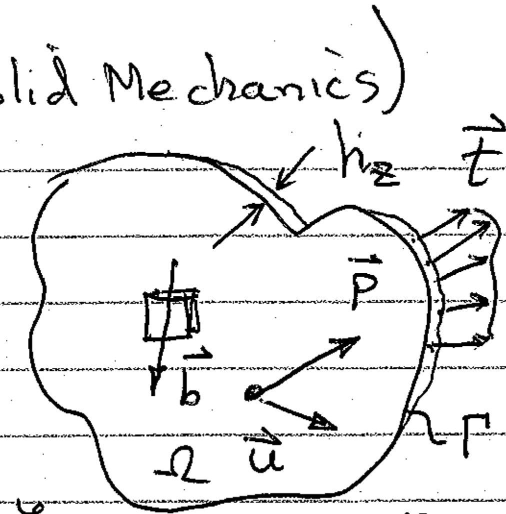
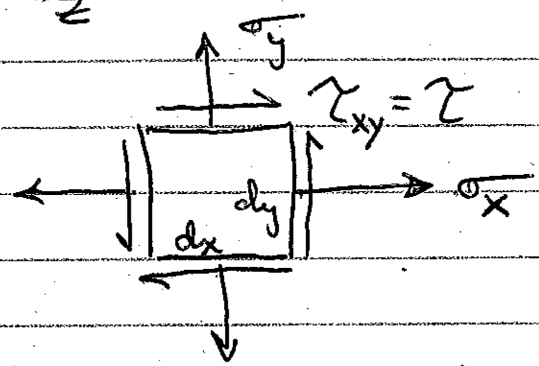
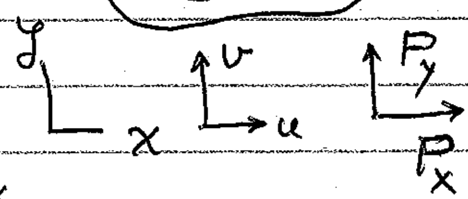
Plane-Stress Analysis (Solid Mechanics)



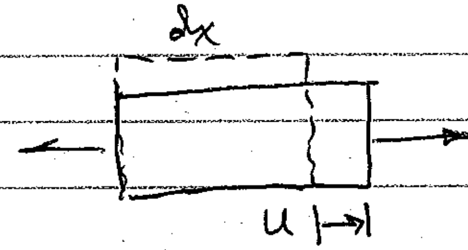
$$\sigma_z \equiv 0 = \tau_{xz} = \tau_{yz}$$

$$h_z << 1$$



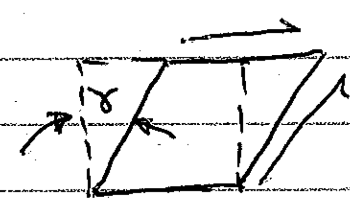


strain-displacement:



$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



$$\vec{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{\sim}{u}$$

displacement

$$\underset{\sim}{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma \end{Bmatrix} \qquad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix}$$

strain stress

Material

$$\underset{\sim}{\sigma} = \underset{\approx}{E} (\underset{\sim}{\epsilon} - \underset{\sim}{\epsilon}_0) + \underset{\sim}{\sigma}_0$$

$$\Pi = U - W, \quad U = \frac{1}{2} \int_\Omega \underset{\sim}{\epsilon}^T \underset{\approx}{E} \, \underset{\sim}{\epsilon} \, d\Omega \quad , d\Omega = h\,dA$$

$$W = \sum_j \vec{u}_j \cdot \vec{P}_j + \int_\Omega \vec{u} \cdot \vec{b} \, d\Omega + \int_\Gamma \vec{t} \cdot \vec{u} \, d\Gamma$$

Equilibrium: $\Pi(\vec{u}) \Rightarrow$ minimum

Plane-Stress FEA approach: $n_s \equiv 2$

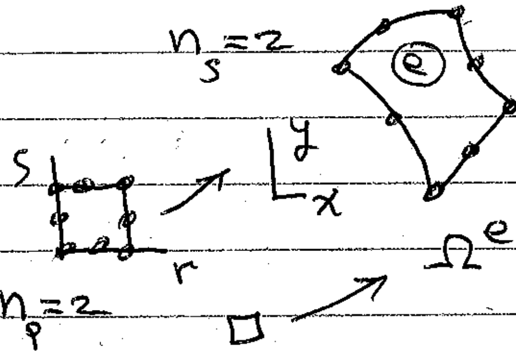
Unknowns: $\vec{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}$ at every node. $n_g \equiv 2$

Mesh: $\int_{\Omega} \sim d\Omega = \sum_e \int_{\Omega^e} \sim d\Omega$

$$\int_{\Gamma} \sim d\Gamma = \sum_b \int_{\Gamma^b} \sim d\Gamma \,, \quad \Pi = \sum_e \Pi^e$$

