Relations
Binary Relations

Ordered Pair -- (x,y)

- \( (x, y) = (x^*, y^*) \) means that \( x = x^* \) and \( y = y^* \)
- \( (x, y) \neq (y, x) \) -- order matters

Cross Product

- \( A \times B = \{(a, b) \mid (a \in A) \land (b \in B)\} \)

Binary Relation

- Any subset \( R \) of \( A \times B \) is called a binary relation on \( A, B \).
- \( (a, b) \in R \iff aRb \)
Examples

1. \( R = \{(a,b) \mid a < b\} \)

2. \( R = \{(\text{students},\text{courses})\} \)

3. \( R = \{(f,g) \mid f = O(g)\} \)

4. \( R = \{(A,B) \mid |A| = |B|\} \)

5. Functions: \( R = \{(a,f(a)) \mid a \in A, f(a) \in f(A)\} \)

   -- All Functions are Relations

   -- NOT All Relations are Functions
Representations

1. Tables

2. Graphs

3. Matrices
Directed Graphs

Analogy

- Graphs $\approx$ Relations
- Directed graphs are pictures of relations
- $uRv \iff$ there is an edge from $u$ to $v$

Bipartite Graphs

- All edges go from set of vertices A to disjoint set of vertices B
- $R \subset A \times B$ -- general edge relation

Edge Relations

- Contain only topological -- yes/no -- information.
- No other data except connectivity.
Number of Relations

Finite Sets

\[ |A| = m \quad \text{and} \quad |B| = n \]

\[ \Rightarrow |A \times B| = |A| \cdot |B| = mn \]

\[ \Rightarrow \# \text{ relations on } A \times B = \# \text{ subsets of } A \times B = 2^{mn} \]
N-Ary Relations

- $R \subseteq A_1 \times A_2 \times \cdots \times A_n$

- Table = Relational Data Base

- Projections -- Delete some columns

- Joins -- Combine overlapping tables
## Relational Data Base

<table>
<thead>
<tr>
<th>Student</th>
<th>Homework</th>
<th>Midterm</th>
<th>Final</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lydia</td>
<td>90</td>
<td>85</td>
<td>95</td>
<td>A–</td>
</tr>
<tr>
<td>Joe</td>
<td>80</td>
<td>85</td>
<td>90</td>
<td>B</td>
</tr>
<tr>
<td>Ron</td>
<td>60</td>
<td>45</td>
<td>50</td>
<td>F</td>
</tr>
<tr>
<td>Dan</td>
<td>95</td>
<td>98</td>
<td>100</td>
<td>A+</td>
</tr>
<tr>
<td>Sally</td>
<td>70</td>
<td>65</td>
<td>75</td>
<td>C</td>
</tr>
</tbody>
</table>
Most Important Relations

1. Equivalence Relations (on $A \times A$)

2. Transitive Closure

3. Partial Order
Equivalence Relations

Properties

1. Reflexive -- $aRa$

2. Symmetric -- $aRb \Rightarrow bRa$

3. Transitive -- $aRb$ and $b Rc \Rightarrow aRc$
Examples of Equivalence Relations

1. Rice undergraduates in the same college.

2. People of same height.

3. Computers with same amount of memory.

4. Programs that compute the same function.

5. Horses of the same color

6. Sets of the same cardinality.

7. Propositions that are logically equivalent.

8. Functions in the same complexity class.
Examples of Relations that are NOT Equivalence Relations

1. \( a \) is the father of \( b \) -- not reflexive, not symmetric

2. \( a \) is the brother of \( b \) -- not symmetric (sisters)

3. \( a \) has at least one parent in common with \( b \) -- not transitive

4. \( f = O(g) \) -- not symmetric
Reflexive and Symmetric Representations

1. Graphs

2. Matrices
Equivalence Classes for Equivalence Relations

Equivalence Classes

• \([a] = \{x \mid aRx\}\)

Properties

• \([a]=[b] \iff aRb\)

• \([a] \cap [b] = \emptyset \text{ otherwise}\)
Partitions

Definition

1. \( A = \bigcup_{i \in I} A_i \)
   -- every element of \( A \) lies in some \( A_i \)

2. \( A_i \cap A_j = \emptyset \quad i \neq j \)
   -- no element of \( A \) lies in more than one \( A_i \)
**Equivalence Relations ⇔ Partitions**

**Theorem:** Equivalence Relations ⇔ Partitions

**Proof:**

⇒: Let $A_a = [a]$.

