MECH 417

HW 1

HW 1, a

Grade $\boxed{20/20}$

SOLUTION

1. (a) $\int_0^L x^2 dx$

$= \left[\frac{1}{3}x^3\right]_0^L = \frac{1}{3}(L^3 - 0) = \boxed{\frac{1}{3}L^3}$

(b) $\int_0^L x^2 dx = \int_0^1 x(r)^2 \left(\frac{dx}{dr}\right) dr$

for $x = L \cdot r + x_1$

$\frac{dx}{dr} = (L \cdot r)' = L$

$x(r)^2 = (L \cdot r)^2$

$\therefore \int_0^1 x(r)^2 \left(\frac{dx}{dr}\right) dr = \int_0^1 (L \cdot r)^2 \left(\frac{d}{dr}(L \cdot r)\right) dr$

$= \int_0^1 (L^2 r^2) \cdot L \, (dr) = L^3 \int_0^1 r^2 \, dr$

$= L^3 \left[\frac{1}{3} r^3\right]_0^1 = \frac{L^3}{3}(1^3 - 0) = \boxed{\frac{L^3}{3}}$

(c) $\int_0^L x^2 dx = \int_{-1}^1 x(a)^2 \left(\frac{dx}{da}\right) da$ for

$x = L(1+a)/2$ and $-1 \le a \le 1$

$-1 \le a \le 1$ satisfies $0 < x < L$

$\frac{dx(a)}{da} = \frac{d}{da}\left(\frac{L + a \cdot L}{2}\right) = \frac{L}{2}$

$\therefore \int_{-1}^1 x(a)^2 \left(\frac{dx}{da}\right) da = \int_{-1}^1 \left[\frac{L(1+a)}{2}\right]^2 \left(\frac{L}{2}\right) da$

$= \int_{-1}^1 \frac{L^3 (1+a)^2}{4 \cdot 2} da = \frac{L^3}{8} \int_{-1}^1 (1+a)^2 da$

$= \frac{L^3}{8} \int_{-1}^1 (a^2 + 2a + 1) da = \frac{L^3}{4}\left[\frac{1}{3}a^3 + a^2 + a\right]_{-1}^1$

$= \frac{L^3}{8}\left[\frac{1}{3}(1^3 - (-1)^3) + \underbrace{\cancel{(1^2 - (-1)^2)}}_{0} + (1 - (-1))\right]$

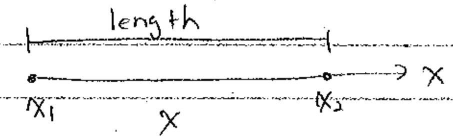
$= \frac{L^3}{8}\left[\frac{2}{3} + 2\right] = \frac{L^3}{8}\left(\frac{8}{3}\right) = \boxed{\frac{L^3}{3}}$

2. Evaluate the three domain "measures"

→ This question is asking to determine the three given domains' physical or non-dimensional length or measures

(a) $\int_0^L dx = [x]_0^L$

This is a physical Ω domain.

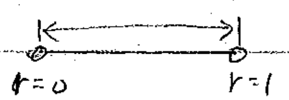


length or measure $= x_2 - x_1 = \underline{L - 0 = \boxed{L}}$

(b) $\int_0^1 dr = [r]_0^1$

This is a parametric domain.

The measure is non-dimensional length



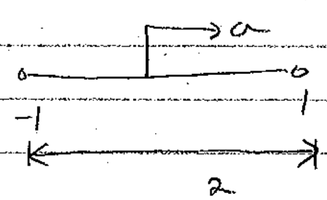
$\underline{\text{measure} = 1 - 0 = \boxed{1}}$

(c) $\int_{-1}^{1} da$

$= [a]_{-1}^{1} = 1 - (-1) = 2$

This is a parametric domain.

