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1.1 SOLUTION OF BOUNDARY VALUE PROBLEMS

The best way to solve any physical problem governed by a differential equation is to obtain the analytical solution. There are many situations, however, where the analytical solution is difficult to obtain. The region under consideration may be so irregular that it is mathematically impossible to describe the boundary. The configuration may be composed of several different materials whose regions are mathematically difficult to describe. Problems involving anisotropic materials are usually difficult to solve analytically, as are equations having nonlinear terms.

A numerical method can be used to obtain an approximate solution when an analytical solution cannot be developed. All numerical solutions produce values at discrete points for one set of the independent parameters. The complete solution procedure is repeated each time these parameters change. Numerical solutions, however, are more desirable than no solution at all. The calculated values provide important information about the physical process even though they are at discrete points.

There are several procedures for obtaining a numerical solution to a differential equation. The methods can be separated into three basic groupings: (1) the finite difference method, (2) the variational method, and (3) the methods that weight a residual. These methods are briefly discussed in the following paragraphs.

Finite Difference Method

The finite difference method approximates the derivatives in the governing differential equation using difference equations. This method is useful for solving heat transfer and fluid mechanics problems and works well for two-dimensional regions with boundaries parallel to the coordinate axes. The method, however, is rather cumbersome when regions have curved or irregular boundaries, and it is difficult to write general computer programs for the method.

Variational Method

The variational approach involves the integral of a function that produces a number. Each new function produces a new number. The function that produces the lowest number has the additional property of satisfying a specific differential equation. To help clarify this concept, consider the integral

$$\Pi = \int_0^H \left[\frac{D}{2}\left(\frac{dy}{dx}\right)^2 - Qy\right] dx \tag{1.1}$$

The numerical value of Π can be calculated given a specific equation $y=f(x)$. The calculus of variations shows, however, that the particular equation $y=g(x)$, which yields the lowest numerical value for Π, is the solution to the differential equation

$$D\frac{d^2y}{dx^2} + Q = 0 \tag{1.2}$$

with the boundary conditions $y(0)=y_0$ and $y(H)=y_H$.

The process can be reversed. Given a differential equation, an approximate solution can be obtained by substituting different trial functions into the appropriate functional. The trial function that gives the minimum value of Π is the approximate solution.

The variational method is the basis for many finite element formulations, but it has a major disadvantage: It is not applicable to any differential equation containing a first derivative term.

Weighted Residual Methods

The weighted residual methods also involve an integral. In these methods, an approximate solution is substituted into the differential equation. Since the approximate solution does not satisfy the equation, a residual or error term results. Suppose that $y=h(x)$ is an approximate solution to (1.2). Substitution gives

$$D\frac{d^2h(x)}{dx^2}+Q=R(x)\neq 0 \tag{1.3}$$

since $y=h(x)$ does not satisfy the equation. The weighted residual methods require that

$$\int_0^H W_i(x)R(x)\,dx=0 \tag{1.4}$$

The residual $R(x)$ is multiplied by a weighting function $W_i(x)$, and the integral of the product is required to be zero. The number of weighting functions equals the number of unknown coefficients in the approximate solution. There are several choices for the weighting functions, and some of the more popular choices have been assigned names.

Collocation Method. Impulse functions $W_i(x)=\delta(x-X_i)$ are selected as the weighting functions. This selection is equivalent to requiring the residual to vanish at specific points. The number of points selected equals the number of undetermined coefficients in the approximate solution.

Subdomain Method. Each weighting function is selected as unity, $W_i(x)=1$, over a specific region. This is equivalent to requiring the integral of the residual to vanish over an interval of the region. The number of integration intervals equals the number of undetermined coefficients in the approximate solution.

Galerkin's Method. Galerkin's method uses the same functions for $W_i(x)$ that were used in the approximating equation. This approach is the basis of the finite element method for problems involving first-derivative terms. This method yields the same result as the variational method when applied to differential equations that are self-adjoint. Galerkin's method is used to develop the finite element equations for the field problems discussed in this book.

