Lecture 11: Some Applications of Vector Geometry

apply thine heart unto my knowledge. Proverbs 22:18

1. Introduction

Vector geometry and vector algebra are powerful tools for deriving geometric relationships and algebraic identities. In this lecture we shall employ vector methods to derive trigonometric laws, metric expressions, intersection formulas, interpolation techniques, and inside-outside tests that have wide ranging applications in Computer Graphics and Geometric Modeling.

2. Trigonometric Laws

Trigonometric laws are often required for the analysis of geometry. As early as Lecture 1, we employed the Law of Cosines to compute the length of the radius in a wheel and the lengths of the diagonals in a rosette. Here we shall provide simple derivations of the Law of Cosines and the Law of Sines using dot product and cross product.

2.1 Law of Cosines. Consider ΔABC with corresponding sides a,b,c (see Figure 1). The Law of Cosines asserts that

$$c^2 = a^2 + b^2 - 2\,ab\,Cos(C). \tag{2.1}$$

The Law of Cosines is a generalization of the Pythagorean Theorem; indeed, when $C = \pi/2$, the Law of Cosines is the Pythagorean Theorem, since $Cos(\pi/2) = 0$.

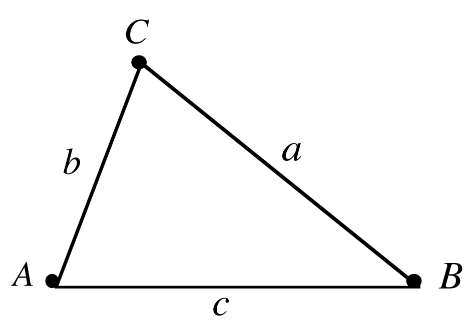


Figure 1: Law of Cosines: $c^2 = a^2 + b^2 - 2\,ab\,Cos(C)$

Whenever you see cosine, think dot product. To derive the Law of Cosines, recall that

$$c^2 = |B - A|^2 = (B - A) \bullet (B - A). \tag{2.2}$$

Since we need to introduce the sides a,b into the formula for c^2, observe that

$$B - A = (B - C) + (C - A). \tag{2.3}$$

Now the rest of the derivation is mechanical. Substituting Equation (2.3) into Equation (2.2), expanding by the distributive law, and invoking the definition of the dot product yields

$$
\begin{aligned}
c^2 &= (A-B)\bullet(A-B) \\
&= \big((B-C)+(C-A)\big)\bullet\big((B-C)+(C-A)\big) \\
&= (B-C)\bullet(B-C) + 2(B-C)\bullet(C-A) + (C-A)\bullet(C-A) \\
&= |B-C|^2 - 2(B-C)\bullet(A-C) + |C-A|^2 \\
&= a^2 - 2\,ab\,Cos(C) + b^2
\end{aligned}
$$

2.2 Law of Sines. Consider ΔABC in Figure 2. The Law of Sines asserts that

$$\frac{a}{Sin(A)} = \frac{b}{Sin(B)} = \frac{c}{Sin(C)}. \tag{2.4}$$

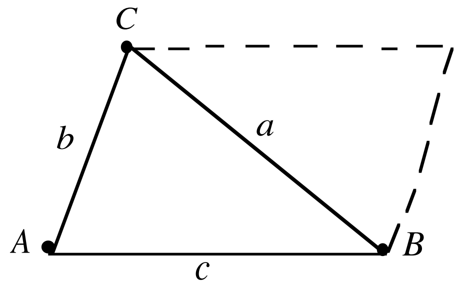


Figure 2: Law of Sines: $\dfrac{a}{Sin(A)} = \dfrac{b}{Sin(B)} = \dfrac{c}{Sin(C)}$.