The measure will be a non-dimensional length

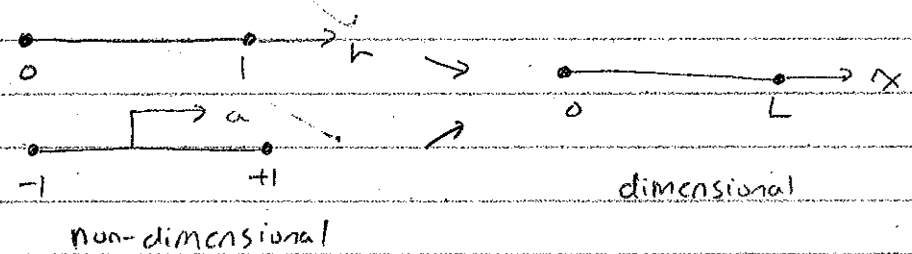


$\boxed{\underline{\text{Measure} = 2}}$

HW 1. a

3. Evaluate integral 1.(c) by Gaussian quadratures

□. parametric $\qquad\qquad$ Ω. physical



Numerical Integration by Gaussian Quadratures

$$I = \int_a^b f(x)\,dx = \int_\alpha^\beta f(a)\,\frac{dx(a)}{da}\,da$$

$$\simeq \sum_{g=1}^{n} f(a_g)\,\frac{dx(a_g)}{da}\,w_g$$

To determine the number n to numerically integrate, we should know that the above equation will exactly integrate a polynomial of degree $(2n-1)$.

Since the function we are integrating is x^2, a polynomial of degree 2, the minimum possible number of points for numerical integration $(=n)$ is

$$2n-1 = 2 \quad \therefore \quad n = 1.5 \;\left(=\tfrac{3}{2}\right) \rightarrow 2$$

So, we must select $n=2$, two points.

1.(c) is

$$\int_0^L x^2\,dx = \int_{-1}^{1} (x(a))^2 \left(\frac{dx}{da}\right) da$$

When we refer to table 3.3.1 'Gaussian Quadratures'

$$\int_a^b f(r)\,dr = \sum_{i=1}^{n} f(r_i)\,W_i$$

and table has corresponding values of r_i and W_i (tabulated values) for given (a, b) and n

For this problem

$a = -1, \quad b = 1, \quad n = 2$

$r_i = \pm \frac{1}{\sqrt{3}} \qquad w_i = 1$

In other word, $r_1 = \frac{1}{\sqrt{3}}, \quad r_2 = -\frac{1}{\sqrt{3}}$

$w_1 = w_2 = 1$

Then, $\int_{-1}^{1} (x(a))^2 \left(\frac{dx(a)}{da}\right) da$

$= \sum_{g=1}^{2} f(a_g) \frac{dx(a_g)}{da} w_g$

where $f(x) = x^2, \quad f(x(a)) = \left(\frac{L(1+a)}{2}\right)^2$

$\frac{dx(a)}{da} = \frac{L}{2}$

$\therefore \sum_{g=1}^{2} f(x(a_g)) \left.\frac{dx}{da}\right|_{a_g} w_g$

$= f(x(a_1)) \left(\frac{L}{2}\right)(w_1) + f(x(a_2)) \left(\frac{L}{2}\right)(w_2)$

Since, $a_1 = +\frac{1}{\sqrt{3}}, \quad a_2 = -\frac{1}{\sqrt{3}}$

$= \left[\frac{L(1+a_1)}{2}\right]^2 \left(\frac{L}{2}\right)(1) + \left[\frac{L(1+a_2)}{2}\right]^2 \left(\frac{L}{2}\right)(1)$

$= \frac{L^3}{8} \left\{ (1+a_1)^2 + (1+a_2)^2 \right\}$

$= \frac{L^3}{8} \left\{ \left(1+\frac{1}{\sqrt{3}}\right)^2 + \left(1-\frac{1}{\sqrt{3}}\right)^2 \right\}$

$= \frac{L^3}{8} (2.67) \simeq \boxed{\frac{1}{3} L^3}$

HW 1.b. linear interpolation Calculus $\boxed{20/20}$

$$u(x) = \underbrace{N(x)}_{1\times2} \ \underbrace{x^e}_{2\times1}$$

To perform linear interpolation, we must first assume (guess) the shape of solution for each element

$u(x) \rightarrow$ polynomial linear

$$= A + Bx$$

$$= \underbrace{P(x)}_{1\times2} \ \underbrace{C^e}_{2\times1} = [1 \quad x] \begin{bmatrix} A \\ B \end{bmatrix}$$

We substitute boundary conditions

$$u(x_1) = u_1 \quad \text{and} \quad u(x_2) = u_2$$

$$\begin{aligned} u_1 &= A + Bx_1 \\ u_2 &= A + Bx_2 \end{aligned} \rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Solving A and B using above equations,

$$A = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \qquad B = \frac{u_2 - u_1}{x_2 - x_1}$$

Substituting A and B into $u = A + Bx$ will give the interpolating polynomial

$$u = u_1\left(\frac{x_2 - x}{x_2 - x_1}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$= u_1 N_1(x) + u_2 N_2(x)$$

