R. R. Craig, "Mechanics of Materials"
Wiley, 1996.

TABLE 1. Unit-Load Deflection Calculations

(1) Member	(2) L_i	(3) F_i	(4) F_{ui}	(5) $F_i F_{ui} L_i$
AB	$\sqrt{3}L$	$\frac{-2P\sqrt{3}}{3}$	$\frac{2}{3}$	$-\frac{4}{3}PL$
AC	L	$\frac{4P}{3}$	$\frac{2\sqrt{3}}{9}$	$\frac{8\sqrt{3}}{27}PL$
BC	L	$\frac{2P}{3}$	$\frac{-2\sqrt{3}}{9}$	$\frac{-4\sqrt{3}}{27}PL$
BD	L	$\frac{-2P}{3}$	$\frac{2\sqrt{3}}{9}$	$-\frac{4\sqrt{3}}{27}PL$
CD	$\sqrt{3}L$	$\frac{P\sqrt{3}}{3}$	$\frac{2}{3}$	$\frac{2}{3}PL$
				$\sum_{i=1}^{5} = -\frac{2}{3}PL$

Review the Solution The answer is dimensionally correct. It also makes sense that pulling down on node C would tend to pull node D to the left, so the sign of the answer is reasonable.

*11.9 DYNAMIC LOADING; IMPACT

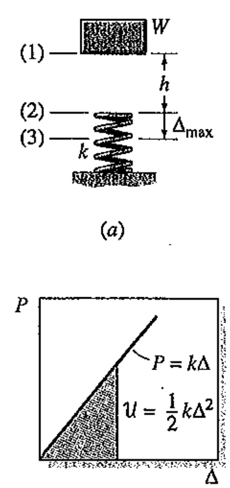

FIGURE 11.21 Impact loading—Falling-weight case.

So far we have assumed that all loads are applied slowly until each reaches its maximum value and then remains at this *static-load* value. When forces are applied more rapidly, like wave loading of an offshore oil platform or impact loading of an automobile during a collision, it is necessary to turn to the topic of structural dynamics to determine the time-dependent behavior of the dynamically loaded deformable body.[12] Here we make simplifying assumptions that enable us to see that there is definitely a difference between the response of a deformable body to static loading and its response to dynamic loading.

We begin by employing a "massless" linear spring as the deformable member. Later we will generalize to bars, beams, and other deformable bodies. In the first case we consider a weight W that falls from height h onto the massless, linear spring with spring constant k (i.e., $P = k\Delta$). In the second case we consider a mass that is moving with speed v when it strikes the spring.

Gravitational Potential Energy Converted to Strain Energy of a "Massless" Linear Spring.

Let weight W be dropped from height h onto a massless, linear spring, and assume that no energy is lost during the initial contact of the weight with the spring. Three positions are identified in Fig. 11.21*a*: (1) the position where the weight is released from rest, (2) the position where the weight initially contacts the spring, and (3) the lowest position reached by the weight, where the spring has its maximum compression, Δ_{max}, and where the weight is (momentarily) stopped.

[12]For example, see *Structural Dynamics—An Introduction to Computer Methods*, by Roy R. Craig, Jr. [Ref. 11-3].

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If no energy is lost during the impact, we can equate the gravitational potential energy lost by the weight in falling through the distance $(h + \Delta_{max})$ to the increase in strain energy in the spring when it is compressed by an amount Δ_{max}. Then,

$$W(h + \Delta_{max}) = \frac{1}{2}k\Delta_{max}^2 \tag{11.60}$$

Solving for the positive root of this quadratic equation in Δ_{max}, we get

$$\Delta_{max} = \left(\frac{W}{k}\right) + \left[\left(\frac{W}{k}\right)^2 + 2h\left(\frac{W}{k}\right)\right]^{1/2} \tag{11.61}$$

But W/k is the static compression, Δ_{st}, that would occur if the weight W were slowly lowered onto the spring. Therefore, Eq. 11.61 can be recast in the convenient form

$$\boxed{\Delta_{max} = \Delta_{st}\left[1 + \left(1 + \frac{2h}{\Delta_{st}}\right)^{1/2}\right]} \tag{11.62}$$

This equation says that $\Delta_{max} \gtrsim 2\Delta_{st}$, with

$$\Delta_{max} = 2\Delta_{st} \tag{11.63}$$

if the weight is suddenly released when it is just touching the spring, that is, at position (2) in Fig. 11.21*a*.