Then by the properties of equivalence classes the sets $A_a$ form a partition of $A$.

⇐: Let $\{A_i\}$ be a partition of $A$, and define

\[ a R b \iff \exists i \ a, b \in A_i. \]

Then it is easy to check that $R$ is reflexive, symmetric, and transitive, so $R$ is an equivalence relation.

**QED**
Functions on Equivalence Classes

Subtlety

- Let $f([a]) = g(a)$

- To show $f$ is well-defined, must show that $a R b \implies g(a) = g(b)$

3. If $a R b \implies g(a) = g(b)$, then we say that $g$ respects equivalence classes
Example

Rice Undergraduates

• \( aRb \iff a \text{ and } b \text{ are in the same college} \)

Functions

• \( f([\text{Mary}]) = \text{Mary's Last Name} \)
  -- \( f \) does not respect equivalence classes

• \( f([\text{Mary}]) = \text{Mary's College} \)
  -- \( f \) respects equivalence classes
Closures

Closure
• Smallest relation $S \supseteq R$ with property $P$

Reflexive Closure
• $S = R \cup \Delta$, where $\Delta = \{(a,a)\}$

Symmetric Closure
• $S = R \cup R^{-1}$, where $(b,a) \in R^{-1} \iff (a,b) \in R$

Transitive Closure
• $S = R^*$ (see next lecture)
Transitive Closure
Composition

Functions
• If \( f : A \rightarrow B \) and \( g : B \rightarrow C \), then \( g \circ f : A \rightarrow C \)
• \((g \circ f)(a) = g(f(a))\)

Relations
• If \( R \subseteq A \times B \) and \( S \subseteq B \times C \), then \( S \circ R \subseteq A \times C \)
• \( a(S \circ R)c \iff \exists b \in B \) such that \( aRb \) and \( bSc \)
Examples of Composition

Definitions

• \( a R b \) means \( b = \) parent of \( a \)

• \( b S c \) means \( b = \) sibling of \( c \)

Composition

• \( a(S \circ R)c \) means
Examples of Composition

Definitions

• \(a R b\) means \(b = \) parent of \(a\)

• \(b S c\) means \(b = \) sibling of \(c\)

Composition

• \(a(S \circ R)c\) means \(c = \) aunt/uncle of \(a\)

• \(a(R \circ R)c\) means
Examples of Composition

Definitions

- \( a R b \) means \( b = \text{parent of } a \)
- \( b S c \) means \( b = \text{sibling of } c \)

Composition

- \( a(S \circ R)c \) means \( c = \text{aunt/uncle of } a \)
- \( a(R \circ R)c \) means \( c = \text{grandparent of } a \)
- \( a(R^{-1} \circ S \circ R)c \) means
Examples of Composition

Definitions

• \( a R b \) means \( b = \text{parent of } a \)

• \( b S c \) means \( b = \text{sibling of } c \)

Composition

• \( a(S \circ R)c \) means \( c = \text{aunt/uncle of } a \)

• \( a(R \circ R)c \) means \( c = \text{grandparent of } a \)

• \( a(R^{-1} \circ S \circ R)c \) means \( c = \text{cousin of } a \)
Composition and Matrix Multiplication

Notation

- $M = \text{Matrix for } R$
- $N = \text{Matrix for } S$

Composition

- $M \ast N = \text{Matrix for } S \circ R$
- $\ast = \text{Boolean Matrix Multiplication}$
- $- + = \text{or}$
- $- \times = \text{and}$
Powers and Closure