Whenever you see sine, think cross product. To derive the Law of Sines, observe that by the definition of the cross product

$$
\begin{aligned}
2Area(\Delta ABC) &= \big|(A-B)\times(C-B)\big| = c\,a\,Sin(B) \\
2Area(\Delta ABC) &= \big|(C-A)\times(B-A)\big| = bc\,Sin(A) \\
2Area(\Delta ABC) &= \big|(B-C)\times(A-C)\big| = ab\,Sin(C)\,.
\end{aligned}
$$

Therefore

$$c\,a\,Sin(B) = bc\,Sin(A) = ab\,Sin(C)\,,$$

so

$$\frac{a}{Sin(A)} = \frac{b}{Sin(B)} = \frac{c}{Sin(C)}.$$

3. Representations for Lines and Planes

We are going to derive formulas for distances and intersections between simple geometric objects. We shall focus primarily on lines and planes, so a word about representations for lines and planes is in order here before we proceed any further.

3.1 Lines. Two points determine a line, but a point and a vector also determine a line (see Figure 3, left). Generally it will be more convenient to represent a line L in terms of a point P on the line and a vector v parallel to the line, since the formulas for distances and intersections are readily represented in terms of dot and cross products, which apply to vectors but not to points. The line L determined by the point P and the vector v can be expressed parametrically by the linear equation

$$L(t) = P + tv\,, \tag{3.1}$$

which is equivalent to the three linear parametric equations for the coordinates

$$x(t) = p_1 + tv_1,\;\; y(t) = p_2 + tv_2,\;\; z(t) = p_3 + tv_3.$$

2.2 Planes. There are two convenient ways to represent a plane in 3-dimensions: parametrically and implicitly. Just like a line L can be represented by a point P and a nonzero vector v, a plane S can be represented by a point P and two linearly independent vectors u,v (see Figure 3, right). The plane S determined by the point P and the vectors u,v can be expressed parametrically by the linear equation

$$S(u,v) = P + su + tv\,, \tag{3.2}$$

which is equivalent to the three linear parametric equations for the coordinates

$$x(s,t) = p_1 + su_1 + tv_1,\;\; y(s,t) = p_2 + su_2 + tv_2,\;\; z(s,t) = p_3 + su_3 + tv_3.$$

Alternatively, a plane S can be represented by a point P on S and a vector N normal to S. A point Q lies on the plane S if and only if the vector $Q - P$ is perpendicular to the normal vector N -- that is, if and only if

$$N \bullet (Q - P) = 0. \tag{3.3}$$

Equation (3.3) is called the *implicit equation* of the plane S. Notice that the implicit equation is also a linear equation. Indeed if $N = (a,b,c)$,$Q = (x,y,z)$, and $N \bullet P = d$, then

$$N \bullet (Q - P) = 0 \;\Leftrightarrow\; ax + by + cz + d = 0\,. \tag{3.4}$$

Equation (3.4) is called an implicit equation because there are no explicit expressions for the coordinates x,y,z as there are in the parametric representation. Notice that if we have a parametric representation for the plane S in terms of a point P and two linearly independent vectors u,v , then we can easily generate an implicit representation, since $N = u \times v$ is normal to the plane S.

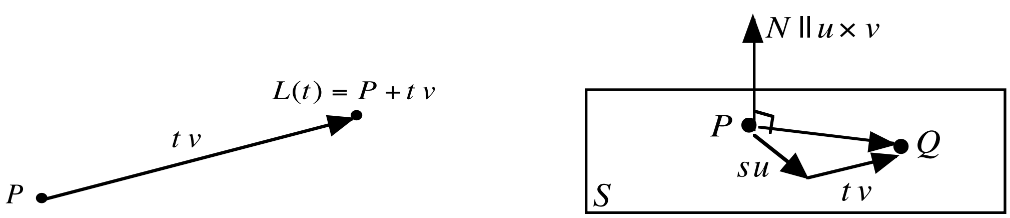


Figure 3: A line L determined by a point P and a direction vector v (left). A plane S determined either by a point P and two linearly independent vectors u,v or by a point P and a normal vector N (right). Notice that $N \,||\, u \times v$.