At position (3) in Fig. 11.21*a*, where the spring is at its maximum compression, Δ_{max}, the force exerted on the spring by the weight (and vice versa) has a magnitude

$$P_{max} = k\Delta_{max} \tag{11.64}$$

where Δ_{max} is given by Eq. 11.61 or 11.62.

Kinetic Energy Converted to Strain Energy of a "Massless" Linear Spring. Figure 11.22 shows a mass that is moving with speed v at the instant when it makes contact with a massless, linear spring. If we assume that no energy is lost in the impact process, then energy is conserved, and the kinetic energy of the mass at position (1) in Fig. 11.22 is converted to strain energy stored in the spring in position (2). Then,

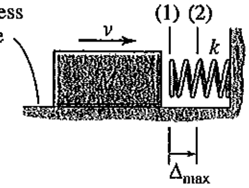

FIGURE 11.22 Impact loading—moving-mass case.

$$\frac{1}{2}Mv^2 = \frac{1}{2}k\Delta_{max}^2 \tag{11.65}$$

or

$$\boxed{\Delta_{max} = \sqrt{\frac{Mv^2}{k}}} \tag{11.66}$$

As in Case A, the maximum force exerted on the spring by the mass is given by Eq. 11.64, with Δ_{max}, in the present case, given by Eq. 11.66.

Impact on Deformable Bodies. Figure 11.23 depicts a sliding collar that drops from height h, makes contact with a flange on the end of a linearly elastic rod, and stretches the rod by an amount Δ_{max}. The analysis of Case A can be applied to this system and other deformable bodies if we make the following assumptions:

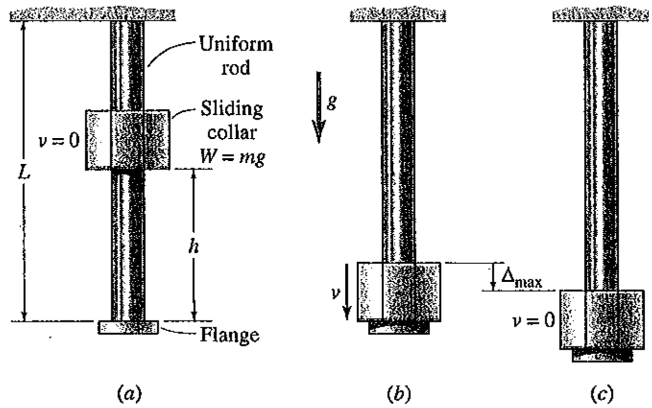

FIGURE 11.23 Impact loading of a uniform rod.

- The mass of the impacted deformable body is negligible in comparison with the impacting mass.
- The impacting mass is rigid.
- No energy is lost in the impact.

These assumptions lead to conservative answers. That is, the deformation and stresses calculated are greater than the actual values would be if energy losses and other factors hinted at above were completely accounted for.

With the preceding assumptions, we do not have to consider conservation of momentum upon impact or consider stress waves in the impacted body; we can just apply the results obtained previously for a "massless" spring. Equations 11.62 and 11.66 can be used to determine Δ_{max} for the respective two types of impact, with an equivalent stiffness, k, determined for the particular elastic body impacted. However, to determine the maximum stress caused by the impact, we must apply the dynamic load P_{max} of Eq. 11.64 to the particular body impacted (e.g., rod, beam, etc.). The next two example problems illustrate the effect of impact loading on deformable bodies.

