

### Introduction

Friction acts parallel to a contact surface and has a maximum amplitude. If motion impends, or exists, the friction force (or torque) acts in a direction opposing the motion (or rotation). The new challenge due to the presence of friction is that you have to make assumptions about the problem, solve the problem, and then verify the original assumption. If the original assumption was wrong, then you must make a new assumption and repeat the process. Eventually, you get an answer that agrees with your assumption. Remember, for friction problems you must first state your assumptions, and then later verify them.

Up to this point, you have made an implied assumption that there was no impending slipping or tipping (because you were not given a coefficient of friction). When you have a coefficient of friction given you have to consider three options: 1 there is no impending motion (as before), 2 there is an impending slip in the positive tangential direction, or 3 there is impending slip in the negative tangential direction.

In belt friction problems, it is useful to note that the maximum moment that can be developed by belt friction is  $M_{max} = R (e^{\mu\beta} - 1)T_{min}$  for a constant radius contact. That relation can be used to check rotational impending slip conditions, like  $f_{max}$  is used to check for impending translation.

### Example 9.1

The stages of this problem are slightly incomplete because the assumptions were not fully stated, nor checked in some parts. That omission is not present in the TK Solver model because it introduces logic rules for the possible behavior and the validation of the results. The initial data are that the angle of incline is 20 degrees, the mass of the crate is 80 kg, and the surface coefficient of friction is 0.4. Figure 1 shows the system and the FBD of the crate for the initial *assumptions* that there is *no impending slip* when the cable tension force is 400 N. The task is to find the necessary friction force. In most friction problems, experience shows it is useful to pick a coordinate system that is tangent and normal to a potential slid surface. That is also shown in Figure 1 (right).

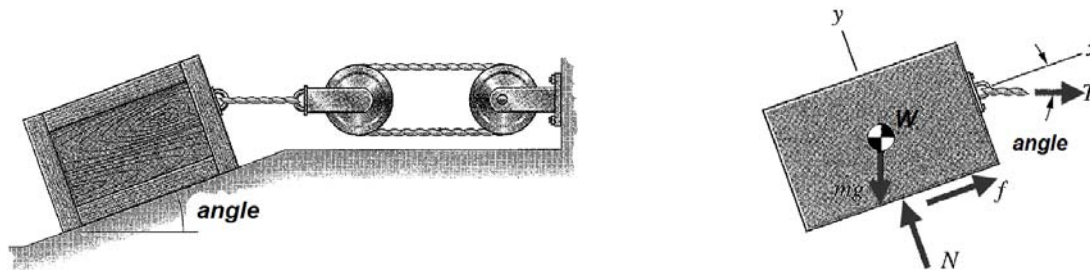


Figure 1 Crate system for assumption of no impending slip

For those assumptions, you solve the problem as before by simply writing the equilibrium and gravity relations. However, you must also check that assumption (which was omitted in the text) by finding the maximum static friction force and showing that it is greater than the required friction.

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Figure 2 shows the TK rules for a single body concurrent system involving friction. There is a text string input variable named *assume* to remind you that you must make a choice for the behavior of the object. The text inputs to TK are preceded with a single quote mark as the first character. Here the choices are 'none for standard equilibrium, 'up for impending motion up the plane, or 'down for impending slip down the plane. For either impending slip assumption, you usually have to give the normal force a Guess starting value. The rules also have a text string output named *status* that you must compare to what you assumed before you accept the output force values. A *no slip* output means that a default equilibrium state exists. A *status* result of *impending slip* means the friction force is at its maximum value. A result of *dynamic* is an error response meaning you need dynamic equations covered later.

Status	Rule
Comment	; Bedford-Fowler Example 9.1, 3rd Ed.
Comment	; friction properties and their status
Satisfied	$f_{max} = \mu * \text{abs}(N)$
Satisfied	if $\text{abs}(f) < f_{max}$ then status = "no slip"
* Ignored	if $\text{abs}(f) = f_{max}$ then status = "impending slip"
* Ignored	if $\text{abs}(f) > f_{max}$ then status = "dynamic"
Comment	; Equilibrium, x-axis parallel to the plane
Satisfied	$f + T * \cos(\text{angle}) - W * \sin(\text{angle}) = 0$ ; $\Sigma F_x = 0$
Satisfied	$N - T * \sin(\text{angle}) - W * \cos(\text{angle}) = 0$ ; $\Sigma F_y = 0$
Satisfied	$W = m * g$ ; Gravity
Comment	; assumed behaviour to be checked against status
* Ignored	if assume = "up" then $f = -f_{max}$
* Ignored	if assume = "down" then $f = f_{max}$
Satisfied	if not (given ('assume) ) then assume = "none"

**Figure 2 Equilibrium and friction logic for concurrent forces**

Figure 3 shows the variables for a no impending motion assumption, as given in part a) of the text example. It gives the same results as the text ( $f = -107 \text{ N}$ ), plus other important results for checking the original assumption. Namely, it shows  $f_{max} = 350 \text{ N}$  which shows that the surface could develop the value of  $f$  (107 N) required for equilibrium. It also shows a consistent *status* = "no slip" required by the original assumption.

In contrast, if you change the cable tension force to 800 N, then you get an error of *status* = "dynamic" which means that the crate accelerates and static equilibrium is not possible. That is shown in Figure 4. Static equilibrium is impossible because the necessary friction force of 483 N is larger than the surface roughness can develop (404 N), so dynamic motion has to occur.

The rules already allow for the second part of this example in the text. Namely, to assume impending motion up the plane and to solve for the corresponding tension force. Then the friction force has a direction down the plane (opposing the impending slip up the plane) and takes on its maximum static value. The FBD for that assumption is in Figure 5, along with the force results (obtained by giving N a

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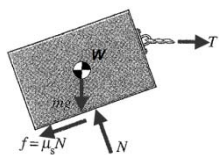
starting guess value). The required tension (702 N) agrees with the text value. The *status* of “impending slip” is consistent with the starting assumption, so the analysis is complete.

