	T_AB		N	Cable force AB (tension)
	700	T_AC		N	Cable force AC (tension)
Guess	100	T_AD		N	Cable force AD (tension)

Status	Input	Name	Output	Unit	Comment
		W	1079.59	N	Hanging weight (negative y-direction)
		m	110.0499	kg	Hanging mass
	9.81	g		m/s^2	Gravitational constant
		T_AB	571.5476	N	Cable force AB (tension)
	700	T_AC		N	Cable force AC (tension)
		T_AD	185.1481	N	Cable force AD (tension)

Figure 30 Using a measured cable force to find the mass

**Example 3.6**

This problem again has more of an analytic geometry equations than equilibrium equations. To find the cable force,  $T$ , you need to determine the coordinates of point  $C$ . Since the rod is frictionless, you know that the normal force,  $N$ , is perpendicular to bar line  $BD$  (that is, the dot product of force vector  $\mathbf{N}$  and relative position  $\mathbf{r}_{BD}$  is zero). While there are infinite many lines perpendicular to  $BD$ , there is only one direction that allows the normal force to satisfy the three-dimensional equilibrium equations. The geometry of the concurrent force system and the FBD are given in Figure 31.



**Figure 31 Slider on a frictionless bar in space**

The analytic geometry relationships can be solved first, so do that before considering the details of the mechanics. The rules for the needed geometric data are given in Figure 32, along with the resulting output from a single direct solve. First the relative position of line  $BD$  was determined, and then the location of point  $C$  on the bar was defined. Note that the later employed the components of unit vector  $\mathbf{e}_{BD}$ , even though they were not yet defined. Knowing the location of point  $C$  the *line vector from  $C$  to  $A$*  was defined, since the assumed cable tension,  $T_{CA}$ , would be in that direction. Finally, the line unit vectors were defined by dividing the relative position vectors by their magnitudes.

Having obtained a successful solve of the geometry problem, you can now turn to the mechanics issues. How many unknowns are there here? Since the cable force by definition acts along line  $CA$  ( $\mathbf{T}_{CA} = T_{CA} * \mathbf{e}_{CA}$ ), we only need its magnitude,  $T_{CA}$ . That is one unknown. The slider force vector from the bar,  $\mathbf{N}$ , is unknown, so you have to find all three of its components ( $N_x, N_y, N_z$ ). Therefore, you have a total of four unknowns and must therefore write four new equations. The FBD of the three-dimensional concurrent force system will yield three equations of equilibrium. Where will you find the fourth equation? The answer is that the force vector  $\mathbf{N}$  is constrained to be perpendicular to line segment  $BD$ . Thus, their dot product is zero,  $\vec{\mathbf{N}} \cdot \vec{\mathbf{r}}_{BD} = 0$ . Of course, that can also be written using the unit vector for the line in question,  $\vec{\mathbf{N}} \cdot \vec{\mathbf{e}}_{BD} = 0$ . These additional rules, along with the gravitational law, are shown in Figure 33.

The gravity law was satisfied in the first direct solve, along with the geometry relations. However, the four force items were not output (probably due to round off errors). Thus, one is give a guess value to start the iterative solver, and the final force outputs are given in Figure 34.

Status	Rule	Input	Name	Output	Unit	Comment
Comment	; Ex 3.6 Equilibrium by J.E. Akin, Rice Univers					Bedford-Fowler, 3-rd Ed, Exa
Comment	; Assume all cables are in tension.					
Comment	; NOTE: TK is non-procedural. Put rules in any					
Comment	; Put '=' anywhere in each rule					
Comment	; Position vector from B to D, and its length					
Satisfied	$r_{BDx} = x_D - x_B$	0	x_A		ft	Point coordinate data
Satisfied	$r_{BDy} = y_D - y_B$	7	y_A		ft	Point coordinate data
Satisfied	$r_{BDz} = z_D - z_B$	4	z_A		ft	Point coordinate data
Satisfied	$L_{BD} = \sqrt{r_{BDx}^2 + r_{BDy}^2 + r_{BDz}^2}$	0	x_B		ft	Point coordinate data
		7	y_B		ft	Point coordinate data
		0	z_B		ft	Point coordinate data
			x_C	2.666667	ft	Point coordinate data
			y_C	2.333333	ft	Point coordinate data
			z_C	2.666667	ft	Point coordinate data
		4	x_D		ft	Point coordinate data
		0	y_D		ft	Point coordinate data
		4	z_D		ft	Point coordinate data
		6	L_BC		ft	Slider position along bar
			L_BD	9	ft	Length of slider BD
			L_CA	5.537749	ft	Length of cable CA
						Line unit vectors
			e_CAx	-.4815434		cable direction cosine w.r.t. x
			e_CAy	.842701		cable direction cosine w.r.t. y
			e_CAz	.2407717		cable direction cosine w.r.t. z
			e_BDx	.4444444		bar direction cosine w.r.t. x
			e_BDy	-.7777778		bar direction cosine w.r.t. y
			e_BDz	.4444444		bar direction cosine w.r.t. z
						Relative position vectors
			r_CAx	-2.666667	ft	cable length increment
			r_CAy	4.666667	ft	cable length increment
			r_CAz	1.333333	ft	cable length increment
			r_BDx	4	ft	bar length increment
			r_BDy	-7	ft	bar length increment
			r_BDz	4	ft	bar length increment