EXAMPLE 11.17

For the rod and sliding collar of Fig. 11.23, (a) determine an expression for Δ_{max} as a function of W, A, E, h, and L. (b) If the weight is dropped from a height of $h = 40\Delta_{st}$, determine the value of the *impact amplification factor* Δ_{max}/Δ_{st}. (c) Determine the maximum impact stress σ_{max} in terms of the static stress σ_{st}, the drop height h, and the rod parameters.

Solution

(a) Determine Δ_{max}, the maximum displacement. We could use either Eq. 11.61 or 11.62, taking Δ to be positive for elongation, rather than for compression as in the original derivation. Let us use Eq. 11.62.

$$\Delta_{max} = \Delta_{st}\left[1 + \left(1 + \frac{2h}{\Delta_{st}}\right)^{1/2}\right] \qquad (1)$$

If the weight were to be lowered slowly onto the flange of the rod, we would get the static elongation

$$\Delta_{st} = \frac{W}{k} = \frac{WL}{AE} \qquad (2)$$

Combining Eqs. (1) and (2), we get the desired expression

$$\Delta_{max} = \frac{WL}{AE}\left[1 + \left(1 + \frac{2AEh}{WL}\right)^{1/2}\right] \qquad \textbf{Ans. (a)} \qquad (3)$$

(b) Determine the impact amplification factor. For $h = 40\Delta_{st}$, Eq. (1) gives

$$\frac{\Delta_{max}}{\Delta_{st}} = 10 \qquad \textbf{Ans. (b)} \qquad (4)$$

(c) Determine the maximum impact stress. Since $\sigma = E\epsilon = E\left(\frac{\Delta}{L}\right)$ for axial deformation, Eq. (1) can be converted directly to the following equation for σ_{max}:

$$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2Eh}{L\sigma_{st}}\right)^{1/2}\right] \qquad \textbf{Ans. (c)} \qquad (5)$$

It is left as an exercise for the reader (Probs. 11.9-8, 11.9-9) to show that impact loading of a rod with enlarged cross section over a portion of its length produces a higher maximum stress than the stress given by Eq. (5) for a uniform rod. Thus, from the stand-point of elastic energy absorption, a rod with uniform cross section is preferable to a rod with nonuniform cross section.

EXAMPLE 11.18

A diver is springing on the diving board shown in Fig. 1. On a particular bounce, the diver reaches a height h above the end of the board. Treat the diver as a rigid mass that falls from a height h, and assume that the diving board is much lighter than the diver (e.g., a fiberglass board might be very light) and is straight when the diver strikes it at the very end. (a) Determine the maximum deflection of the tip of the diving board. Express your answer in terms of W, h, and the parameters of the diving board—E, I, and L. (b) Determine the maximum flexural stress caused by the diver's "impact." Express your answer in terms of the maximum static flexural stress and the parameters of the diving board.

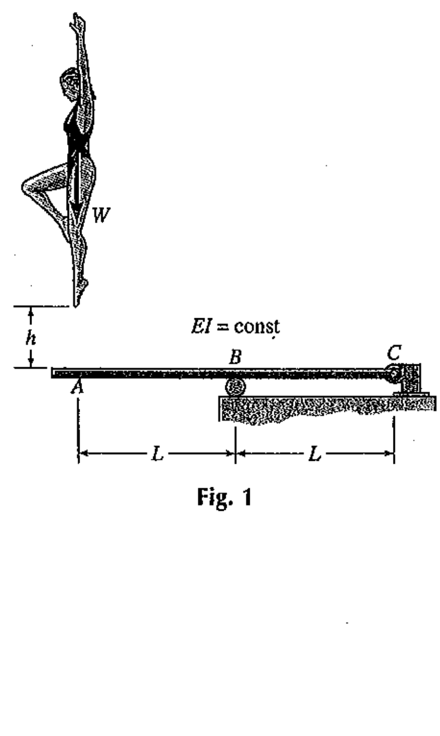

Fig. 1

Neglect shear deformation and neglect any energy loss during impact. Also neglect the mass of the diving board, and assume that EI = const.