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.1, 3rd Ed.
.4	$\mu$			coefficient of friction
20	angle		degrees	plane angle from horizontal
80	m		kg	crate mass
	W	784.8	N	weight of crate
400	T		N	cable force
	assume	'none		assumed impending motion: 'none, 'up, 'down
	N	874.2788	N	plane normal reaction force (guess if not none)
	f	-107.4596	N	plane friction force, + up the plane
	f_max	349.7115	N	maximum static friction force
	status	"no slip"		no slip, impending, or dynamic
9.81	g		m/sec^2	gravitational constant

Figure 3 Results for an assumption of no impending motion

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.1, 3rd Ed.
.4	$\mu$			coefficient of friction
20	angle		degrees	plane angle from horizontal
80	m		kg	crate mass
	W	784.8	N	weight of crate
800	T		N	cable force
	assume	'none		assumed impending motion: 'none, 'up, 'down
	N	1011.087	N	plane normal reaction force (guess if not none)
	f	-483.3367	N	plane friction force, + up the plane
	f_max	404.4348	N	maximum static friction force
	status	'dynamic		no slip, impending, or dynamic
9.81	g		m/sec^2	gravitational constant

Figure 4 A cable force that causes acceleration (no static equilibrium)



Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.1, 3rd Ed.
.4	$\mu$			coefficient of friction
20	angle		degrees	plane angle from horizontal
80	m		kg	crate mass
	W	784.8	N	weight of crate
	T	701.7269252	N	cable force
'up	assume			assumed impending motion: 'none, 'up, 'down
	N	977.4755123	N	plane normal reaction force (guess if not none)
	f	-390.9902049	N	plane friction force, + up the plane
	f_max	390.9902049	N	maximum static friction force
	status	"impending"		no slip, impending, or dynamic
9.81	g		m/sec^2	gravitational constant

Figure 5 Free Body Diagram and outputs for impending slip up the plane

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Since logical rules are included in this model, the Rule Status of which were satisfied and which were ignored changed in the second analysis. For completeness, the changed logic status (contrasting Figure 2) are given in Figure 6. If you made the assumption of impending motion down the plane then the cable force is  $T = -25 \text{ N}$ , compression (try it). We usually assume cables do not transmit compression, but another attachment mechanism could do so.

Status	Rule
	<b>Comment</b> ; Bedford-Fowler Example 9.1, 3rd Ed.
	<b>Comment</b> ; friction properties and their status
Satisfied	$f_{\text{max}} = \mu * \text{abs}(N)$
* Ignored	if abs (f) < f_max then status = "no slip"
Satisfied	if abs (f) = f_max then status = "impending slip"
* Ignored	if abs (f) > f_max then status = "dynamic"
	<b>Comment</b> ; Equilibrium, x-axis parallel to the plane
Satisfied	$f + T * \cos(\text{angle}) - W * \sin(\text{angle}) = 0$ ; $\Sigma F_x = 0$
Satisfied	$N - T * \sin(\text{angle}) - W * \cos(\text{angle}) = 0$ ; $\Sigma F_y = 0$
	<b>Satisfied</b> $W = m * g$ ; Gravity
	<b>Comment</b> ; assumed behaviour to be checked against status
Satisfied	if assume = "up" then f = - f_max
* Ignored	if assume = "down" then f = f_max
* Ignored	if not (given ('assume') ) then assume = "none"

Figure 6 Change in ignored logic rules for this case

### Example 9.2

The general system and its indeterminate FBD are in Figure 7. This problem involves possible sliding (translation) and possible tipping (rotation), or both or neither. Thus, the possible logic checks are more complicated. The new rule set is shown in Figure 8. The stated problem is to find the force  $F$  and its height  $h$  that will cause simultaneous impending slip left and tip left. The text solution is obtained as an analytic expression (included in the rule set for comparison). Here, you will utilize numerical values so that other assumptions can also be treated. To impend to tip left the normal (and friction) force at B must vanish. Then, only one friction force remains, at point A.

For the simultaneous impending tip-slide condition, the one remaining friction force must have its maximum static value. It will also be equal and opposite to  $F$ , and they form a couple. That couple must balance the moment of the weight about point A. In the numerical examples below values have been chosen for the mass  $m$ , base width  $b$ , and coefficient of friction  $\mu$ . Assumptions are input via the string variables *slip* and *tip*. Partial assumption checks are given by the output *status*, like before. The numerical value of the height to the applied force exactly matches the analytic value, for these assumptions. The Variable Sheet for this case is in Figure 9. Figure 10 shows the case where applied force actually tips over the chest. It gives the error status of *dynamic* because the normal force at point B should never become tensile.

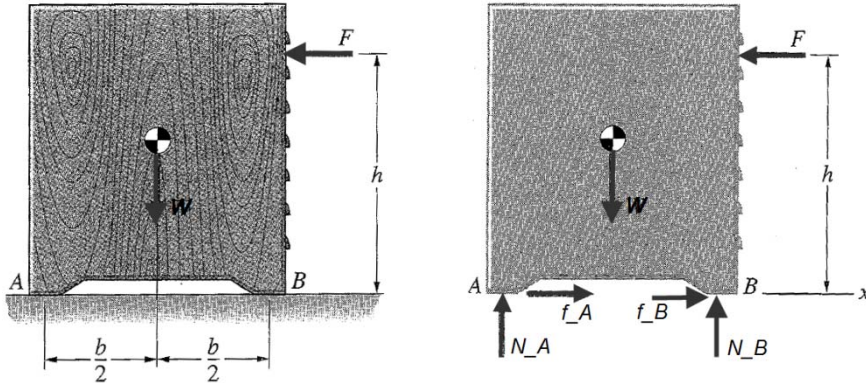


Figure 7 Chest loads and geometry

Status	Rule
Comment	; Bedford-Fowler Example 9.2, 3rd Ed.
Comment	; friction and tip properties and their status
Satisfied	$f_{max} = \mu * abs(N)$
* Ignored	if not (known ('status')) then status = "no slip/tip"
Satisfied	if (or (abs (f) = f_max, N_A = 0, N_B = 0)) then status = "impending"
* Ignored	if (or (abs (f) > f_max, N_A < 0, N_B < 0)) then status = "dynamic"
Comment	; Equilibrium, x-axis parallel to the plane
Satisfied	$N = N_A + N_B$
Satisfied	$f = f_A + f_B$
Satisfied	$f - F = 0$ ; $\Sigma F_x = 0$
Satisfied	$N - W = 0$ ; $\Sigma F_y = 0$
Satisfied	$F * h - W * b/2 + N_B * b = 0$ ; $\Sigma M_A = 0$
Satisfied	$W = m * g$ ; Gravity
Comment	; assumed slip behaviour to be checked against status
* Ignored	if slip = "right" then $f_A = -\mu * abs(N_A)$
* Ignored	if slip = "right" then $f_B = -\mu * abs(N_B)$
Satisfied	if slip = "left" then $f_A = \mu * abs(N_A)$
Satisfied	if slip = "left" then $f_B = \mu * abs(N_B)$
* Ignored	if not (given ('slip')) then slip = "none"
Comment	; assumed tip behaviour to be checked against status
* Ignored	if tip = "right" then $N_A = 0$
* Ignored	if tip = "right" then $f_A = 0$
Satisfied	if tip = "left" then $N_B = 0$
Satisfied	if tip = "left" then $f_B = 0$
* Ignored	if not (given ('tip')) then tip = "none"
Comment	; analytic solution impending tip-slide to left
Satisfied	$h_{LL} = 0.5 * b / \mu$