Figure 32 Slider-bar analytic geometry rules and outputs

Status	Rule
Satisfied	$W = m * g$ ; Gravity
Comment	; Concurrent force system equilibrium equations
Satisfied	$T_{CA} * e_{CAx} + N_x = 0$ ; $\Sigma F_x = 0$
Satisfied	$T_{CA} * e_{CAy} + N_y - W = 0$ ; $\Sigma F_y = 0$
Satisfied	$T_{CA} * e_{CAz} + N_z = 0$ ; $\Sigma F_z = 0$
Comment	; Constraint equation, $N \cdot e_{BD} == 0$
Satisfied	$N_x * e_{BDx} + N_y * e_{BDy} + N_z * e_{BDz} = 0$

Figure 33 Equilibrium and force constraint relationships

Since the three-dimensional FBD is difficult to visualize, you may want to draw the true shape projections of the FBD onto the XY, XZ, and YZ planes. To do that you need additional rules to get the components of tension  $T_{CA}$  (try it). This worksheet is stored with file name Bedford\_Chap\_3\_6.tkw.

Status	Input	Name	Output	Unit	Comment
					Gravity
	100	W		lb	Slider weight
		m	3.108863	slug	Slider mass
	32.17	g		ft/sec^2	Gravitational constant
					Concurrent forces equilibrium
Guess	10	T_CA		lb	Cable force AC (tension)
		N_x		lb	Slider normal force component
		N_y		lb	Slider normal force component
		N_z		lb	Slider normal force component

Name	Output	Unit	Comment
T_CA	102.0112	lb	Cable force AC (tension)
N_x	49.12281	lb	Slider normal force component
N_y	14.03509	lb	Slider normal force component
N_z	-24.5614	lb	Slider normal force component

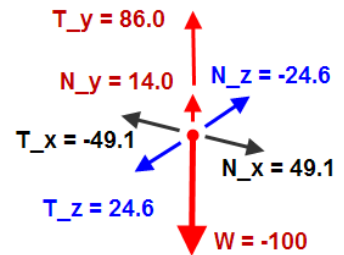


Figure 34 Iterative solve results for the forces

### Example 3.7

The speaker system is supported by two cables of specified length. To determine the cable forces you again begin with some basic geometry data related to the distance between support points,  $b$ . That is, you must employ the Law of Cosines (for a non-right angle triangle) and the Law of Sines. The system sketch and its planar FBD are given in Figure 35, while the trigonometric and equilibrium rules are given in Figure 36. Note that the Law of Cosines is written as you usually find it. You do not have to re-arrange it to find the angle  $\alpha$ , as you would in a hand solution.