Least Squares Method. The least squares method utilizes the residual as the weighting function and obtains a new error term defined by

$$Er = \int_0^H [R(x)]^2 \, dx \tag{1.5}$$

This error is minimized with respect to the unknown coefficients in the approximate solution. The least squares method has been utilized to formulate finite element solutions, but it is not as popular as Galerkin's method and the variational approach.

The variational method and the weighted residual methods each involve an integral. These methods can be grouped under the heading of integral formulations. A numerical solution based on an integral formulation is a new concept for many people; therefore, some of the more common methods are illustrated by solving a simple problem.

1.2 INTEGRAL FORMULATIONS FOR NUMERICAL SOLUTIONS

The immediate objective is to illustrate how each of the integral methods discussed in the previous section can be used to obtain an approximate solution to a physical problem. The example is a simply supported beam subjected to concentrated moments at each end. The beam and its bending moment diagram are shown in Figure 1.1.

The governing differential equation is

$$EI\frac{d^2y}{dx^2} - M(x) = 0 \tag{1.6}$$

with the boundary conditions

$$y(0) = 0 \qquad \text{and} \qquad y(H) = 0 \tag{1.7}$$

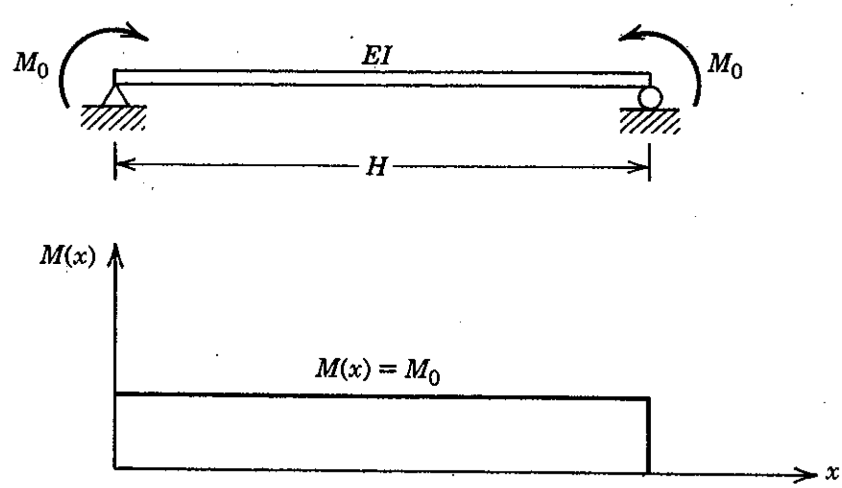

Figure 1.1. A simply supported beam with concentrated end moments.

The coefficient EI represents the resistance of the beam to deflection and $M(x)$ is the bending moment equation. In this example, $M(x)$ has the constant value M_0.

An approximate equation for the beam deflection is

$$y(x)=A \sin \frac{\pi x}{H} \tag{1.8}$$

where A is an undetermined coefficient. This solution is an acceptable candidate because it satisfies the boundary conditions $y(0)=y(H)=0$ and has a shape similar to the expected deflection curve. The exact solution of the differential equation is

$$y(x)=\frac{M_0 x}{2EI}(x-H) \tag{1.9}$$

1.2.1 Variational Method

The integral for the differential equation (1.6) is

$$\Pi=\int_0^H \left[\frac{EI}{2}\left(\frac{dy}{dx}\right)^2+M_0 y\right]dx \tag{1.10}$$

The value of A that makes (1.8) the best approximation to the deflection curve is the value that makes Π a minimum. To evaluate A, Π must be written as a function of A and then minimized with respect to A. Noting that

$$\frac{dy}{dx}=\frac{A\pi}{H}\cos\frac{\pi x}{H}$$

we find that Π becomes

$$\Pi=\int_0^H \left[\frac{EI}{2}\left(\frac{A\pi}{H}\cos\frac{\pi x}{H}\right)^2+M_0 A \sin\frac{\pi x}{H}\right]dx$$

or

$$\Pi=\left(\frac{EI\pi^2}{4H}\right)A^2+\left(\frac{2M_0 H}{\pi}\right)A \tag{1.11}$$