Figure 8 Chest tip-slide-equilibrium rules

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Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.2, 3rd Ed.
.4	$\mu$			coefficient of friction
100	m		kg	chest mass
	W	981	N	weight of chest
1	b		m	base width
'left	slip			assumed impending slip: 'none, 'left, 'right
'left	tip			assumed impending tip: 'none, 'left, 'right
	N	981	N	total normal reaction force (guess if not none)
	f	392.4	N	total friction force, + to right
	f_max	392.4	N	maximum static friction force
	status	'impending		no slip/tip, impending, or dynamic
	F	392.4	N	applied force
	h	1.25	m	height to force F
	h_LL	1.25	m	analytic tip-slide-left height
9.81	g		m/sec^2	gravitational constant
	f_A	392.4	N	base friction
	f_B	0	N	base friction
	N_A	981	N	normal force
	N_B	0	N	normal force

Figure 9 Chest results for both impending tip-slide to the left

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.2, 3rd Ed.
.4	$\mu$			coefficient of friction
100	m		kg	chest mass
	W	981	N	weight of chest
1	b		m	base width
	slip	'none		assumed impending slip: 'none, 'left, 'right
	tip	'none		assumed impending tip: 'none, 'left, 'right
	N	981	N	total normal reaction force (guess if not none)
	f	390	N	total friction force, + to right
	f_max	392.4	N	maximum static friction force
	status	'dynamic		no slip/tip, impending, or dynamic
390	F		N	applied force
1.5	h		m	height to force F
	h_LL	1.25	m	analytic tip-slide-left height
9.81	g		m/sec^2	gravitational constant
	f_A		N	base friction
	f_B		N	base friction
	N_A	1075.5	N	normal force
	N_B	-94.5	N	normal force

Figure 10 Chest has dynamic (tipping) motion when known force is too high

Likewise, if you push with a small force  $F$ , at a high height you should expect the chest to impend to tip, but not to impend to slide. For the input values in Figure 11, the required tipping force  $F$  is less than the maximum friction force. Thus, it is clear that the assumption of tip without slip is validated and that solution is complete.

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.2, 3rd Ed.
.4	$\mu$			coefficient of friction
100	$m$		kg	chest mass
	$W$	981	N	weight of chest
1	$b$		m	base width
	slip	'none		assumed impending slip: 'none', 'left', 'right
'left	tip			assumed impending tip: 'none', 'left', 'right
	$N$	981	N	total normal reaction force (guess if not none)
	$f$	163.5	N	total friction force, + to right
	$f_{max}$	392.4	N	maximum static friction force
	status	'impending		no slip/tip, impending, or dynamic
	$F$	163.5	N	applied force
3	$h$		m	height to force $F$
	$h_{LL}$	1.25	m	analytic tip-slide-left height
9.81	$g$		m/sec <sup>2</sup>	gravitational constant
	$f_A$	163.5	N	base friction
	$f_B$	0	N	base friction
	$N_A$	981	N	normal force
	$N_B$	0	N	normal force

Figure 11 High force to cause tipping only without slip

Text Figure 9.14

For the pair of blocks in text figure 9.14 the logical first assumption is that the lower block is intended to slide to the left, and the top will thus slide up the vertical wall. That figure is repeated here as Figure 12, with changes that anticipate alternate assumptions.

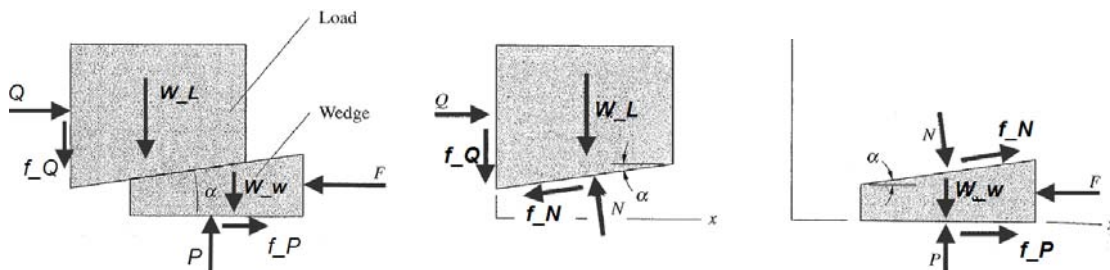


Figure 12 System of load block on a wedge (with possible slip to the left)

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When multiple bodies are in contact at frictional surfaces, you have to make consistent assumptions about the frictional forces and verify them. When you draw the original FBD you should apply the forces in equal and opposite pairs and make them consistent with one assumed possible behavior. There are three possible assumptions for impending sliding. Two of them are detailed in the rules in Figure 13. It is clear that for the wedge to impend to slide left, then the load block must impend to slide up the wall. For impending slip to the right, there are two possible behaviors for the load block (discussed later).