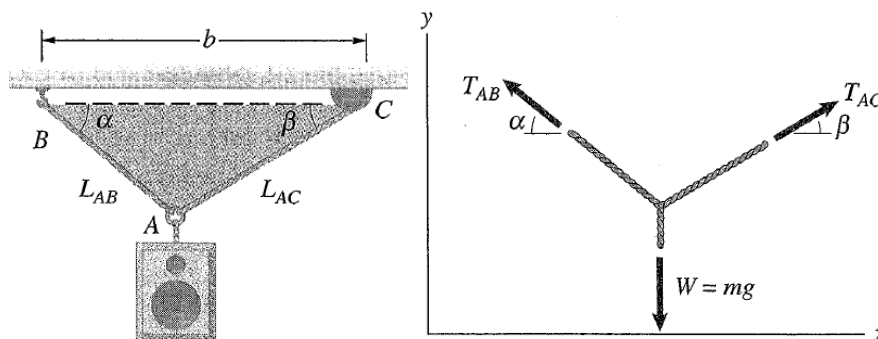


Figure 35 A weight hanging on two cables

Status	Rule
Comment	; Ex 3.7 Equilibrium by J.E. Akin, Rice University
Comment	; Assume all cables are in tension
Comment	; NOTE: TK is non-procedural. Put rules in any order.
Comment	; Put '=' anywhere in each rule
Comment	; Trigonometry
Satisfied	$L_{AC}^2 = b^2 + L_{AB}^2 - 2 * b * L_{AB} * \cos(\alpha)$ ; Law of Cosines
Satisfied	$L_{AC} / \sin(\alpha) = L_{AB} / \sin(\beta)$ ; Law of Sines
Satisfied	$L_{total} = L_{AB} + L_{AC}$ ; Total length
Comment	; From Free Body Diagram (FBD) #1
Satisfied	$W = m * g$ ; Gravity
Satisfied	$T_{AC} * \cos(\beta) - T_{AB} * \cos(\alpha) = 0$ ; $\Sigma F_x = 0$
Satisfied	$T_{AC} * \sin(\beta) + T_{AB} * \sin(\alpha) - W = 0$ ; $\Sigma F_y = 0$
Satisfied	$Ratio_{AB} = T_{AB} / W$ ; < 1 imposed limit
Satisfied	$Ratio_{AC} = T_{AC} / W$ ; < 1 imposed limit

Figure 36 Equilibrium and trigonometry relations

You should observe that as the angles approach zero the cable forces would have to approach infinity. No material is that strong. Here it is given that cable forces may not exceed the value of the weight. As a reminder of that design limit, two extra variables (Ratio\_AB, Ratio\_AC) were defined to give the ratio of each cable force and the weight. To find the minimum angles (and lengths) that could be used, the two ratios were set equal to unity and the problem was solved, as seen in Figure 37. The two angles were 30 degrees, the length were about 1.16 m, and the two cable forces indeed equaled the weight.

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 3.7, 3rd Ed
	W	9810	N	Weight (in negative y-direction)
1000	m		kg	Mass
9.81	g		m/s^2	Gravitational constant
	T_AC	9810	N	Right cable (tension)
	T_AB	9810	N	Left cable force (tension)
2	b		m	Distance between supports
	L_AB	1.1547005	m	Right cable length (guess)
	L_AC	1.1547005	m	Left cable length (guess)
	$\alpha$	30	degrees	Angle ABC
	$\beta$	30	degrees	Angle ACB
	L_total	2.3094011	m	Sum of cable lengths > b
1	Ratio_AB			Required to be less than 1
1	Ratio_AC			Required to be less than 1

Figure 37 Find the minimum design angles

When the example cable lengths are specified to be 1 and 1.2 m, as in Figure 38, then the cable forces are above the given design limit. By varying the length of AC you can graph the the cable force to ratios as in Figure 39.

Input	Name	Output	Unit	Comment
	T_AC	11489.021	N	Right cable (tension)
	T_AB	11940.836	N	Left cable force (tension)
2	b		m	Distance between supports
1	L_AB		m	Right cable length (guess)
1.2	L_AC		m	Left cable length (guess)
	$\alpha$	27.126753	degrees	Angle ABC
	$\beta$	22.331645	degrees	Angle ACB
	L_total	2.2	m	Sum of cable lengths > b
	Ratio_AB	1.2172106		Required to be less than 1
	Ratio_AC	1.171154		Required to be less than 1

