Calculating the actual positions of two balls after an elastic collision when the collision is detected after the balls should have collided.

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The equations of motion for two objects at constant velocity are:

$$\boldsymbol{\vec{x}_1(t)} = \vec{v}_1 t + \vec{x}_{1,0}$$

$$\boldsymbol{\vec{x}_2(t)} = \vec{v}_2 t + \vec{x}_{2,0}$$

At time t=0, the collision is detected. Therefore

$$\boldsymbol{R^2} \equiv (\mathbf{r_1} + \mathbf{r_2})^2 \geq \left|\vec{x}_{2,0} - \vec{x}_{1,0}\right|^2$$

Solve for the time of the collision

$$\boldsymbol{R^2} = \left|(\vec{v}_2 t + \vec{x}_{2,0}) - (\vec{v}_1 t + \vec{x}_{1,0})\right|^2 = \left|(\vec{v}_2 - \vec{v}_1)t + (\vec{x}_{2,0} - \vec{x}_{1,0})\right|^2 \equiv \left|\overrightarrow{\Delta v}t + \overrightarrow{\Delta x}\right|^2$$

$$\boldsymbol{R^2} = \left|\overrightarrow{\Delta v}t + \overrightarrow{\Delta x}\right|^2 = \Delta v^2 t^2 + 2\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)t + \Delta x^2$$

Rearranging

$$\mathbf{0} = \Delta v^2 t^2 + 2\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)t + \left(\Delta x^2 - \boldsymbol{R^2}\right)$$

Solving for the time of the collision:

$$t = \frac{-2\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right) \pm \sqrt{\left(2\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)\right)^2 - 4\Delta v^2(\Delta x^2 - \boldsymbol{R^2})}}{2\Delta v^2}$$
$$= \frac{-\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right) \pm \sqrt{\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)^2 - \Delta v^2(\Delta x^2 - \boldsymbol{R^2})}}{\Delta v^2}$$

Note that since the collision has already occurred and the above inequality holds, there are always two real solutions for the collision time, one positive and one negative. We are interested in the negative solution because the collision has already happened by the time we detect it.

Sanity check: Suppose we are looking right at the moment that the collision occurs, then

$$0 = \Delta x^2 - \boldsymbol{R^2}$$

$$t = \begin{cases} 0 \\ -\dfrac{2\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)}{\Delta v^2} \end{cases}$$

The first value is the current time, as expected. The second value is the time of collision for the other sides of the balls, which is either in the past or the present, depending on which way the balls are going with respect to each other.

We are only interested in the general case, negative time solution:

$$t = \frac{-\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right) - \sqrt{\left(\overrightarrow{\Delta v} \cdot \overrightarrow{\Delta x}\right)^2 - \Delta v^2(\Delta x^2 - \boldsymbol{R^2})}}{\Delta v^2}$$

Given this negative value for time, the positions of each ball at the moment of collision can be calculated.

Given the velocities after the collision (calculated using the impulse transferred), the correct position after the collision can be calculated from those positions forward by amount t.