Status	Rule
Comment	; Bedford-Fowler Figure 9.14 example, 3rd Ed
Comment	; ASSUME no wall contact for slip = 'right. $\alpha < \text{no\_wall}$
Comment	; Wedge FBD
Satisfied	$f_P + N \sin(\alpha) + f_N \cos(\alpha) - F = 0$ ; $\Sigma F_x = 0$
Satisfied	$P - W_w - N \cos(\alpha) + f_N \sin(\alpha) = 0$ ; $\Sigma F_y = 0$
Satisfied	if slip = "left" then $f_N = f_{Nmax}$
Satisfied	if slip = "left" then $f_P = f_{Pmax}$
* Ignored	if slip = "right" then $f_P = -f_{Pmax}$
* Ignored	if slip = "right" then $f_Q = 0$
* Ignored	if slip = "right" then $Q = 0$
* Ignored	if not (given ('slip')) then slip = "none"
Comment	; Load FBD
Satisfied	$Q - N \sin(\alpha) - f_N \cos(\alpha) = 0$ ; $\Sigma F_x = 0$
Satisfied	$-f_Q + N \cos(\alpha) - f_N \sin(\alpha) - W_L = 0$ ; $\Sigma F_y = 0$
Satisfied	if slip = "left" then $f_Q = f_{Qmax}$
Comment	; System FBD (optional)
Satisfied	$Q + f_P - F = 0$ ; $\Sigma F_x = 0$
Satisfied	$-f_Q - W_L - W_w + P = 0$ ; $\Sigma F_y = 0$
Comment	; analytic forces to impend slip left or right
Satisfied	$F_{left} = \mu W_w + W_L * ((1-\mu^2)\tan(\alpha) + 2\mu) / (1-\mu^2 - 2\mu \tan(\alpha))$
Satisfied	$F_{right} = -\mu (W_w + W_L)$ ; if no wall contact
Satisfied	$f_{Nmax} = \mu \text{ abs}(N)$ ; for checking
Satisfied	$f_{Pmax} = \mu \text{ abs}(P)$ ; for checking
Satisfied	$f_{Qmax} = \mu \text{ abs}(Q)$ ; for checking
Satisfied	$\tan(\text{no\_wall}) = \mu$ ; for checking
* Ignored	if and (slip='right, $\mu > \text{no\_wall}$ ) then status = "dynamic"
* Ignored	if $\text{abs}(f_P) > f_{Pmax} * 1.001$ then status = "dynamic"
* Ignored	if and (slip='none, $\text{abs}(f_P) = f_{Pmax}$ ) then status = "impending"
* Undeterm	if not (known ('status')) then status = "ok"
* Ignored	if $\mu = 0$ then $f_N = 0$ ; frictionless no slip solution
* Ignored	if $\mu = 0$ then $f_P = 0$ ; frictionless no slip solution
* Ignored	if $\mu = 0$ then $f_Q = 0$ ; frictionless no slip solution

Figure 13 Rules impending sliding left, or right (without vertical wall contact)

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First impending slip to the left of the wedge is considered. The text analytically solves four equations to obtain an expression for the corresponding force as a percentage of the weight of the load block. Figure 13 gives the same rules (plus two for system equilibrium), but here numerical values are chosen using the same guidelines as the text. The angle of the top of the wedge is small at 10 degrees. The coefficient of friction on all surfaces is taken in turn to be 0, 0.2, and 0.4. The text example requires  $W_w = 0.2 W_L$ , so the load block is assigned a weight of 1,000 N and the wedge weight is 200 N. The assumed *slip* behavior is input to be to the *left*. Those inputs for smooth surfaces ( $\mu = 0$ ) are given in Figure 14, along with the corresponding outputs. For smooth surface equilibrium, the force F is 0.176 of the load block weight. Any larger value would cause dynamic motion to the left, and any smaller value would result in dynamic motion to the right.

Input	Name	Output	Unit	Comment
				<b>Bedford-Fowler Figure 9.14 example, 3rd Ed</b>
0	$\mu$			<b>coefficient of friction</b>
1000	$W_L$		<b>N</b>	<b>weight of load</b>
200	$W_w$		<b>N</b>	<b>weight of wedge</b>
10	$\alpha$		<b>degrees</b>	<b>angle between load and wedge</b>
	no_wall	0	<b>degrees</b>	<b>largest angle for no wall contact for slip 'right</b>
'none	slip			<b>wedge impending slip: 'none, 'left, 'right</b>
	status	'impending		<b>error if status = dynamic</b>
	F	176.327	<b>N</b>	<b>applied force, + left</b>
	N	1015.427	<b>N</b>	<b>wedge-load interface normal force</b>
	f_N	0	<b>N</b>	<b>wedge-load interface friction force</b>
	f_Nmax	0	<b>N</b>	<b>wedge-load maximum friction force</b>
	P	1200	<b>N</b>	<b>wedge-floor interface normal force</b>
	f_P	0	<b>N</b>	<b>wedge-floor interface friction force</b>
	f_Pmax	0	<b>N</b>	<b>wedge-floor maximum friction force</b>
	Q	176.327	<b>N</b>	<b>wedge-wall interface normal force, &gt;= 0</b>
	f_Q	0	<b>N</b>	<b>wedge-wall interface friction force</b>
	f_Qmax	0	<b>N</b>	<b>maximum allowed wall friction force</b>
	F_left	176.327	<b>N</b>	<b>analytic force for impending left slip</b>
	F_right	0	<b>N</b>	<b>impending right slip force, iff <math>\alpha \leq \text{no\_wall}</math></b>

**Figure 14 Equilibrium forces for smooth surfaces**

The second text case is for  $\mu = 0.2$  and assumed slip impending to the left. In that case, you should see that all surfaces impend to slip at the same time, and all friction forces take their maximum value, and oppose the direction of relative impending slip. To obtain the results in Figure 15, you must assign guess values to all the normal forces. As in the text, the required applied horizontal force is 0.680 times the weight of the load block. Doubling the friction coefficient, for impending slip left, yields a require force of about 1.44 times the load, as found in the text example. Those outputs are in Figure 16.

Input	Name	Output	Unit	Comment
				Bedford-Fowler Figure 9.14 example, 3rd Ed
.2	$\mu$			coefficient of friction
1000	W_L		N	weight of load
200	W_w		N	weight of wedge
10	$\alpha$		degrees	angle between load and wedge
	no_wall	11.30993	degrees	largest angle for no wall contact for slip 'right
'left	slip			wedge impending slip: 'none, 'left, 'right
	status			error if status = dynamic
	F	680.0153	N	applied force, + left
	N	1141.61	N	wedge-load interface normal force
	f_N	228.3219	N	wedge-load interface friction force
	f_Nmax	228.3219	N	wedge-load maximum friction force
	P	1284.618	N	wedge-floor interface normal force
	f_P	256.9237	N	wedge-floor interface friction force
	f_Pmax	256.9237	N	wedge-floor maximum friction force
	Q	423.0916	N	wedge-wall interface normal force, $\geq 0$
	f_Q	84.61833	N	wedge-wall interface friction force
	f_Qmax	84.61833	N	maximum allowed wall friction force
	F_left	680.0153	N	analytic force for impending left slip
	F_right	-240	N	impending right slip force, iff $\alpha \leq \text{no\_wall}$

Figure 15 Results for impending left slip and  $\mu = 0.2$

It is useful to remember that if a surface is inclined at an angle greater than the arctangent of the coefficient of friction, objects on that surface will always slip (and create a dynamic problem). For inclined angles less than that, the object sits on the surface in equilibrium. In the rules in Figure 13 that angle is given the name *no\_wall*. For the given angle, and  $\mu \geq 0.2$ , the two objects stick together and there is no wall force Q. That is the case for the given data. Such a result, for the larger friction value, is in Figure 17. Note in Figure 17 that the interface friction force is less than its maximum value. That validates the assumption that the load sits on the wedge for high friction.