Minimizing Π yields

$$\frac{\partial\Pi}{\partial A}=2\left(\frac{EI\pi^2}{4H}\right)A+\frac{2M_0 H}{\pi}=0$$

and

$$A=-\frac{4M_0 H^2}{\pi^3 EI} \tag{1.12}$$

The approximate solution is

$$y(x)=-\frac{4M_0 H^2}{\pi^3 EI}\sin\frac{\pi x}{H} \tag{1.13}$$

1.2.2 Collocation Method

The collocation method requires that the residual equation for the approximate solution be zero at as many points as there are undetermined coefficients. The residual is obtained by substituting (1.8) into (1.6). The result is

$$R(x) = -EI\frac{A\pi^2}{H^2}\sin\frac{\pi x}{H} - M_0 \tag{1.14}$$

since

$$\frac{d^2y}{dx^2} = -\frac{A\pi^2}{H^2}\sin\frac{\pi x}{H}$$

There is only one undetermined coefficient; therefore, $R(x)$ is equated to zero at one point between 0 and H. Selecting $x = H/2$ for convenience, we obtain

$$R\left(\frac{H}{2}\right) = -EI\frac{A\pi^2}{H^2}\sin\frac{\pi}{2} - M_0 = 0$$

and

$$A = -\frac{M_0H^2}{EI\pi^2} \tag{1.15}$$

The approximate solution is

$$y(x) = -\frac{M_0H^2}{EI\pi^2}\sin\frac{\pi x}{H} \tag{1.16}$$

Had a point other than $x = H/2$ been selected, a different approximate solution would have been obtained.

1.2.3 Subdomain Method

The subdomain method requires that $\int R(x)\,dx = 0$ over as many subintervals as there are undetermined coefficients. The user can choose how long to make each subinterval. In the present example there is only one unknown coefficient; thus the interval must be $[0, H]$. The residual equation is (1.14); thus

$$\int_0^H R(x)\,dx = \int_0^H \left[-EI\frac{A\pi^2}{H^2}\sin\frac{\pi x}{H} - M_0\right]dx = 0$$

Integration yields

$$-\left(\frac{2EI\pi}{H}\right)A - M_0H = 0$$

and

$$A = -\frac{M_0H^2}{2\pi EI} \tag{1.17}$$

The approximate solution is

$$y(x) = -\frac{M_0 H^2}{2\pi EI} \sin \frac{\pi x}{H} \tag{1.18}$$

1.2.4 Galerkin's Method

When using Galerkin's method, $\int W_i(x)R(x)\,dx$ is evaluated using the same functions for $W_i(x)$ that were used in the approximate solution. In this example there is only one weighting function, $W_i(x) = \sin \pi x/H$. The residual equation is (1.14) and the integral is

$$\int_0^H \sin\frac{\pi x}{H}\left[-EI\frac{A\pi^2}{H^2}\sin\frac{\pi x}{H} - M_0\right]dx = 0$$

Integrating yields

$$-\frac{EI\pi^2 A}{2H} + \frac{2M_0 H}{\pi} = 0$$

Solving gives

$$A = -\frac{4M_0 H^2}{\pi^3 EI} \tag{1.19}$$

and the approximate solution is

$$y(x) = -\frac{4M_0 H^2}{\pi^3 EI}\sin\frac{\pi x}{H} \tag{1.20}$$

This solution is identical to the solution obtained using the variational method.

