Concepts of Mechanical Vibrations

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Introduction

Consider the single degree of freedom (dof) system in Figure 1 that is usually introduced in a first course in physics or ordinary differential equations. There, k is the spring constant, or stiffness, and m is the mass, and c is a viscous damper. If the system is subjected to a horizontal force, say f(t), then Newton's law of motion leads to the differential equation of motion in terms of the displacement as a function of time, x(t):

$$m\, d^2x / dt^2 + c\, dx / dt + k\, x(t) = f(t)$$

which requires the initial conditions on the displacement, x(0), and velocity, v(0) = dx / dt(0). When there is no external force and no damping, then it is called free, undamped motion, or simple harmonic motion (SHM):

$$m\, d^2x / dt^2 + k\, x(t) = 0.$$

The usual simple harmonic motion assumption is $x(t) = a \sin(\omega t)$ where a is the amplitude of motion and ω is the circular frequency of the motion. Then the motion is described by

$$[k - \omega^2 m]\, a \sin(\omega t) = 0, \text{ or } [k - \omega^2 m] = 0.$$

This is solved for the circular frequency, ω, which is related to the so called natural frequency, F_n, by $F_n = \omega / 2\pi$.

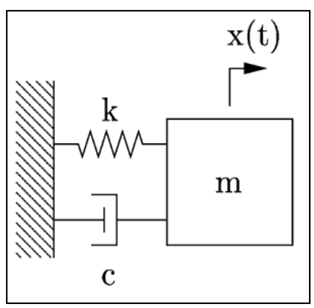

Figure 1 A spring-mass-damper system

Natural Frequencies

A spring and a mass interact with one another to form a system that resonates at their characteristic natural frequency. If energy is applied to a spring-mass system, it will vibrate at its natural frequency. The level of a general vibration depends on the strength of the energy source as well as the damping inherent in the system. The natural frequency of an undamped (c = 0), free (f=0), single spring-mass system is given by the following equation:

$$\omega = 2\pi F_n = \sqrt{(k/m)}$$

where F_n is the natural frequency. From this, it is seen that if the stiffness increases, the natural frequency also increases, and if the mass increases, the natural frequency decreases. If the system has damping, which all physical systems do, its frequency of response is a little lower, and depends on the amount of damping.

Finite element vibrations

Any physical structure vibration can be modeled by springs (stiffnesses), masses, and dampers. In elementary models you use line springs and dampers, and point masses. In finite element models, the continuous nature of the stiffness and mass leads to the use of square matrices for stiffness, mass, and damping. They can still contain special cases of line element springs and dampers, as well as point masses. Dampers dissipate energy, but springs and masses do not.

If you have a finite element system with many dof then the above single dof system generalizes to a displacement vector, **X**(t) interacting with a square mass matrix, **M**, stiffness matrix, **K**, damping matrix **C**, and externally applied force vector, **F**(t), but retains the same general form:

$$\mathbf{M}\, d^2\mathbf{X} / dt^2 + \mathbf{C}\, d\mathbf{X} / dt + \mathbf{K}\, \mathbf{X}(t) = \mathbf{F}(t)$$

plus the initial conditions on the displacement, **X**(0), and velocity, **v**(0) = d**X** / dt(0). Integrating these equations in time gives a *time history solution*. The solution concepts are basically the same, they just have to be done using matrix algebra. The corresponding SHM, or free vibration mode (**C** = **0**, **F** = **0**) for a finite element system is

$$\mathbf{M}\, d^2\mathbf{X} / dt^2 + \mathbf{K}\, \mathbf{X}(t) = \mathbf{0}.$$

The SHM assumption generalizes to **X**(t) = **A** sin (ωt) where the amplitude, **A**, is usually called the mode shape vector at circular frequency ω. This leads to the general matrix *eigenvalue problem*

$$|\, \mathbf{K} - \omega^2\, \mathbf{M}\, | = 0.$$

There is a frequency, say ω_k, and mode shape vector, $\mathbf{A}_k$, for each degree of freedom, k. A matrix eigenvalue-eigenvector solution is much more computationally expensive that a matrix time history solution. Therefore most finite element systems usually solve for the first few natural frequencies. Depending on the available computer power, that may mean 10 to 100 frequencies. CosmosWorks includes natural frequency and mode shape calculations as well as time history solutions.

Usually you are interested only in the first few natural frequencies. A zero natural frequency corresponds to a rigid body motion. If a shell model is used the rotational dof exist and the mass matrix is generalized to include the mass moments of inertia. For every natural frequency there is a corresponding vibration mode shape. Most mode shapes can generally be described as being an axial mode, torsional mode, bending mode, or general mode

Like stress analysis models, probably the most challenging part of getting accurate finite element natural frequencies and mode shapes is to get the type and locations of the restraints correct. A crude mesh will give accurate frequency values, but not accurate stress values. TK Solver contains equations for most known analytic solutions for the frequencies of mechanical systems. They can be quite useful in validating the finite element frequency results.