Plan the Solution We can use the results of Example Problem 11.4 to determine an expression for k to use in Eq. 11.61 for Δ_{max}. In Part (b) we can use Eq. 11.64 to determine the maximum force that the diver exerts on the beam. Then we can use the flexure formula, Eq. 6.13, with the maximum bending moment produced by P_{max}.

Solution

(a) Determine the maximum deflection at the tip of the diving board. This deflection can be determined from Eq. 11.61, which we can write in the form

$$\Delta_{max} = \frac{W}{k}\left[1 + \left(1 + \frac{2kh}{W}\right)^{1/2}\right] \qquad (1)$$

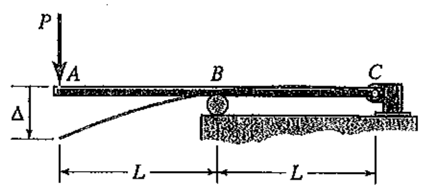

Fig. 2 The deflection caused by a tip load.

From Example Problem 11.4, the deflection Δ at the tip of the beam (diving board) in Fig. 2 is

$$\Delta = \frac{2}{3}\frac{PL^3}{EI} \tag{2}$$

Since k is defined by $P = k\Delta$, for this beam configuration and loading

$$k = \frac{3EI}{2L^3} \tag{3}$$

Therefore, Eqs. (1) and (3) may be combined, giving the desired answer

$$\Delta_{\max} = \frac{2WL^3}{3EI}\left[1 + \left(1 + \frac{3EIh}{WL^3}\right)^{1/2}\right] \qquad \textbf{Ans. (a)} \tag{4}$$

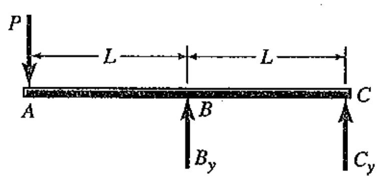

Fig. 3 Free-body diagram.

(b) Determine the maximum flexural stress. For the free body in Fig. 3,

$$\left(\sum M\right)_B = 0: \qquad C_y = -P$$

Therefore, the beam loading is symmetric about B, and the maximum bending moment along the beam occurs at B, where $M_B = -PL$. We obtain the maximum impact force at the tip, A, by combining Eq. 11.64 and Eq. (3), giving

$$P_{\max} = \left(\frac{3EI}{2L^3}\right)\Delta_{\max} \tag{5}$$

so the maximum bending moment is

$$(M_B)_{\max} = -\left(\frac{3EI}{2L^2}\right)\Delta_{\max} \tag{6}$$

Let c be the distance from the neutral axis to the top fibers of the beam. Then, from the flexure formula, Eq. 6.13,

$$\sigma_{\max} = \frac{-(M_B)_{\max}c}{I} = \left(\frac{3Ec}{2L^2}\right)\Delta_{\max} \tag{7}$$

Let σ_{st} be the maximum flexural stress in the beam under static loading (e.g., with the diver standing on the end of the diving board). Then, $(M_B)_{st} = -WL$, so

$$\sigma_{st} = \frac{WLc}{I} \tag{8}$$

Finally, we can combine Eqs. (4), (7), and (8) to get the following expression relating $\sigma_{\max}$ and σ_{st}:

$$\sigma_{\max} = \sigma_{st}\left[1 + \left(1 + \frac{3Ehc}{\sigma_{st}L^2}\right)^{1/2}\right] \qquad \textbf{Ans. (b)} \tag{9}$$

(Note that this expression differs from the answer for axial impact on a rod, whereas the same expression, Eq. 11.62, relates $\Delta_{\max}$ to Δ_{st} for both problems.)