5.2 LINEAR TRIANGULAR ELEMENT

The linear triangular element shown in Figure 5.7 has straight sides and three nodes, one at each corner. A consistent labeling of the nodes is a necessity and the labeling in this book proceeds counterclockwise from node i, which is arbitrarily specified. The nodal values of ϕ are Φ_i, Φ_j, and Φ_k whereas the nodal coordinates are (X_i, Y_i), (X_j, Y_j), and (X_k, Y_k).

The interpolation polynomial is

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{5.4}$$

with the nodal conditions

$$\begin{aligned} \phi &= \Phi_i \quad \text{at} \quad x = X_i,\ y = Y_i \\ \phi &= \Phi_j \quad \text{at} \quad x = X_j,\ y = Y_j \\ \phi &= \Phi_k \quad \text{at} \quad x = X_k,\ y = Y_k \end{aligned}$$

Substitution of these conditions into (5.4) produces the system of equations

$$\begin{aligned} \Phi_i &= \alpha_1 + \alpha_2 X_i + \alpha_3 Y_i \\ \Phi_j &= \alpha_1 + \alpha_2 X_j + \alpha_3 Y_j \\ \Phi_k &= \alpha_1 + \alpha_2 X_k + \alpha_3 Y_k \end{aligned} \tag{5.5}$$

which yields

$$\alpha_1 = \frac{1}{2A}\left[(X_j Y_k - X_k Y_j)\Phi_i + (X_k Y_i - X_i Y_k)\Phi_j + (X_i Y_j - X_j Y_i)\Phi_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(Y_j - Y_k)\Phi_i + (Y_k - Y_i)\Phi_j + (Y_i - Y_j)\Phi_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[(X_k - X_j)\Phi_i + (X_i - X_k)\Phi_j + (X_j - X_i)\Phi_k\right]$$

where the determinant

$$\begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix} = 2A \tag{5.6}$$

and A is the area of the triangle.

Substituting for α_1, α_2, and α_3 in (5.4) and rearranging produces an equation for ϕ in terms of three shape functions and Φ_i, Φ_j, and Φ_k that is

$$\phi = N_i\Phi_i + N_j\Phi_j + N_k\Phi_k \tag{5.7}$$

where

$$N_i = \frac{1}{2A}\left[a_i + b_i x + c_i y\right] \tag{5.8}$$

$$N_j = \frac{1}{2A}\left[a_j + b_j x + c_j y\right] \tag{5.9}$$

$$N_k = \frac{1}{2A}\left[a_k + b_k x + c_k y\right] \tag{5.10}$$

and

$$\begin{aligned} a_i &= X_j Y_k - X_k Y_j, \quad & b_i &= Y_j - Y_k, \quad \text{and} \quad & c_i &= X_k - X_j \\ a_j &= X_k Y_i - X_i Y_k, \quad & b_j &= Y_k - Y_i, \quad \text{and} \quad & c_j &= X_i - X_k \\ a_k &= X_i Y_j - X_j Y_i, \quad & b_k &= Y_i - Y_j, \quad \text{and} \quad & c_k &= X_j - X_i \end{aligned}$$

The scalar quantity ϕ is related to the nodal values by a set of shape functions that are linear in x and y. This means that the gradients $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are constant within the element. For example,

$$\frac{\partial\phi}{\partial x} = \frac{\partial N_i}{\partial x}\Phi_i + \frac{\partial N_j}{\partial x}\Phi_j + \frac{\partial N_k}{\partial x}\Phi_k \tag{5.11}$$

but

$$\frac{\partial N_\beta}{\partial x} = \frac{b_\beta}{2A} \qquad \beta = i, j, k$$

Therefore,

$$\frac{\partial\phi}{\partial x} = \frac{1}{2A}\left[b_i\Phi_i + b_j\Phi_j + b_k\Phi_k\right] \tag{5.12}$$

Since b_i, b_j, and b_k are constants (they are fixed once the nodal coordinates are specified) and Φ_i, Φ_j, and Φ_k are independent of the space coordinates, the derivative has a constant value. A constant gradient within any element means that many small elements have to be used to accurately approximate a rapid change in ϕ.

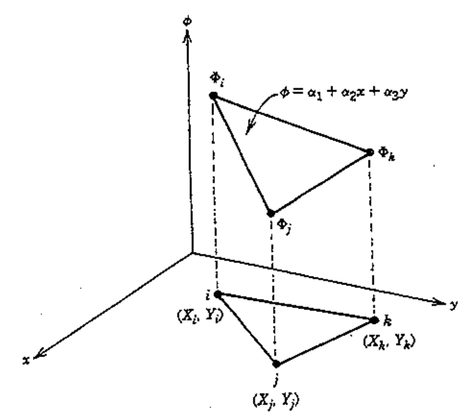

Figure 5.7 Parameters for the linear triangular element.