1.2.5 Least Squares Method

A new error term, $Er = \int [R(x)]^2\,dx$, is formed when using the least squares method. Substitution of the residual equation gives

$$Er = \int_0^H \left[-\frac{EI\pi^2}{H^2} A \sin\frac{\pi x}{H} - M_0\right]^2 dx$$

Integration gives

$$Er = \frac{A^2 H}{2}\left(\frac{EI\pi^2}{H^2}\right)^2 + \frac{4M_0 EI\pi}{H} A + M_0^2 H$$

The error is minimized with respect to A producing

$$\frac{\partial Er}{\partial A} = AH\left(\frac{EI\pi^2}{H^2}\right)^2 + \frac{4M_0 EI\pi}{H} = 0 \tag{1.21}$$

After solving for A, the approximate solution is

$$y(x) = -\frac{4M_0 H^2}{\pi^3 EI}\sin\frac{\pi x}{H} \tag{1.22}$$

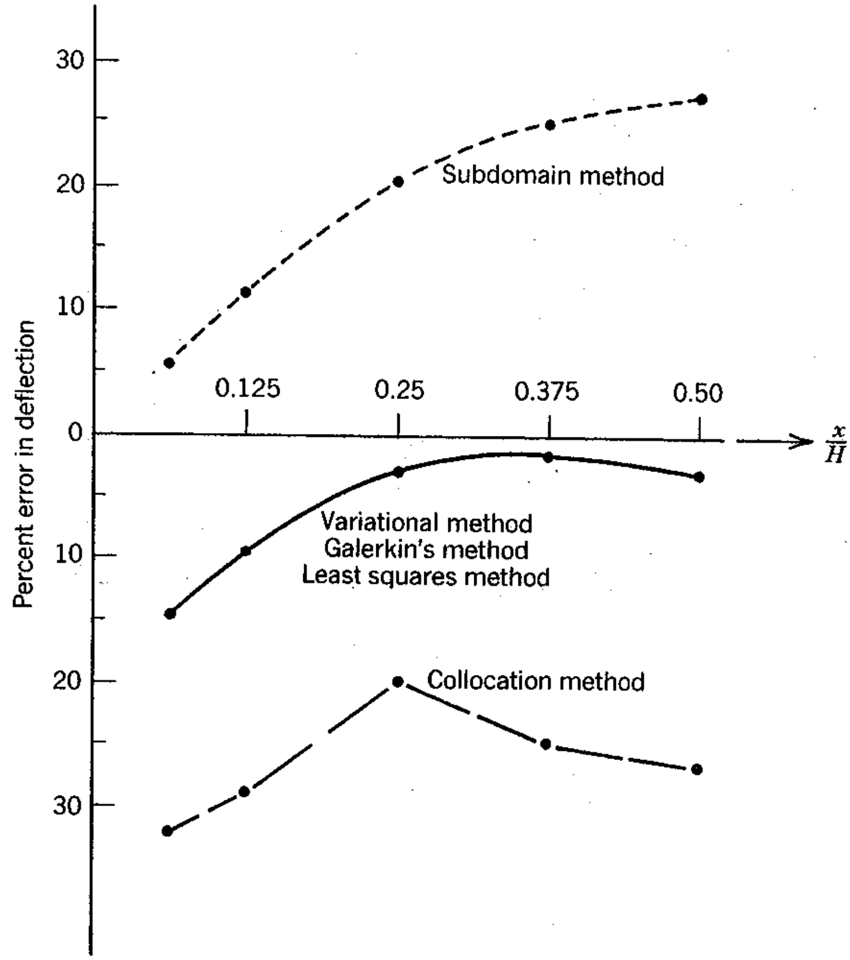

Figure 1.2. Percentage error for the five approximate solutions for the simply supported beam.

and is identical to the solutions obtained using the variational method and Galerkin's method, (1.13) and (1.20).

It is not possible to state which method is the most accurate. The error depends on the approximating function and the equation being solved. Percentage error curves for the different methods are given in Figure 1.2. It appears that equation (1.22) is more accurate than either equation (1.16) or (1.18). However, it is possible to find a collocation point which produces a maximum deflection that agrees with the exact value (see Problem 1.16). The collocation points or subregions selected affect the accuracy of the approximate solutions.

The important point to be gained from these examples is that the numerical solution of a differential equation can be formulated in terms of an integral. The integral formulation is a basic characteristic of the finite element method.

1.3 POTENTIAL ENERGY FORMULATIONS

The solution of solid mechanics problems, which includes the solution of two- and three-dimensional elasticity problems as well as plate and shell structures, can be

APPLIED FINITE ELEMENT ANALYSIS

Second Edition

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