A different result, for low friction, is given in Figure 18 for impending slip to the right. The status output reports an error (dynamic). That is because the interface friction (342 N) is larger than the maximum (188 N) that can develop there. Also, note that the *no\_wall* angle is less than the given incline, so the wall force Q must be present because the load would slip down relative to the wedge.

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Input	Name	Output	Unit	Comment
				Bedford-Fowler Figure 9.14 example, 3rd Ed
.4	$\mu$			coefficient of friction
1000	W_L		N	weight of load
200	W_w		N	weight of wedge
10	$\alpha$		degrees	angle between load and wedge
	no_wall	21.80141	degrees	largest angle for no wall contact for slip 'right
'left	slip			wedge impending slip: 'none, 'left, 'right
	status			error if status = dynamic
	F	1436.507	N	applied force, + left
	N	1452.813	N	wedge-load interface normal force
	f_N	581.1251	N	wedge-load interface friction force
	f_Nmax	581.1251	N	wedge-load maximum friction force
	P	1529.83	N	wedge-floor interface normal force
	f_P	611.932	N	wedge-floor interface friction force
	f_Pmax	611.932	N	wedge-floor maximum friction force
	Q	824.5748	N	wedge-wall interface normal force, $\geq 0$
	f_Q	329.8299	N	wedge-wall interface friction force
	f_Qmax	329.8299	N	maximum allowed wall friction force
	F_left	1436.507	N	analytic force for impending left slip
	F_right	-480	N	impending right slip force, iff $\alpha \leq \text{no\_wall}$

Figure 16 Results for impending left slip and  $\mu = 0.4$

Input	Name	Output	Unit	Comment
				Bedford-Fowler Figure 9.14 example, 3rd Ed
.4	$\mu$			coefficient of friction
1000	W_L		N	weight of load
200	W_w		N	weight of wedge
10	$\alpha$		degrees	angle between load and wedge
	no_wall	21.80141	degrees	largest angle for no wall contact for slip 'right
'right	slip			wedge impending slip: 'none, 'left, 'right
	status	'dynamic		error if status = dynamic
	F	-480	N	applied force, + left
	N	984.8078	N	wedge-load interface normal force
	f_N	-173.6482	N	wedge-load interface friction force
	f_Nmax	393.9231	N	wedge-load maximum friction force
	P	1200	N	wedge-floor interface normal force
	f_P	-480	N	wedge-floor interface friction force
	f_Pmax	480	N	wedge-floor maximum friction force
	Q	0	N	wedge-wall interface normal force, $\geq 0$
	f_Q	0	N	wedge-wall interface friction force
	f_Qmax	0	N	maximum allowed wall friction force
	F_left	1436.507	N	analytic force for impending left slip
	F_right	-480	N	impending right slip force, iff $\alpha \leq \text{no\_wall}$

Figure 17 Results for impending right slip and  $\mu = 0.4$

Input	Name	Output	Unit	Comment
				Bedford-Fowler Figure 9.14 example, 3rd Ed
.2	$\mu$			coefficient of friction
1000	$W_L$		N	weight of load
200	$W_w$		N	weight of wedge
20	$\alpha$		degrees	angle between load and wedge
	no_wall	11.30993	degrees	largest angle for no wall contact for slip 'right
'right	slip			wedge impending slip: 'none, 'left, 'right
	status	'dynamic		error if status = dynamic
	F	-240	N	applied force, + left
	N	939.6926	N	wedge-load interface normal force
	f_N	-342.0201	N	wedge-load interface friction force
	f_Nmax	187.9385	N	wedge-load maximum friction force
	P	1200	N	wedge-floor interface normal force
	f_P	-240	N	wedge-floor interface friction force
	f_Pmax	240	N	wedge-floor maximum friction force
	Q	0	N	wedge-wall interface normal force, $\geq 0$
	f_Q	0	N	wedge-wall interface friction force
	f_Qmax	0	N	maximum allowed wall friction force
	F_left	960.1872	N	analytic force for impending left slip
	F_right	-240	N	impending right slip force, iff $\alpha \leq \text{no\_wall}$

Figure 18 Invalid dynamic response occurs for bad assumptions in slipping right

Example 9.10

The two-pulley system of Figure 19 is externally indeterminate, as it has six unknowns. That means that you must prepare FDBs for the individual pulleys also. That is true even if there is no impending belt slip that requires consideration of belt friction. In this case, the system is rendered solvable by measuring (specifying) the external horizontal force on one pulley. There are four possible assumptions regarding impending belt slip. Two assumptions are covered in the text example while three are covered in the TK rules of Figure 20. The four options are no slip occurs, slip occurs on the small pulley only, or on the large pulley only, or on both pulleys at once.