Powers of a Relation -- Recursive Definition

• \( R^0 = I \) (Identity)

• \( R^1 = R \)

• \( R^2 = R \circ R \)

• \( R^{n+1} = R \circ R^n = R \circ \cdots \circ R \) \( n+1 \) factors

Explicit Definition

• \( a R^n b \iff a = x_0 R x_1 R x_2 \cdots x_{n-1} R x_n = b \) (by induction on \( n \))

Transitive Closures

• \( R^+ = \bigcup_{k \geq 1} R^k \)

• \( R^* = \bigcup_{k \geq 0} R^k \)
Closures

Transitive Closure

- $R^+$ = transitive closure of $R$
- $a R^+ b \iff a = x_0 R x_1 R x_2 \cdots R x_{n-1} R x_n = b \quad n \geq 1$

Reflexive and Transitive Closure

- $R^* =$ transitive closure of $R$
- $a R^* b \iff a = x_0 R x_1 R x_2 \cdots R x_{n-1} R x_n = b \quad n \geq 0$
- $R^*$ is often called just the transitive closure

Observations

- * means 0 or more
- + means 1 or more
- $R^*$ is reflexive
- $R^+$ need not be reflexive
Examples

1. \( R = \{(a,b) \mid a \text{ is a parent of } b\} \)
   -- \( R^+ = ? \)
   -- \( R^* = ? \)

2. \( R = \{(a,b) \mid a \text{ shares a common border with } b\} \)
   -- \( R^+ = ? \)
   -- \( R^* = ? \)

3. \( R = \{(a,b) \mid \text{computer } a \text{ is connected to computer } b\} \)
   -- \( R^+ = ? \)
   -- \( R^* = ? \)

4. \( R = \{(a,b) \mid \text{instruction } a \text{ precedes instruction } b\} \)
   -- \( R^+ = ? \)
   -- \( R^* = ? \)
More Examples

Graphs
• \(\rightarrow\) means edge
• \(\rightarrow^*\) means path

Trees
• \(\rightarrow\) means child
• \(\rightarrow^*\) means descendant

Computers
• \(\Rightarrow\) means can get from one configuration (instantaneous description, snapshot) to another in one move (1 machine cycle, 1 instruction)
• \(\Rightarrow^*\) means an entire computation
Closures

Matrix Definition

• \( M = \text{Matrix for the relation } R \)
• \( R^+ = \sum_{k \geq 1} M^k \)
• \( R^* = \sum_{k \geq 0} M^k \)
• Matrix multiply and add = boolean multiply and add

Graph Definition

• \( G = (V, E) \)
  -- \( V = A \) (set on which \( R \) is defined)
  -- \( E = \{ a \rightarrow b \mid a R b \} \)
• \( a R^+ b \iff a \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \rightarrow b \quad n \geq 1 \)
• \( a R^* b \iff a \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \rightarrow b \quad n \geq 0 \)
Simple Theorems on Transitivity

Theorem 1: $R$ is transitive if and only if $R \supseteq R^n$ for all $n \geq 1$.

Proof: $\Rightarrow$: By induction on $n$.

$\Leftarrow$: $R \supseteq R^n \Rightarrow R \supseteq R^2 \Rightarrow R$ transitive

Theorem 2: 1. $R^*$ is reflexive

2. $R^+, R^*$ are transitive

3. $R^+, R^* \supseteq R$

Proof: Obvious from Definitions
Fundamental Theorem

Theorem 3: \( R^* \) is the smallest reflexive and transitive relation that contains \( R \).

In particular, \( R^* = \bigcap Q \), where the intersection is over all reflexive and transitive relations \( Q \) that contain \( R \).

Proof: By Theorem 2, \( R^* \) is clearly a reflexive and transitive relation that contains \( R \). Now suppose that \( Q \) is any reflexive and transitive relation that contains \( R \). Then

\[
a R^* b \Rightarrow a = x_0 R x_1 R x_2 \cdots x_{n-1} R x_n = b
\]

\[
\Rightarrow a = x_0 Q x_1 Q x_2 \cdots x_{n-1} Q x_n = b
\]

\[
\Rightarrow a Q b
\]

because \( Q \) is reflexive and transitive.