$n_s = 2$

Interpolate Geometry



$$x(r,s) = \underset{\sim}{G}(r,s)\,\{x^e\}$$

$$= \sum_{k=1}^{n_x} G_k(r,s)\, x_k^e$$

$n_p = 2$

$n_x \sim$ # geometry nodes

etc. $y(r,s)$ $\qquad$ $n_p \sim$ # parametric space

Parametric Derivative: $\dfrac{\partial x}{\partial r} = \dfrac{\partial \underset{\sim}{G}(r,s)}{\partial r}\, \underset{\sim}{x}^e$, etc

Element Jacobian

$$\underset{\approx}{J}^e(r,s) = \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{Bmatrix} \left[x(r,s) \quad y(r,s) \right]^e$$

$$\underset{2\times 2}{\underset{\approx}{J}^e} = \overset{2\times n_x}{\left(\underset{\sim}{\nabla}_{\square}\, \underset{\sim}{G}(r,s) \right)} \; \overset{n_x \times 2}{\left[x^e \,\vdots\, y^e \right]}$$

Calculus: $d\Omega = |\underset{\approx}{J}^e|\, d\square$, $\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{\approx}{J}^{e^{-1}} \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{Bmatrix}$

Plane-Stress FEA

Order the unknowns in the mesh:

n_m = # of nodes in the mesh

$n_g = 2$ = # dof per node

$n_d = n_g * n_m$ = # of unknowns in system

$$\underset{n_d \times 1}{\underset{\sim}{\delta}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_{n_m} \end{Bmatrix}$$

connection list
sub-set
gather

Element unknowns: $\underset{\sim}{\delta}^e \subseteq_e \underset{\sim}{\delta}$

$$\underset{n_i \times 1}{\underset{\sim}{\delta}^e} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ v_{n_n} \end{Bmatrix}^e$$

$n_n \sim$ # solution nodes on the element

$n_i = n_g * n_n$ = # element unknowns

Interpolate scalar x-component:

$$u(r,s) = \sum_{j=1}^{n_n} N_j(r,s)\, u_j^e$$

etc for y-component

$$v(r,s) = \sum_j N_j(r,s)\, v_j^e$$

Extract $\epsilon_x = \frac{\partial u}{\partial x}$ and $\epsilon_y = \frac{\partial v}{\partial y}$:

$$\epsilon_x^e(r,s) = \frac{\partial}{\partial x}\left(\sum_j N_j(r,s)\, u_j^e \right)$$

$$= \sum_j \frac{\partial N_j(r,s)}{\partial x}\, u_j^e$$

likewise,
$$\epsilon_y^e(r,s) = \sum_j \frac{\partial N_j(r,s)}{\partial y}\, v_j^e$$

We need

$$\underset{n_s \times n_n}{\underset{\sim}{\nabla}\,\underset{\sim}{N}} = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \left[N_1(r,s)\; N_2 \cdots N_{n_n}(r,s) \right]$$

$$= \underset{\approx}{J}^{e^{-1}} \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{Bmatrix} \left[N_1(r,s) \cdots N_{n_n}(r,s) \right]$$

to form the strains. Also

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_j \frac{\partial N_j}{\partial y} u_j^e + \sum_j \frac{\partial N_j}{\partial x} v_j^e .$$

Re-write these terms as matrix expressions related to the element unknowns, $\underset{\sim}{\delta}^e$

$$u(r,s) = \begin{bmatrix} N_1(r,s) & 0 & N_2(r,s) & 0 & \dots & N_{n_n}(r,s) & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_{n_n} \end{Bmatrix}^e$$

so

$$\vec{u} = \begin{Bmatrix} u(r,s) \\ v(r,s) \end{Bmatrix}_{2\times 1} = \left[\begin{array}{cc|cc|c|cc} N_1(r,s) & 0 & N_2(r,s) & 0 & \dots & N_{n_n}(r,s) & 0 \\ 0 & N_1 & 0 & N_2 & & 0 & N_{n_n} \end{array}\right] \underset{\sim}{\delta}^e$$

Likewise, for the strains

$$\underset{\sim}{\epsilon}^e(r,s) = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma \end{Bmatrix}^e = \left[\begin{array}{cc|c|cc} N_{1,x} & 0 & & N_{n_n,x} & 0 \\ 0 & N_{1,y} & \dots & 0 & N_{n_n,y} \\ N_{1,y} & N_{1,x} & & N_{n_n,y} & N_{n_n,x} \end{array}\right] \underset{\sim}{\delta}^e$$

3×1 $\quad 3\times(n_g * n_n)$ $\quad n_i \times 1$

Standard notation:

$$\underset{\sim}{\epsilon}^e(r,s) = \underset{\approx}{B}^e(r,s) \; \underset{\sim}{\delta}^e$$

3×1 $\quad 3\times n_i$ $\quad n_i\times 1$

combinations of the gradient components of the solution interpolation

$$U^e = \frac{1}{2} \underset{\sim}{\delta}^{e^T} \int_{\Omega^e} \underset{\approx}{B}^{e^T} \underset{\approx}{E}^e \underset{\approx}{B}^e \, d\Omega \; \underset{\sim}{\delta}^e = \frac{1}{2} \underset{\sim}{\delta}^e \underset{\approx}{K}^e \underset{\sim}{\delta}^e$$

1×1 $\quad 1\times n_i$ $\quad n_i \times n_i$ $\quad n_i\times 1$