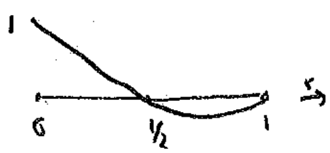
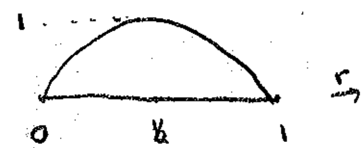
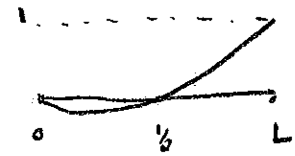
①

$$\underset{0 \quad L/2 \quad L}{\text{———}} \xrightarrow{x} \qquad \xrightarrow{x = Lr} \qquad \underset{0 \quad 1/2 \quad 1}{\text{———}} \xrightarrow{r}$$

a) $N_1(r) = 1 - 3r + 2r^2 \qquad N_2(r) = 4r - 4r^2 \qquad N_3(r) = 2r^2 - r$







b) $\sum_k N_k(x) = 1 - 3r + 2r^2 + 4r - 4r^2 + 2r^2 - r$

$= 1$

c) $\underline{B} = \frac{dN}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix}$

$= \begin{bmatrix} \frac{dN_1}{dr}\frac{dr}{dx} & \frac{dN_2}{dr}\frac{dr}{dx} & \frac{dN_3}{dr}\frac{dr}{dx} \end{bmatrix}$

$= \begin{bmatrix} (4r-3)\frac{1}{L} & (-8r+4)\frac{1}{L} & (4r-1)\frac{1}{L} \end{bmatrix}$

$= \frac{1}{L}\begin{bmatrix} 4r-3 & -8r+4 & 4r-1 \end{bmatrix}$

d) $\sum_k B_k(x) = \frac{1}{L}\,[0]$

$= 0$

e) $\underline{F} = \int_L \underline{N}(x)\,dx = \begin{bmatrix} \int_L N_1(x) & \int_L N_2(x)\,dx & \int_L N_3(x)\,dx \end{bmatrix}$

$= \begin{bmatrix} L\int_0^1 1 - 3r + 2r^2\,(dr) & L\int_0^1 4r - 4r^2\,dr & L\int_0^1 2r^2 - r\,dr \end{bmatrix}$

$= L\begin{bmatrix} r - \frac{3}{2}r^2 + \frac{2}{3}r^3 \Big|_0^1 & 2r^2 - \frac{4}{3}r^3 \Big|_0^1 & \frac{2}{3}r^3 - \frac{1}{2}r^2 \end{bmatrix}_0^1$

f) $\boxed{\sum_k F_k(x)} = L\left[1 - \frac{3}{2} + \frac{2}{3} + 2 - \frac{4}{3} + \frac{2}{3} - \frac{1}{2}\right]$

$= \boxed{L}$

g) $N_1(x)\,N_1(x) = (1 - 3r + 2r^2)^2$

$= 1 - 3r + 2r^2 - 3r + 9r^2 - 6r^3 + 2r^2 - 6r^3 + 4r^4$

$= 1 - 6r + 13r^2 - 12r^3 + 4r^4$

$N_1(x)\,N_2(x) = (1 - 3r + 2r^2)(4r - 4r^2)$

$= 4r - 12r^2 + 8r^3 - 4r^2 + 12r^3 - 8r^4$

$= 4r - 16r^2 + 20r^3 - 8r^4$

$N_1(x)\,N_3(x) = (1 - 3r + 2r^2)(2r^2 - r)$

$= 2r^2 - 6r^3 + 4r^4 - r + 3r^2 - 2r^3$

$= -r + 5r^2 - 8r^3 + 4r^4$

$N_2(x)\,N_2(x) = (4r - 4r^2)^2$

$= 16r^2 - 32r^3 + 16r^4$

$N_2(x)\,N_3(x) = (4r - 4r^2)(2r^2 - r)$

$= 8r^3 - 8r^4 - 4r^2 + 4r^3$

$= -4r^2 + 12r^3 - 8r^4$

$N_3(x)\,N_3(x) = (2r^2 - r)^2$

$= 4r^4 - 4r^3 + r^2$

$\underline{\underline{M}} = \int_L \underline{N}^T(x)\,\underline{N}(x)\,dx$

$\int_0^L N_1(x)\,N_1(x)\,dx = L\int_0^1 1 - 6r + 13r^2 - 12r^3 + 4r^4\,dr$

$= L\left[r - 3r^2 + \frac{13}{3}r^3 - 3r^4 + \frac{4}{5}r^5 \Big|_0^1\right]$

$= \boxed{\frac{2}{15}}\,L$

$$\int_0^L N_1(x) N_2(x)\, dx = L \int_0^1 4r - 16r^2 + 20r^3 - 8r^4$$

$$= L \left[2r^2 - \frac{16}{3}r^3 + 5r^4 - \frac{8}{5}r^5 \right]_0^1$$

$$= \frac{1}{15} L$$

$$\int_0^L N_1(x) N_3(x)\, dx = L \int_0^1 -r + 5r^2 - 8r^3 + 4r^4 \, dr$$

$$= L \left[-\frac{1}{2}r^2 + \frac{5}{3}r^3 - 2r^4 + \frac{4}{5}r^5 \right]_0^1$$

$$= -\frac{1}{30} L$$

$$\int_0^L N_2(x) N_2(x) = L \int_0^1 16r^2 - 32r^3 + 16r^4 \, dr$$

$$= L \left[\frac{16}{3}r^3 - 8r^4 + \frac{16}{5}r^5 \right]$$

$$= \frac{8}{15} L$$

$$\int_0^L N_2(x) N_3(x) = L \left[-\frac{4}{3}r^3 + 3r^4 - \frac{8}{5}r^5 \right]$$

$$= \frac{1}{15} L$$

$$\int_0^L N_3(x) N_3(x) = L \left[\frac{4}{5}r^5 - 1r^4 + \frac{1}{3}r^3 \right]$$

$$= \frac{2}{15} L$$

$$M = L \begin{bmatrix} \frac{2}{15} & \frac{1}{15} & -\frac{1}{30} \\ \frac{1}{15} & \frac{8}{15} & \frac{1}{15} \\ -\frac{1}{30} & \frac{1}{15} & \frac{2}{15} \end{bmatrix}$$ ✓

h.) $$\sum_j \sum_k M_{jk} = \frac{L}{15} \left[2 + 1 - \frac{1}{2} + 1 + 8 + 1 - \frac{1}{2} + 1 + 2 \right]$$

$$= L$$ ✓