Figure 38 Cable forces for specified cable lengths

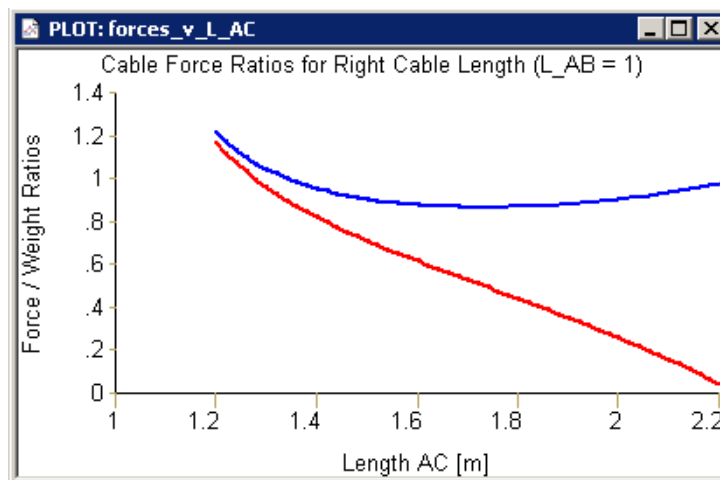


Figure 39 Cable force ratios for  $L_{AB} = 1$

**Example 3.8**

In this problem (Figure 40), the spring was initially un-stretched and horizontal. For a linear spring the force between its two end points is equal to the change in length times the spring stiffness. Clearly that initial position would not satisfy equilibrium. An iterative solution is required to find the vertical displacement,  $h$ , needed to develop enough vertical spring force to balance the applied weight.

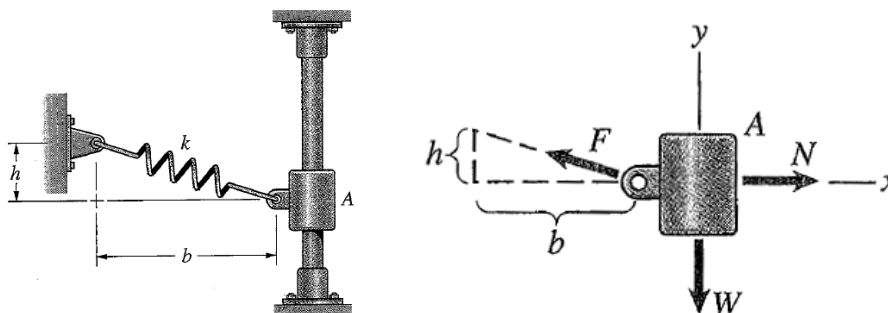


Figure 40 A spring-slider system and its FBD

The simple geometry and equilibrium rules are in Figure 41. To start an iterative solution for the vertical deflection of the weight just assign a guess value to  $h$ . Then you obtain the results in Figure 42.

Status	Rule
Comment	; Bedford-Fowler Ex 3.8 3-rd Ed
Comment	; Initial unloaded horizontal spring
Satisfied	$L^2 = h^2 + b^2$ ; right triangle
Satisfied	$F_S = k * (L - b)$ ; linear spring force
Satisfied	$F_{Sx} = F_S * b / L$ ; similar triangles
Satisfied	$F_{Sy} = F_S * h / L$ ; similar triangles
Comment	; Equilibrium of concurrent forces
Satisfied	$N - F_{Sx} = 0$ ; $\Sigma F_x = 0$
Satisfied	$F_{Sy} - W = 0$ ; $\Sigma F_y = 0$

Figure 41 Spring-slider equilibrium

Status	Input	Name	Output	Unit	Comment
					Spring-Slider Example 3.8
	1	b		ft	Free spring length
Guess	.1	h		ft	Vertical displacement of spring
		L		ft	Stretched spring length
	300	k		lb/ft	Spring stiffness
					System forces
		F_S		lb	Total spring force
		F_Sx		lb	x-spring force
		F_Sy		lb	y-spring force
		N		lb	Normal force from bar
	12	W		lb	Weight of slider

Input	Name	Output	Unit	Comment
				Spring-Slider Example 3.8
1	b		ft	Free spring length
	h	.4515051	ft	Vertical displacement of spring
	L	1.097204	ft	Stretched spring length
300	k		lb/ft	Spring stiffness
				System forces
	F_S	29.16124	lb	Total spring force
	F_Sx	26.57777	lb	x-spring force
	F_Sy	12	lb	y-spring force
	N	26.57777	lb	Normal force from bar
12	W		lb	Weight of slider

Figure 42 Starting a iterative solution and obtaining the deflection and forces

You probably recall from physics that you can describe the energy stored in the spring and the mechanical work done by lowering the weight. You could add rules for those items to the TK model, and plot them for a range of displacement values. Those actions are given in Figure 43. Note that the value of  $h$  (0.451 ft) obtained above as the equilibrium position is also the  $h$  value for the minimum position on the graph. The important preview here is that the state of equilibrium minimizes the Total Potential Energy of a system. That principle will be used in later chapters as an alternative to the direct application of Newton's Laws. The Principle of Minimum Total Potential Energy has the advantage that it can be applied to systems that are statically indeterminate. Such systems require consideration of the deformation of bodies in the system, like the spring in this example.