5.3 BILINEAR RECTANGULAR ELEMENT

The bilinear rectangular element has a length $2b$ and a height $2a$. The nodes are labeled i, j, k, and m with node i always at the lower left-hand corner. The element and the important coordinate systems are shown in Figure 5.10.

The interpolation equation (5.2) is written in terms of local coordinates s and t. There are at least three choices with

$$\phi = C_1 + C_2 s + C_3 t + C_4 st \tag{5.13}$$

being the most useful. The other choices would replace the st term by either s^2 or t^2. Equation (5.13) is used because ϕ is linear in s along any line of constant t and linear in t along any line of constant s. Because of these properties, the element is often said to be bilinear.

Equation (5.13) is written relative to a local coordinate system, whose origin is at node i because the shape functions are easier to evaluate in this reference frame. Another popular coordinate system is qr, which has its origin located at the center of the element (Figure 5.10).

The coefficients C_1, C_2, C_3, and C_4 in (5.13) are obtained by using the nodal values of ϕ and the nodal coordinates (in the st system) to generate four equations. These equations are

$$\begin{aligned}
\Phi_i &= C_1 \\
\Phi_j &= C_1 + (2b)C_2 \\
\Phi_k &= C_1 + (2b)C_2 + (2a)C_3 + (4ab)C_4 \\
\Phi_m &= C_1 + (2a)C_3
\end{aligned}$$

Solving gives

$$\begin{aligned}
C_1 &= \Phi_i \\
C_2 &= \frac{1}{2b}(\Phi_j - \Phi_i) \\
C_3 &= \frac{1}{2a}(\Phi_m - \Phi_i) \\
C_4 &= \frac{1}{4ab}(\Phi_i - \Phi_j + \Phi_k - \Phi_m)
\end{aligned} \tag{5.15}$$

Substitution of (5.15) into (5.13) and rearranging gives

$$\phi = N_i\Phi_i + N_j\Phi_j + N_k\Phi_k + N_m\Phi_m \tag{5.16}$$

where

$$\begin{aligned}
N_i &= \left(1 - \frac{s}{2b}\right)\left(1 - \frac{t}{2a}\right) \\
N_j &= \frac{s}{2b}\left(1 - \frac{t}{2a}\right) \\
N_k &= \frac{st}{4ab} \\
N_m &= \frac{t}{2a}\left(1 - \frac{s}{2b}\right)
\end{aligned} \tag{5.17}$$

The shape functions for the bilinear rectangular element have properties similar to those possessed by the triangular element. Each shape function varies linearly along the edges between its node and the two adjacent nodes. For example, N_i varies linearly along sides ij and im. Each shape function is also zero along the sides its node does not touch, that is, N_i is zero along sides jk and km. The linear variation of ϕ along an edge of the rectangular element and an edge of the triangular element means that these two elements are compatible and can be used adjacent to one another.

The transformation equations between the qr and st coordinate systems are

$$s = b + q \quad \text{and} \quad t = a + r \tag{5.18}$$

Substitution of (5.18) into (5.17) gives the shape functions in terms of q and r.

$$\begin{aligned}
N_i &= \frac{1}{4}\left(1 - \frac{q}{b}\right)\left(1 - \frac{r}{a}\right) \\
N_j &= \frac{1}{4}\left(1 + \frac{q}{b}\right)\left(1 - \frac{r}{a}\right) \\
N_k &= \frac{1}{4}\left(1 + \frac{q}{b}\right)\left(1 + \frac{r}{a}\right) \\
N_m &= \frac{1}{4}\left(1 - \frac{q}{b}\right)\left(1 + \frac{r}{a}\right)
\end{aligned} \tag{5.19}$$

The shape functions defined by (5.19) are useful because they lead to a natural coordinate system that allows the rectangle to be deformed into a general quadrilateral.

A contour line in a rectangular element is generally curved. The intersection of the contour line with the edges can be obtained using linear interpolation. The easiest method for obtaining a third point is to set s or t to zero in the shape function equations and solve (5.16) for the other coordinate value. This procedure is illustrated in the following example.

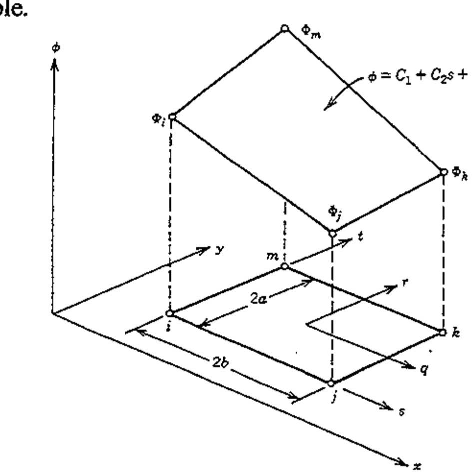

Figure 5.10. Parameters for the bilinear rectangular element.