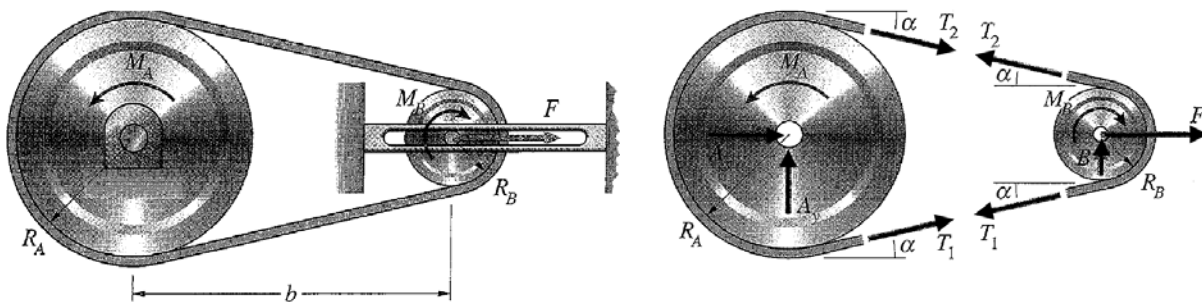


Figure 19 Two pulley system with belt force FBD

Status	Rule
Comment	; Bedford-Fowler Example 9.10, 3rd Ed
Comment	; Large pulley FBD
Satisfied	$A_x + T_1 \cos(\alpha) + T_2 \cos(\alpha) = 0$ ; $\Sigma F_x = 0$
Satisfied	$A_y + T_1 \sin(\alpha) - T_2 \sin(\alpha) = 0$ ; $\Sigma F_y = 0$
Satisfied	$M_a + R_a T_1 - R_a T_2 = 0$ ; $\Sigma M_a = 0$
Comment	; Small pulley FBD
Satisfied	$F - T_1 \cos(\alpha) - T_2 \cos(\alpha) = 0$ ; $\Sigma F_x = 0$
Satisfied	$B - T_1 \sin(\alpha) + T_2 \sin(\alpha) = 0$ ; $\Sigma F_y = 0$
Satisfied	$-M_b - R_b T_1 + R_b T_2 = 0$ ; $\Sigma M_b = 0$
Comment	; System FBD
Satisfied	$A_x + F = 0$ ; $\Sigma F_x = 0$
Satisfied	$A_y + B = 0$ ; $\Sigma F_y = 0$
Satisfied	$M_a - M_b + b B = 0$ ; $\Sigma M_a = 0$
Comment	; Belt friction and geometry
Satisfied	$b \sin(\alpha) = R_a - R_b$
Satisfied	$\beta_B = \pi - 2\alpha$
Satisfied	$\beta_A = \pi + 2\alpha$
Satisfied	$M_{b\_max} = R_b (\exp(\mu \beta_B) - 1) \text{ MIN}(T_1, T_2)$
Satisfied	$M_{a\_max} = R_a (\exp(\mu \beta_A) - 1) \text{ MIN}(T_1, T_2)$
Satisfied	if slip = "small" then $\beta = \beta_B$
* Ignored	if slip = "large" then $\beta = \beta_A$
Satisfied	if slip = "small" then $T_2 = T_1 \exp(\mu \beta)$
* Ignored	if slip = "large" then $T_1 = T_2 \exp(\mu \beta)$
Satisfied	if not ( $T_1 = 0$ ) then ratio = $T_2 / T_1$
Comment	; Redundant rule from pulley moments
Satisfied	$M_a R_b = M_b R_a$

Figure 20 Belt friction and equilibrium rules for the two pulley system

At the bottom of Figure 20 it is noted that the ratio of the moment values is related to the ratio of the pulley radii. If you had not noticed that relation, you would obtain the same numerical results as given below. It often speeds the TK iterative solver convergence to have redundant rules.

For this problem, it can be useful to consider some of the equilibrium solutions without the assumption of impending slip. That can give insight into starting values that you may need for iterations when impending slip is present. The simplest case of no external moments, in Figure 21, yields the pre-tension belt force values of about 102 N. Also note that the maximum moments that can be developed is obtained for each pulley. Giving a no zero value to moment  $M_a$ , as in Figure 22, requires unequal tensions to develop. That in turn reduces the maximum frictional moment (defined in the Introduction above). The Figure 23 plot of that relationship predicts impending slip at about  $M_a = 33 \text{ N}\cdot\text{m}$ .

Input	Name	Output	Unit	Comment
				<b>Bedford-Fowler Example 9.10, 3rd Ed</b>
.5	b		m	distance between pulley centers
200	F		N	horizontal force on small pulley center
.2	Ra		m	large pulley radius
.1	Rb		m	small pulley radius
.8	$\mu$			coefficient of friction
'none	slip			assumed slip 'small or 'large
	$\alpha$	.2013579	radians	angle of belt(s) from horizontal
	$\beta$		radians	belt contact angle on impending slip pulley
	$\beta_A$	3.544308	radians	contact angle on large pulley
	$\beta_B$	2.738877	radians	contact angle on small pulley
	Ax	-200	N	horizontal force on large pulley center
	Ay	0	N	vertical force on large pulley center
	B	0	N	vertical force on small pulley center
0	Ma		N-m	CW moment on large pulley
	Ma_max	327.3745	N-m	max no slip moment on large pulley
	Mb	0	N-m	CCW moment on small pulley
	Mb_max	81.08893	N-m	max no slip moment on small pulley
	ratio	1		ratio of belt tensions
	T1	102.0621	N	bottom belt tension
	T2	102.0621	N	top belt tension

Figure 21 Belt tensions with no external moments

10	Ma		N-m	CW moment on large pulley
	Ma_max	247.1845	N-m	max no slip moment on large pulley
	Mb	5	N-m	CCW moment on small pulley
	Mb_max	61.22628	N-m	max no slip moment on small pulley
	ratio	1.648828		ratio of belt tensions
	T1	77.06207	N	bottom belt tension
	T2	127.0621	N	top belt tension
33	Ma		N-m	CW moment on large pulley
	Ma_max	62.74735	N-m	max no slip moment on large pulley
	Mb	16.5	N-m	CCW moment on small pulley
	Mb_max	15.54218	N-m	max no slip moment on small pulley
	ratio	9.434689		ratio of belt tensions
	T1	19.56207	N	bottom belt tension
	T2	184.5621	N	top belt tension

Figure 22 Increasing the external moment reduces the available friction moment

The outputs for the assumption of small pulley impending slip are in Figure 24. In addition to the given data, it was necessary to assign guess starting values to the pulley support forces to obtain a solution. The results in that figure agree with the text and yield an impending rotational slip value of  $Mb = 16.3 \text{ N}\cdot\text{m}$ . Figure 25 gives the outputs for the assumption that only the large pulley impends to slip. Then the

moment on the smaller pulley is  $M_b = 18.1 \text{ N}\cdot\text{m}$ . That is larger than the moment that causes the smaller pulley to slip. Thus, you must conclude that the assumption that only the large pulley slips is wrong.

