Finite Element Concepts

Exact Differential Equations + BC to
Approximate Matrix Equations + BC

Integral Analytic Approaches

- 1750 System energy + BC →
  Energy methods + BC (Euler)→
  Integral form + BC
- 1750 Differential equations + BC →
  Variational calculus + BC (Euler) →
  Integral form + BC
- 1950 Differential equations + BC →
  Galerkin's method + BC →
  Integral form + BC

Equivalent Forms

- Exactly solving the integral form, + BC, yields the same unique solution as exactly solving the differential equation, + BC (for linear problems).
- **Galerkin's method** can be used to build integral forms for non-linear differential equations as well.

Energy Methods

- (Static) **system energy** = potential energy (strain energy) – external mechanical work
- All are defined with respect to the **displacements** (the unknowns)
- Strain energy uses the material properties
- Work uses the external forces (sources)
- **Equilibrium** →Energy is minimized by the displacements that satisfy the BC (on displacements)

Example Linear Spring

- Linear spring of "stiffness" **k** is fixed at one end (BC) and displaced an amount **x** by end force **f**, find the equilibrium displacement.
- Energy: $E = \frac{1}{2}\, \mathbf{k}\, \mathbf{x}^2 - \mathbf{f}\, \mathbf{x}$
- Minimize w.r.t. **x**: $\partial E / \partial \mathbf{x} = \mathbf{0}$
- Equilibrium: $\mathbf{2/2\ k\ x - f = 0}$, so $\mathbf{x = f / k}$

Galerkin's method

- Let **(DE = 0)** be the differential equation of equilibrium over volume V, subject to **BC**. The **DE** involves the solution, **u** and/or its gradient $\partial \mathbf{u} / \partial \mathbf{x}$, and source terms
- Let **W** be a weighting function over space
- Weighted residual: **IE =** $\int_V$ **W*(DE)** dV **= 0**, subject to **BC**
- Galerkin: Pick **W = u**, so error is "orthogonal" to the solution

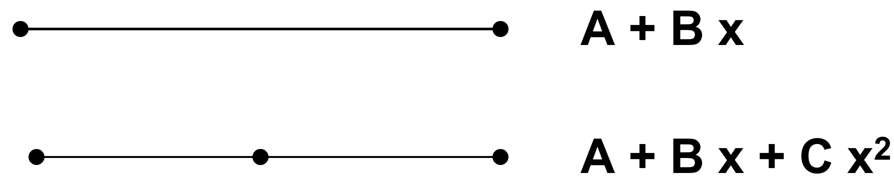
Finite Element Method (1960)

- Requires a known integral form, & BC.
- Assumes the spatial form of an approximate solution to **u**(**x**), and thus to its gradient ∂**u** / ∂**x** (and for **W**)
- Forms a mesh of volume elements and nodal points over V (and its boundary)
- In each element interpolates **u**(**x**) from its values (unknown) at the element's nodes, say $\mathbf{u}_e$, a subset of all unknowns, say **U**

Piecewise Interpolation

- The FE approximation is only piecewise continuous (C^0) at the nodes and element faces, but continuous (C^∞) inside the element.

- Count nodes on element edge: 2 for linear (constant gradient), 3 for quadratic (linear gradient)



FE Algebraic Equations (1)

- **U** are unknowns at all points and we need an algebraic system to compute them
- Split the integral into the sum of the integrals over the elements:

  $$\mathbf{IE} = \int_V = \sum_e \int_V{}^e = \mathbf{0}$$

- Substitute the approx. sol. **u**(**x**, **U**) into the integral form, factor out the **U**, integrate the spatial form for the element matrix and vectors, $\mathbf{k}^e * \mathbf{u}^e = \mathbf{f}^e$

FE Algebraic Equations (2)

- Add up (assemble) all the element matrices and vectors into the large system algebraic equations of equilibrium: **K * U = F**
- Apply the **BC** to the equations to get a smaller algebraic system: $\underline{\mathbf{K}} * \underline{\mathbf{U}} = \underline{\mathbf{F}}$
- Solve this final system: $\underline{\mathbf{U}} = \underline{\mathbf{K}}^{-1} * \underline{\mathbf{F}}$
- Plot **U**. Recover the element gradients and plot them

Local FE Error

- Local error at a point is the product of the element size, h, and the rate of change of the gradient of the solution
- Need a fine mesh where the stress concentrations gradients or the heat flux gradients are high
- Mesh is controlled by "splitting" the solid model

Conduction through a wall

- Thickness L, conduction k, area A = 1, heat source per unit volume Q, inside temperature u_1 = 0, find outside temp u_2.

$$\mathbf{k^e * u^e = f^e}$$

$$\frac{kA}{L}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \frac{QAL}{2}\begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$kA / L\ [1]\ u_2 = QAL / 2\ \{1\}$$

$$u_2 = QL^2 / 2k$$