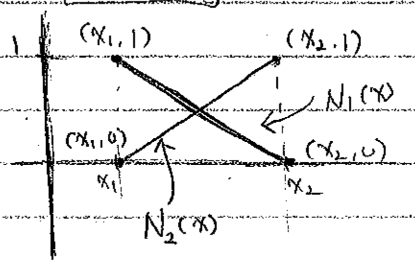
$$= \underset{\sim}{N}(x) \underset{\sim}{u}^e$$

1. For $x \in L^e$,

$L^e \Rightarrow$ range of x between x_1 and x_2

$N_1(x) = \boxed{\dfrac{x_2 - x}{x_2 - x_1}}$ at $x = x_1 \Rightarrow N_1(x_1) = 1$

at $x = x_2 \Rightarrow N_1(x_2) = 0$



$N_2(x) = \boxed{\dfrac{x - x_1}{x_2 - x_1}}$

at $x = x_1$, $\Rightarrow N_2(x_1) = 0$

$x = x_2$, $\Rightarrow N_2(x_2) = 1$

2. Integrate $\int_{Le} N_j(x)\,dx$, $j=1,2$

$$\int_{Le} N_1(x)\,dx = \int_{x_1}^{x_2} \frac{x_2 - x}{x_2 - x_1}\,dx$$

$$= \frac{1}{x_2 - x_1}\left[x_2 \cdot x - \frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$= \frac{1}{x_2 - x_1}\left[x_2(x_2 - x_1) - \frac{x_2^2 - x_1^2}{2}\right) = \frac{1}{\cancel{x_2 - x_1}}\left[\cancel{(x_2 - x_1)}\left(x_2 - \frac{x_2 + x_1}{2}\right)\right.$$

$$= \boxed{\frac{x_2 - x_1}{2}}$$ → area of triangle made by the curve $= \frac{L}{2}$ ✓

$$\int_{Le} N_2(x)\,dx = \int_{x_1}^{x_2} \frac{x - x_1}{x_2 - x_1} = \frac{1}{x_2 - x_1}\left[\frac{x^2}{2} - x_1 x\right]_{x_1}^{x_2}$$

$$= \frac{1}{x_2 - x_1}\left[\frac{x_2^2 - x_1^2}{2} - x_1(x_2 - x_1)\right] = \frac{1}{\cancel{x_2 - x_1}}\left[\cancel{(x_2 - x_1)}\left(\frac{x_2 + x_1}{2} - x_1\right)\right]$$

$$= \boxed{\frac{x_2 - x_1}{2}}$$ → area of triangle made by the curve $= \frac{L}{2}$ ✓

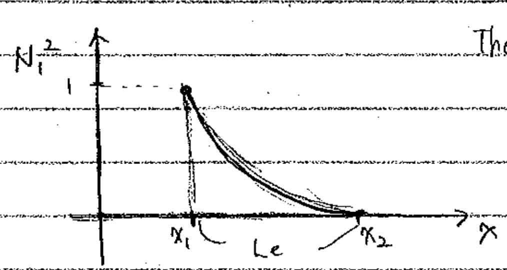
3. For $x \in L^e$, sketch the products $N_1(x)^2$ and $N_1(x) N_2(x)$

$$(N_1(x))^2 = \left(\frac{x_2 - x}{x_2 - x_1}\right)^2 = \frac{x^2 - 2x_2 x + x_2^2}{(x_2 - x_1)^2}$$

at $x = x_1$, $(N_1(x))^2 = 1$

at $x = x_2$, $(N_1(x)^2) = 0$ → $N_1(x)^2$ will be minimum at $x = x_2$

The curve's min value $= 0$



$$N_1(x) N_2(x) = \left(\frac{x_2 - x}{x_2 - x_1}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) = \frac{-(x - x_2)(x - x_1)}{(x_2 - x_1)^2}$$