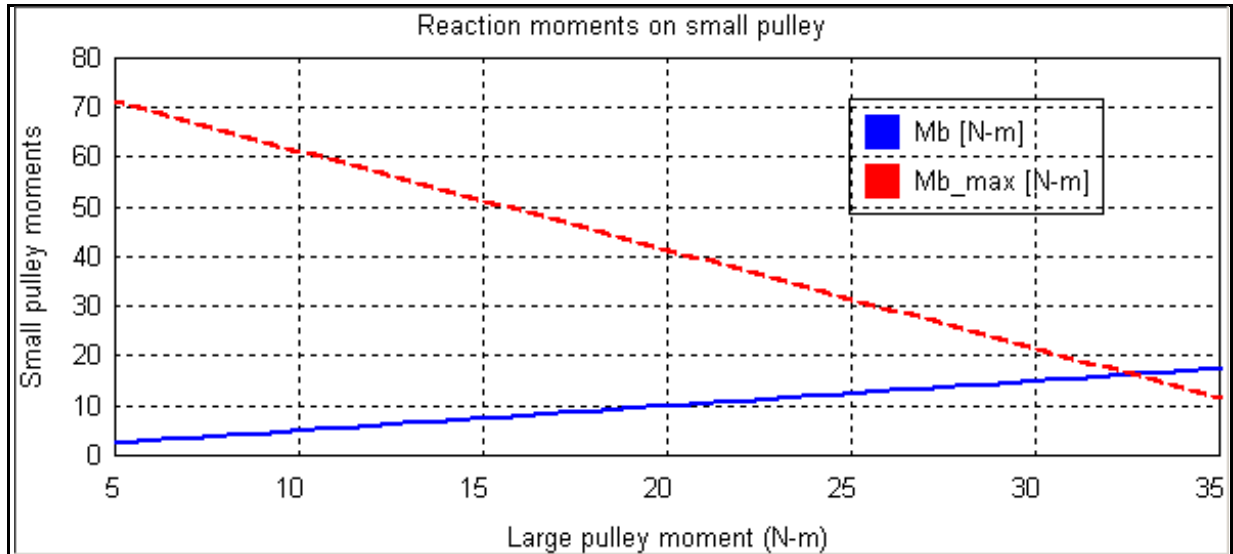


Figure 23 Comparing available friction moment (dashed) to moment required for equilibrium

Input	Name	Output	Unit	Comment
				<b>Bedford-Fowler Example 9.10, 3rd Ed</b>
.5	b		m	distance between pulley centers
200	F		N	horizontal force on small pulley center
.2	Ra		m	large pulley radius
.1	Rb		m	small pulley radius
.8	$\mu$			coefficient of friction
'small	slip			assumed slip 'small or 'large
	$\alpha$	.2013579	radians	angle of belt(s) from horizontal
	$\beta$	2.738877	radians	belt contact angle on impending slip pulley
	$\beta_A$	3.544308	radians	contact angle on large pulley
	$\beta_B$	2.738877	radians	contact angle on small pulley
	Ax	-200	N	horizontal force on large pulley center
	Ay	32.61476	N	vertical force on large pulley center
	B	-32.61476	N	vertical force on small pulley center
	Ma	32.61476	N-m	CW moment on large pulley
	Ma_max	65.83661	N-m	max no slip moment on large pulley
	Mb	16.30738	N-m	CCW moment on small pulley
	Mb_max	16.30738	N-m	max no slip moment on small pulley
	ratio	8.94506		ratio of belt tensions
	T1	20.52518	N	bottom belt tension
	T2	183.599	N	top belt tension

Figure 24 Output for assumption that the small pulley impends to slip

Input	Name	Output	Unit	Comment
				<b>Bedford-Fowler Example 9.10, 3rd Ed</b>
.5	b		m	distance between pulley centers
200	F		N	horizontal force on small pulley center
.2	Ra		m	large pulley radius
.1	Rb		m	small pulley radius
.8	$\mu$			coefficient of friction
'large	slip			assumed slip 'small or 'large
	$\alpha$	.2013579	radians	angle of belt(s) from horizontal
	$\beta$	3.544308	radians	belt contact angle on impending slip pulley
	$\beta\_A$	3.544308	radians	contact angle on large pulley
	$\beta\_B$	2.738877	radians	contact angle on small pulley
-200	Ax		N	horizontal force on large pulley center
	Ay	-36.2983	N	vertical force on large pulley center
	B	36.2983	N	vertical force on small pulley center
	Ma	-36.2983	N-m	CW moment on large pulley
	Ma_max	36.2983	N-m	max no slip moment on large pulley
	Mb	-18.14915	N-m	CCW moment on small pulley
	Mb_max	8.990895	N-m	max no slip moment on small pulley
	ratio	.0586923		ratio of belt tensions
	T1	192.8078	N	bottom belt tension
	T2	11.31633	N	top belt tension

**Figure 24 Output for the incorrect assumption that only the large pulley impends to slip**

The possibility that both pulleys could impend to slip together was not included in the rule set. For that to be possible requires  $\mu_A \beta_A = \mu_B \beta_B$ . Here, the friction coefficient was taken to be the same. Thus the simultaneous slip assumption requires equal contact angles which means equal radii, which are not present in this example.

**Example 9.11**

For the spring-block system shown in Figure 25 the goal is to find the forces developed as the top of the spring is moved horizontally (either left or right). Specifically, you need to find the displacement that causes sliding motion to impend. This is similar to example 3.8 given earlier, but it is extended to include friction effects. The rules for this study are given in Figure 26. The first goal of finding the spring position where slip impends is found by specifying impending slip as an assumption and then guessing a starting value for that position. The outputs in Figure 27 show that the required displacement is about  $x = 0.428$  m, and that the maximum friction available at that position is about 34.6 N.

In general, you should find it educational to see how the friction force and its maximum possible static value varies with position. To do that you can input the assumption of no slip, and a list of spring displacement positions. Executing a list solve lets you plot the friction required for equilibrium versus the available friction. That assumption and the desired list (status) variables are in Figure 28, while the

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graphs of the frictional forces are in Figure 29. The normal force decreases as the spring force increases. Likewise, the (dashed line) available friction force,  $f_{max}$ , decreases in direct proportion to the normal force. The actual friction force required for equilibrium as the spring force angle from the vertical increases. Eventually, the required friction force reaches the maximum available, at about  $x = 0.43$  m. Beyond that position slip will occur and the problems converts to a dynamic solution of the sort to be considered later.