4. Metric Formulas

Here we collect some simple formulas for distance, area, and volume, which will be useful in a variety of future applications.

4.1 Distance. We are interested mainly in the distance between points, lines, and planes. Distance is often related to projection and projection is computed using dot product, so distance formulas are typically expressed in terms of dot products.

4.1.1 Distance Between Two Points. The distance between two points P,Q is the same as the length of the vector from P to Q (see Figure 4). Therefore

$$Dist^2(P,Q) = |Q-P|^2 = (Q-P) \bullet (Q-P). \tag{4.1}$$

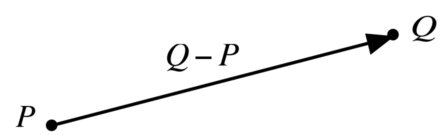


Figure 4: $Dist^2(P,Q) = |Q-P|^2 = (Q-P) \bullet (Q-P)$.

4.1.2 Distance Between a Point and a Line. Consider a point Q and a line L determined by a point P and a direction vector v (see Figure 5). The distance between the point Q and the line L is the length of the perpendicular component of the vector $Q-P$ relative to the vector v. Therefore by the Pythagorean Theorem

$$Dist^2(Q,L) = |(Q-P)_\perp|^2 = |Q-P|^2 - |(Q-P)_{||}|^2,$$

so

$$Dist^2(Q,L) = (Q-P) \bullet (Q-P) - \frac{((Q-P) \bullet v)^2}{v \bullet v}. \tag{4.2}$$

Notice that if v is a unit vector, then $v \bullet v = 1$, so in this case

$$Dist^2(Q,L) = (Q-P) \bullet (Q-P) - ((Q-P) \bullet v)^2. \tag{4.3}$$

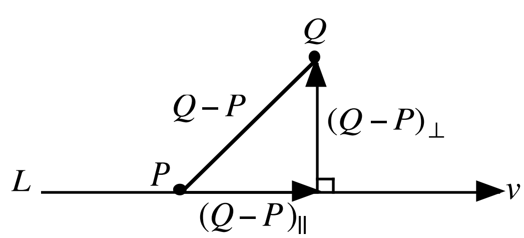


Figure 5: $Dist^2(P,L) = (Q-P) \bullet (Q-P) - \dfrac{((Q-P) \bullet v)^2}{v \bullet v}$.

4.1.3 Distance Between a Point and a Plane. Consider a point Q and a plane S determined by a point P and a normal vector N (see Figure 6). The distance between the point Q and the plane S is the length of the parallel component of the vector $Q-P$ relative to the vector N. Therefore by the definition of the dot product:

$$Dist(Q,S) = |(Q-P)_{||}| = |Q-P|\,Cos(\theta) = \frac{|Q-P|\,|N|\,Cos(\theta)}{|N|} = \frac{|(Q-P)\bullet N|}{|N|}. \quad (4.4)$$

Notice that if N is a unit vector, then $|N| = 1$, so in this case

$$Dist(Q,S) = |(Q-P)\bullet N|\,. \quad (4.5)$$

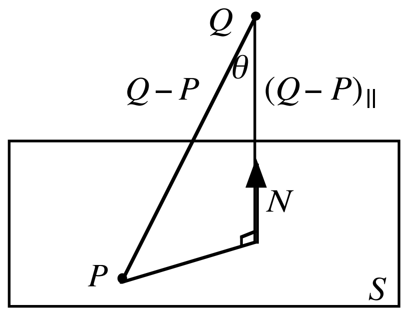


Figure 6: $Dist(Q,S) = |(Q-P)_{||}| = \dfrac{|(Q-P)\bullet N|}{|N|}$.