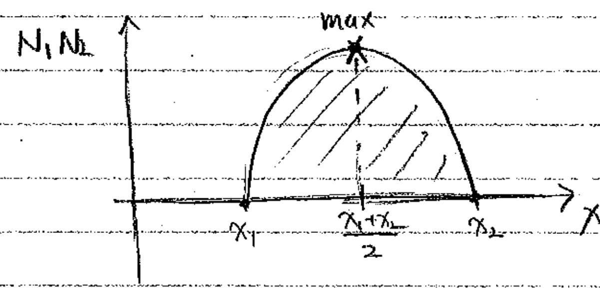
at $x = x_1$, $N_1(x) N_2(x) = 0$

at $x = x_2$, $N_1(x) N_2(x) = 0$

$$= \frac{-\left(x^2-(x_1+x_2)x+x_1x_2\right)}{(x_2-x_1)^2}$$

$$= -\left[\left(x-\frac{x_1+x_2}{2}\right)^2 + x_1x_2 - \frac{(x_1+x_2)^2}{4}\right]$$

This function attains the maximum at $x = \frac{x_1+x_2}{2}$



@ $\frac{L}{2}$ ✓

4. ① The area under $N_1^2(x)$ curve

$$\int_{Le} N_1^2(x) = \int_{x_1}^{x_2} \frac{(x-x_2)^2}{(x_2-x_1)^2}\,dx$$

$$= \frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2} (x-x_2)^2\,dx$$

$$= \frac{1}{(x_2-x_1)^2}\left[\frac{(x-x_2)^3}{3}\right]_{x_1}^{x_2}$$

$$= \frac{1}{(x_2-x_1)^2}\left[\frac{\cancel{(x_2-x_2)^3}-(x_1-x_2)^3}{3}\right]$$

$$= \frac{1}{\cancel{(x_2-x_1)^2}}\,\frac{(x_2-x_1)^3}{3} = \boxed{\frac{x_2-x_1}{3}} = \frac{L}{3}$$ ✓

(2) The area under $N_1(x)N_2(x)$ curve

$$\int_{Le} N_1(x)N_2(x)\,dx = \int_{x_1}^{x_2} -\frac{(x-x_2)(x-x_1)}{(x_2-x_1)^2}\,dx$$

$$= -\frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2} x^2-(x_1+x_2)x+x_1x_2\,dx$$

$$= -\frac{1}{(x_2-x_1)^2}\left[\frac{x^3}{3}-\frac{(x_1+x_2)}{2}x^2+x_1x_2x\right]_{x_1}^{x_2}$$

$$= -\frac{1}{(x_2-x_1)^2}\left[\frac{x_2^3-x_1^3}{3} - \frac{(x_1+x_2)}{2}(x_2^2-x_1^2) + x_1x_2(x_2-x_1)\right]$$

$$= -\frac{1}{(x_2-x_1)^2}\left[\frac{(x_2-x_1)(x_2^2+x_1x_2+x_1^2)}{3} - \frac{(x_1+x_2)^2(x_2-x_1)}{2} + x_1x_2(x_2-x_1)\right]$$

$$= -\frac{1}{\cancel{(x_2-x_1)^2}\,(x_2-x_1)}\left[\cancel{(x_2-x_1)}\left(\frac{x_2^2+x_1x_2+x_1^2}{3} - \frac{x_2^2+2x_1x_2+x_1^2}{2} + x_1x_2\right)\right]$$

$$\Downarrow$$

$$\frac{2(x_2^2+x_1x_2+x_1^2) - 3(x_2^2+2x_1x_2+x_1^2) + 6x_1x_2}{6}$$

$$\Downarrow$$

$$\frac{2x_2^2+2x_1x_2+2x_1^2-3x_2^2-\cancel{6x_1x_2}-3x_1^2+\cancel{6x_1x_2}}{6}$$

$$= -\frac{1}{(x_2-x_1)}\;\frac{-x_2^2+2x_1x_2-x_1^2}{6} = +\frac{1}{\cancel{(x_2-x_1)}}\;\frac{+(x_2-x_1)^{\cancel{2}}}{6}$$

$$= \boxed{\frac{x_2-x_1}{6}} = \frac{L}{6}$$

5. Evaluate the gradient, du/dx, for $x \in L^e$ in terms of the nodal values, $\underset{\sim}{u}^e$

$$u(x) = \underset{\sim}{N}(x)\,\underset{\sim}{u}^e = \begin{bmatrix} \frac{x_2-x}{x_2-x_1} & \frac{x-x_1}{x_2-x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{du(x)}{dx} = \frac{d\underset{\sim}{N}(x)}{dx}\,\underset{\sim}{u}^e$$

$$= \begin{bmatrix} -\frac{1}{x_2-x_1} & \frac{1}{x_2-x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -\frac{1}{x_2-x_1} & \frac{1}{x_2-x_1} \end{bmatrix} \underset{\sim}{u}^e}$$