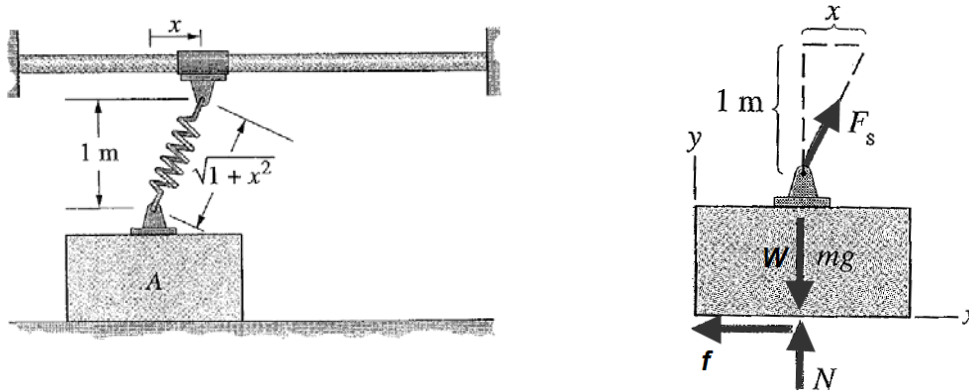


Figure 26 A block-spring system before and at impending slip

Status	Rule
Comment	; Bedford-Fowler Example 9.11, 3rd Ed
Comment	; spring data
Satisfied	$L^2 = h^2 + x^2$ ; spring length
Satisfied	$F_s = k * (L - h)$ ; spring force
Satisfied	$F_{sX} = F_s * x / L$ ; component of spring force
Satisfied	$F_{sY} = F_s * h / L$ ; component of spring force
Comment	; Equilibrium
Satisfied	$N - W + F_{sY} = 0$ ; $\Sigma F_y = 0$
Satisfied	$-f + F_{sX} = 0$ ; $\Sigma F_x = 0$
Comment	; Friction
Satisfied	if slip = "right" then $f = \mu * N$
* Ignored	if slip = "left" then $f = -\mu * N$
* Ignored	if not (given('slip')) then 'slip = "none"
Satisfied	$f_{max} = \mu * \text{abs}(N)$
Satisfied	$W = m * g$ ; gravity

Figure 26 block-spring rules for equilibrium and impending slip

Input	Name	Output	Unit	Comment
				Bedford-Fowler Example 9.11, 3rd Ed
20	m		kg	mass of block
9.81	g		m/sec^2	gravitational constant
	W	196.2	N	weight of block
.3	$\mu$			static coefficient of friction
'right	slip			assumed slip behavior: 'left, 'none, 'right
	f	34.61852	N	friction force
	f_max	34.61852	N	maximum friction amplitude
1	h		m	height, and unstretched length, of spring
1000	k		N/m	spring constant
	L	1.087908	m	length of spring
	x	.428421	m	horizontal displacement of top of spring
	F_s	87.90834	N	force in spring
	N	115.3951	N	normal force on block from plane
	F_sX	34.61852	N	x-component of spring force
	F_sY	80.80492	N	y-component of spring force

Figure 27 Assuming impending slip and guessing the position, x

Status	Input	Name	Output	Unit	Comment
	9.81	g		m/sec^2	gravitational constant
		W	196.2	N	weight of block
	.3	$\mu$			static coefficient of friction
	'none	slip			assumed slip behavior: 'left, 'none, 'right
L		f	.496281	N	friction force
L		f_max	57.37116	N	maximum friction amplitude
	1	h		m	height, and unstretched length, of spring
	1000	k		N/m	spring constant
		L	1.004988	m	length of spring
L	.1	x		m	horizontal displacement of top of spring
		F_s	4.987562	N	force in spring
		N	191.2372	N	normal force on block from plane
		F_sX	.496281	N	x-component of spring force
		F_sY	4.96281	N	y-component of spring force

Figure 28 Assuming no slip for various positions gives list of actual and maximum friction

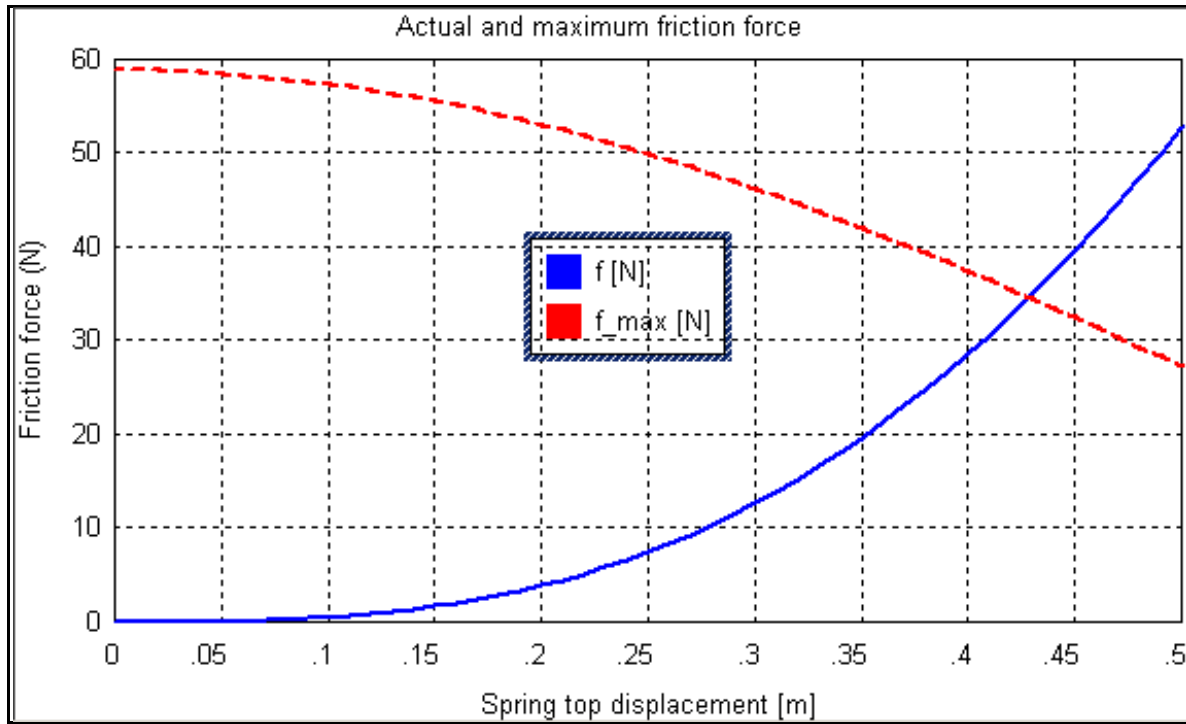


Figure 29 Required friction exceeds available friction at about  $x = 0.43$  m