Alternatively, instead of representing the plane S by a point P and a normal vector N, suppose S is represented by a point P and a pair of vectors u,v (see Figure 7). Since

volume of a parallelepiped = area of the base × height of the parallelepiped

it follows that

$$Vol(u,v,Q-P) = Area(u,v) \times Dist(Q,S)\,.$$

Therefore solving for $Dist(Q,S)$ and invoking the definitions of cross product and determinant, we find that

$$Dist(Q,S) = \frac{Vol(u,v,Q-P)}{Area(u,v)} = \frac{|Det(u,v,Q-P)|}{|u \times v|}\,. \quad (4.6)$$

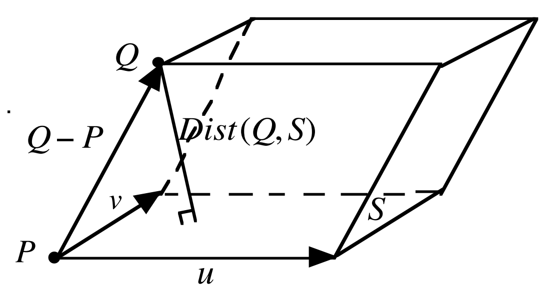


Figure 7: $Dist(Q,S) = \dfrac{Vol(u,v,Q-P)}{Area(u,v)} = \dfrac{|Det(u,v,Q-P)|}{|u \times v|}$.

4.1.4 Distance Between Two Lines. If two lines fail to intersect, then either the lines are parallel or the lines are skew. We shall consider each of these cases in turn.

4.1.4.1 Distance Between Two Parallel Lines. Consider a pair of lines, L_1, L_2, parallel to a vector v. Let P_1 be a point on L_1 and let P_2 be a point on L_2 (see Figure 8). Then

$$Dist(L_1, L_2) = Dist(P_2, L_1).$$

Therefore by Equation (4.2)

$$Dist^2(L_1, L_2) = (P_2 - P_1) \bullet (P_2 - P_1) - \frac{\big((P_2 - P_1) \bullet v\big)^2}{v \bullet v}. \qquad (4.7)$$

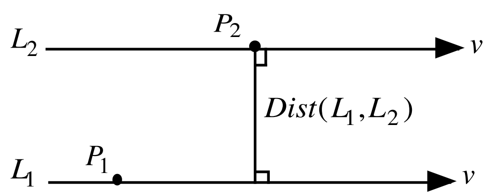


Figure 8: $Dist^2(L_1, L_2) = (P_2 - P_1) \bullet (P_2 - P_1) - \dfrac{\big((P_2 - P_1) \bullet v\big)^2}{v \bullet v}$.

4.1.4.2 Distance Between Two Skew Lines. Consider a pair of skew lines, L_1, L_2, where L_1 is parallel to the vector v_1 and L_2 is parallel to the vector v_2. Let P_1 be a point on L_1, let P_2 be a point on L_2, and let S be the plane determined by the point P_1 and the normal vector $v_1 \times v_2$ (see Figure 9). Then since $v_1 \times v_2$ is perpendicular to both L_1 and L_2,

$$Dist(L_1, L_2) = Dist(P_2, S).$$

Therefore by Equation (4.4)

$$Dist(L_1, L_2) = \frac{\big|(P_2 - P_1) \bullet v_1 \times v_2\big|}{|v_1 \times v_2|} = \frac{\big|Det\big(P_2 - P_1, v_1, v_2\big)\big|}{|v_1 \times v_2|}. \qquad (4.8)$$

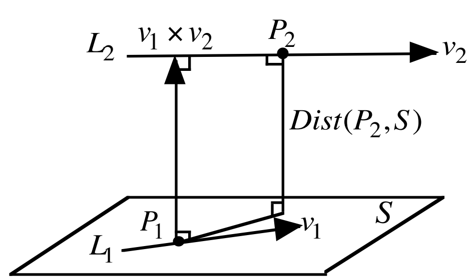


Figure 9: $Dist(L_1, L_2) = \dfrac{\big|Det\big((P_2 - P_1), v_1, v_2\big)\big|}{|v_1 \times v_2|}$.

