Buchanan

2.9. For a rod similar to the one described in Prob. 2.8 write the variational function in terms of linear shape functions. Derive a general model or a *local stiffness matrix* for the rod using the variational function.

The variational function can be written as follows for one element of length L:

$$J(u) = \frac{1}{2}\int_0^L \frac{du}{dx} E \frac{du}{dx} A\, dx - \int_0^L ufA\, dx \tag{a}$$

Use the matrix definition of the linear shape function that was established in Prob. 2.6 and rewrite Eq. (*a*):

$$J(u) = \frac{A}{2}\int_0^L \{u\}^T \left[\frac{dN}{dx}\right]^T [E] \left[\frac{dN}{dx}\right]\{u\}\, dx - A\int_0^L \{u\}^T [N]^T f\, dx \tag{b}$$

Rewrite Eq. (*b*) to illustrate the matrix multiplications:

$$J(u) = \frac{A}{2}\int_0^L [u_1 \quad u_2]\begin{bmatrix} dN_1/dx \\ dN_2/dx \end{bmatrix}[E][dN_1/dx \quad dN_2/dx]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx - A\int_0^L [u_1 \quad u_2]\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} f\, dx \tag{c}$$

At this point the shape functions and their derivatives could be substituted into Eq. (*c*), the various matrix multiplications be carried out, the integration completed, and the result would be similar to the procedure developed in Prob. 2.8. However, Eq. (*c*) merely illustrates the matrix equation. The minimization will be carried out using Eq. (*b*). Refer to Chap. 1 for concepts pertaining to the derivative of a matrix equation.

$$\frac{\partial J(u)}{\partial \{u\}} = A\int_0^L \left[\frac{dN}{dx}\right]^T [E]\left[\frac{dN}{dx}\right]\{u\}\, dx - A\int_0^L [N]^T f\, dx = 0 \tag{d}$$

The matrices are evaluated using the results of Prob. 2.8: for L^e

$$A\int_0^{L^e} \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix}[E][-1/L \quad 1/L]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} dx - A\int_0^L \begin{Bmatrix} (L-x)/L \\ x/L \end{Bmatrix} f\, dx = 0 \qquad L \to L^e$$

Perform the indicated matrix multiplications:

$$A\int_0^{L^e} \begin{bmatrix} E/L^2 & -E/L^2 \\ -E/L^2 & E/L^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} dx = A\int_0^L \begin{Bmatrix} (L-x)/L \\ x/L \end{Bmatrix} f\, dx \qquad L \to L^e$$

After integration the final result appears as (for single element)

$$\underbrace{\begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix}}_{K^e}\underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{u^e} = \underbrace{\begin{Bmatrix} AfL/2 \\ AfL/2 \end{Bmatrix}}_{F^e} \qquad L \to L^e \tag{e}$$

Note that the area term is not factored out of the equation since area may vary from one finite element to the next. The term on the right of the equal sign is interpreted to mean that one-half of the body force is distributed to each node. Equation (*e*) is the local stiffness matrix (stiffness matrix for an individual element) for an axially loaded rod. The stiffness matrix for any similar differential equation would be the same with different material parameters. Finally, Eq. (*e*) is written in terms of matrices as

$$[K]\{u\} = \{f\} \qquad \text{or sometimes as} \qquad [K^e]\{u\}^e = \{f^e\} \tag{f}$$

where the superscript e indicates the *element* stiffness matrix and *element* force matrix.

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Rewrite Eq. (*b*) to illustrate the matrix multiplications:

$$J(u) = \frac{A}{2}\int_0^L [u_1 \quad u_2] \begin{bmatrix} dN_1/dx \\ dN_2/dx \end{bmatrix} [E][dN_1/dx \quad dN_2/dx] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx - A\int_0^L [u_1 \quad u_2] \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} f\, dx \qquad (c)$$

At this point the shape functions and their derivatives could be substituted into Eq. (*c*), the various matrix multiplications be carried out, the integration completed, and the result would be similar to the procedure developed in Prob. 2.8. However, Eq. (*c*) merely illustrates the matrix equation. The minimization will be carried out using Eq. (*b*). Refer to Chap. 1 for concepts pertaining to the derivative of a matrix equation.

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Perform the indicated matrix multiplications:

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