Status	Rule
Satisfied	Potential = $0.5 * k * (L - b)^2$ ; Strain energy in spring
Satisfied	Work = $h * W$ ; Work from displacement
Satisfied	Total = Potential - Work ; Total Potential Energy

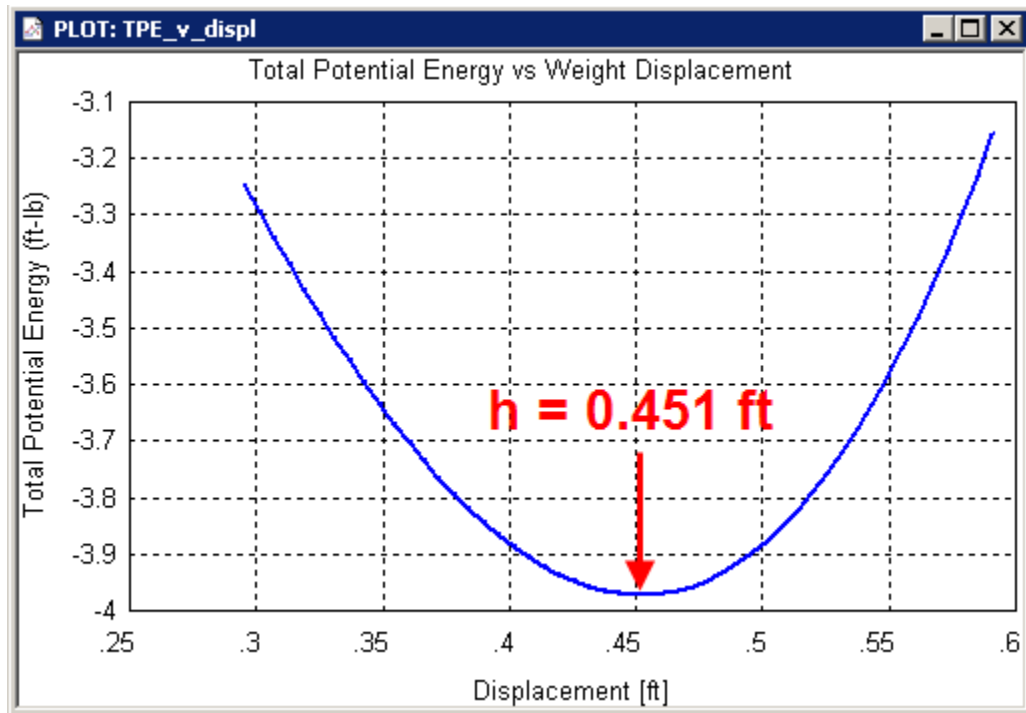



Figure 43 Equilibrium minimizes the total potential energy



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## TK Solver

### Car on an incline

[Rules Sheet](#)

[Rules](#)

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```
; Ex 3.1 Equilibrium by J.E. Akin, Rice University
; The plane angle is 20 degrees up from horizontal,
; thus, the normal force is 20 degrees left of vertical
; Assume the cable is in tension and normal force is compression
; NOTE: TK is non-procedural. Put rules in any order.
;       Put '=' anywhere in each rule
; From Free Body Diagram (FBD) #1
W = m * g ; Gravity
T - N * sin(angle) = 0 ; ΣF_x = m * a_x == 0 Horizontal
N * cos(angle) - W = 0 ; ΣF_y = m * a_y == 0 Vertical
```

Variables Sheet				
Input	Name	Output	Unit	Comment
	W	14126.4	N	Car weight (in negative y-direction)
1440	m		kg	Car mass
9.81	g		m/s <sup>2</sup>	Gravitational constant
	T	5141.589117	N	Cable force (tension)
	N	15033.000885	N	Plane normal force (compression)
20	angle		degrees	Angle of plane up from horizontal
				<b>; set cable tension tension to 5000 N</b>
				<b>; and solve for required angle</b>
				<b>; with a starting Guess angle</b>