Hence \( Q \supset R^* \). QED
Relations on Finite Sets

Theorem 4: Let $|A|=n$, and let $R$ be a relation on $A$. If there is a path in $R$ from $a$ to $b$, then there is a path in $R$ from $a$ to $b$ of length at most $n$ ($n-1$ if $a \neq b$).


Corollary: $|A|=n \Rightarrow R^* = R \cup R^2 \cup \cdots \cup R^n$
Partial Order
Orders

Partial Order
- Reflexive -- \( aRa \)
- Antisymmetric -- \( aRb \) and \( bRa \) \( \Rightarrow \) \( a=b \)
- Transitive -- \( aRb \) and \( bRc \) \( \Rightarrow \) \( aRc \)

Note: There may be elements that are NOT comparable

Total Order
- Partial order where every two elements are comparable

Well Order
- Total order where every nonempty subset has a smallest element
- Induction works only on well ordered sets
- Base case = smallest element
Examples

• \{N, \leq\}

• \{\mathbb{Z}, \leq\}

• \{P(S), \supseteq\}

• \{\mathbb{Z}^+, |\}

• Orders on \(N \times N\)
  -- Lexicographic -- \((a,b) < (c,d) \iff a < c \text{ or } a = c \text{ and } b < d\)
  -- Product -- \((a,b) < (c,d) \iff a < c \text{ and } b < d\)

• Order on Strings \(\Sigma^*\)
  -- Lexicographic order = Dictionary order

• Graphs and Trees
  -- Subgraphs and Subtrees
Hasse Diagrams

- Graphical representation of a poset
- Relation graph without reflexive and transitive edges
- See pictures
Definitions

Maximal and Minimal Elements
• in the set
• not unique

Greatest (Maximum) and Least (Minimum) Elements
• in the set
• unique

Upper and Lower Bounds
• not necessarily in the set
• not unique

Lub and Glb
• not necessarily in the set
• unique
Examples

Hasse Diagram

• Maximal and Minimal Elements
• Upper and Lower Bounds

\{P(S), \supset\}\n
• Greatest = S \quad Least = \emptyset

\{Z, \leq\}\n
• no greatest or least element

\{(0,1), \leq\}\n
• no greatest or least element
Lattices

Definition

• Poset where every pair has a lub and a glb

Examples

• Total Orders

• \( \{ \mathbb{Z}^+, \mid \} \)
  -- \( \text{glb}(a,b) = \gcd(a,b) \)
  -- \( \text{lub}(a,b) = \lcm(a,b) \)

• \( \{ P(S), \supset \} \)
  -- \( \text{glb}(A,B) = A \cap B \)
  -- \( \text{lub}(A,B) = A \cup B \)

• \( \{(N \times N, \text{Product})\} \)
  -- \( \text{glb}\{(a,c), b, d\} = (\min(a, c), \min(b, d)) \)
  -- \( \text{lub}\{(a,c), b, d\} = (\max(a, c), \max(b, d)) \)
Topological Sort

Purpose

• Convert a partial order into a total order

Applications

• Scheduling -- Engineering
• Hidden Surface Algorithms -- Computer Graphics
Topological Sort on Finite Sets

Lemma
• Every finite poset has a minimal element
• Proof by induction on $|S|$

Algorithm
• Choose a minimal element $a$ of $S$
• Choose a minimal element $b$ of $S - \{a\}$
• Continue until all elements of $S$ are exhausted
• Rank elements in order chosen

Result of Topological Sort is NOT Unique!
• Examples -- Hasse diagrams
Scheduling Comp 280 for Spring 2011

Chapter 12

Chapter 7

Chapter 8  Chapter 6  Chapter 10

Chapter 2  Chapter 5  Chapter 9

Chapter 1  Chapter 4  Chapter 3