Euler's Theorem of Variational Calculus

The transition from a variational statement to an equivalent governing differential equation is relatively simple and will be demonstrated here. The reverse process, however, is more involved and any generalized processes restrictive—for the very reason that frequently no variational principle can be established.

Let us take a problem in which

$$\chi = \int_V f(x, y, z, \phi, \phi_x, \phi_y, \phi_z)\,\mathrm{d}V + \int_C (q\phi + \alpha\phi^2/2)\,\mathrm{d}S \quad \text{(A.6.1)}$$

is to be minimized. In this f is an arbitrary function, $\phi_x = \partial\phi/\partial x$, etc., and C is a portion of the boundary surface on which prescribed values of ϕ are not imposed. On remainder $\phi = \phi_B$.

Considering an arbitrary small variation of the unknown function and its derivatives we have

$$\delta\chi = \int_V \left(\frac{\partial f}{\partial \phi}\,\delta\phi + \frac{\partial f}{\partial \phi_x}\,\delta\phi_x + \frac{\partial f}{\partial \phi_y}\,\delta\phi_y + \frac{\partial f}{\partial \phi_z}\,\delta\phi_z\right)\mathrm{d}V + \int_C (q\,\delta\phi + \alpha\phi\,\delta\phi)\,\mathrm{d}S. \quad \text{(A.6.2)}$$

As

$$\delta\phi_x = \delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}(\delta\phi), \text{ etc.}$$

we can write Eq. (A.6.2) as

$$\delta\chi = \int_V \left(\frac{\partial f}{\partial\phi}\,\delta\phi + \frac{\partial f}{\partial\phi_x}\frac{\partial}{\partial x}(\delta\phi) + \cdots\right)\mathrm{d}V + \int_C (q\,\delta\phi + \alpha\phi\,\delta\phi)\,\mathrm{d}S = 0. \quad \text{(A.6.3)}$$

In the above we have equated $\delta\chi$ to zero, as at the minimum (or stationary point) the 'variation' becomes zero.

Now putting $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and integrating the second term of above equation by parts (as in Eq. (3.30) of Chapter 3) with respect to x we have

$$\int_V \frac{\partial f}{\partial\phi_x}\frac{\partial}{\partial x}(\delta\phi)\,\mathrm{d}V = \int_S \frac{\partial f}{\partial\phi_x}\,\delta\phi\, l_x\,\mathrm{d}S - \int_V \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\phi_x}\right)\delta\phi\,\mathrm{d}V$$

in which l_x is the direction cosine of the normal to the outer surface with the x axis. Performing similar operations on the other similar terms of Eq. (A.6.3) and substituting we have finally

$$\delta\chi = \int_V \delta\phi\left\{\frac{\partial f}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial\phi_y}\right) - \frac{\partial}{\partial z}\frac{\partial f}{\partial\phi_z}\right\}\mathrm{d}V + \int_C \delta\phi\left\{q + \alpha\phi + l_x\frac{\partial f}{\partial\phi_x} + l_y\frac{\partial f}{\partial\phi_y} + l_z\frac{\partial f}{\partial\phi_z}\right\}\mathrm{d}S. \quad \text{(A.6.4)}$$

The second integral is only taken over the boundary C as on the remainder of surface S we have prescribed values of ϕ and therefore $\delta\phi = 0$.

For Eq. (A.6.4) to be true for any arbitrary variation $\delta\phi$ we must have

$$\frac{\partial f}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial\phi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\phi_z}\right) = 0 \quad \text{(A.6.5a)}$$

everywhere within the region V, and on the boundary C

$$l_x\frac{\partial f}{\partial\phi_x} + l_y\frac{\partial f}{\partial\phi_y} + l_z\frac{\partial f}{\partial\phi_z} + q + \alpha\phi = 0. \quad \text{(A.6.5b)}$$

These two equations, if satisfied by ϕ, minimize χ. If the solution is unique then formulations (A.6.1) and (A.6.5) are equivalent. The above differential equations are known as the Euler equations of the problem.

If the functional depended also on higher derivatives of ϕ a similar procedure would result in appropriate Euler equations. Again, if χ were a functional of several independent functions ϕ, ψ, etc., and their derivatives, a similar variational process would result in a series of Euler equations defining several differential governing equations.

Minimizing a quadratic form:

$$V(\underset{\sim}{x}) = \frac{1}{2}\, \underset{\sim}{x}^T \underset{\approx}{K}\, \underset{\sim}{x} - \underset{\sim}{x}^T \underset{\sim}{P} \rightarrow \text{min}$$

$$V(x_i) = \frac{1}{2} \sum_i \sum_j x_i\, k_{ij}\, x_j - \sum_i x_i P_i \longrightarrow \text{min}$$

k_{ij} & P_i known. $\quad \delta_{ij} \equiv \begin{cases} 1 \text{ if } i=j \\ 0 \text{ else} \end{cases}$

$$\boxed{\frac{\partial V}{\partial x_i} = 0_i} \qquad 1 \le i \le n$$

for a stationary value

$$\frac{\partial V}{\partial x_i} = \frac{1}{2} \sum_i \sum_j 1 \cdot k_{ij}\, x_j + \frac{1}{2} \sum_i \sum_j x_i\, k_{ij} \frac{\partial x_j}{\partial x_i}$$

$$- P_i = 0_i, \quad \text{but } x_i \frac{\partial x_j}{\partial x_i} = x_i\, \delta_{ij} = x_j$$

So $\quad \dfrac{\partial V}{\partial x_i} = 0_i = \dfrac{2}{2} \sum_i \sum_j k_{ij}\, x_j - P_i$

or

$$\boxed{\underset{\approx}{K}\, \underset{\sim}{x} = \underset{\sim}{P}}$$

renders V stationary.