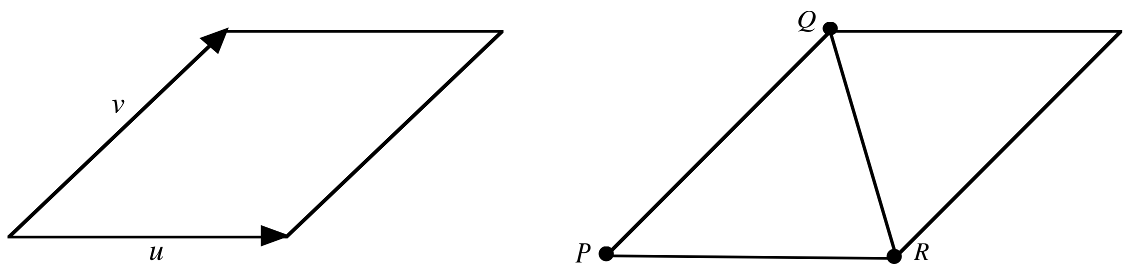
4.2 Area. We are interested mainly in the areas of triangles, parallelograms, and planar polygons. Since the cross product is defined in terms of area, formulas for area are typically expressed in terms of cross products.

4.2.1 Triangles and Parallelograms. Consider a parallelogram determined by two vectors u,v (see Figure 10, left). Then by the definition of the cross product

$$Area(u,v) = |\, u \times v \,| \,. \tag{4.9}$$

Two triangles make a parallelogram (see Figure 10, right). Therefore

$$Area(\Delta PQR) = \frac{Area(Q-P, R-P)}{2} = \frac{|\,(Q-P)\times(R-P)\,|}{2} \,. \tag{4.10}$$



$Area(u,v) = |\, u \times v \,|$. $Area(\Delta PQR) = \dfrac{|\,(Q-P)\times(R-P)\,|}{2}$

Figure 10: Area formulas for the parallelogram and the triangle.

4.2.2 Polygons -- Newell's Formula. Consider a closed convex planar polygon P with vertices $P_1,\ldots,P_{n+1}$, where $P_{n+1} = P_1$ (see Figure 11, left). Let Q be any point in the interior of the polygon P, and define

$$N(P) = \frac{1}{2}\sum_{k=1}^{n}(P_k - Q)\times(P_{k+1} - Q) \,. \tag{4.11}$$

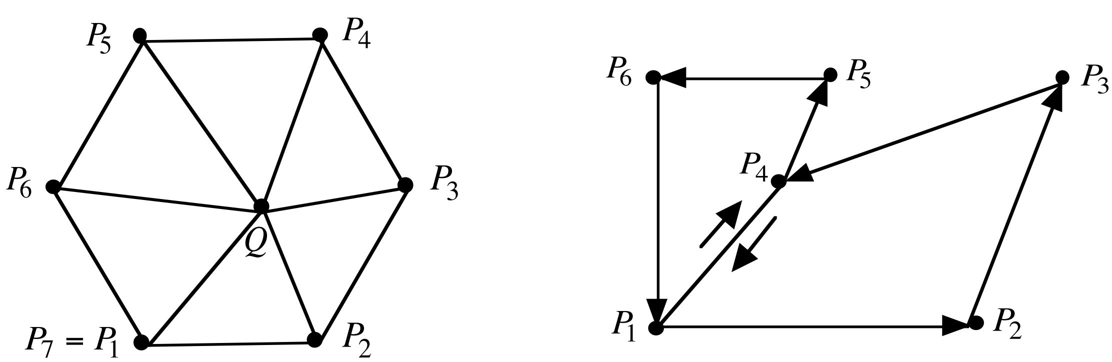


Figure 11: A convex polygon (left) and a non-convex polygon (right). The non-convex polygon on the right can be split into two convex polygons by inserting the edge P_1P_4. Notice that the new edge P_1P_4 has opposite orientations in the polygons $P_1P_2P_3P_4$ and $P_1P_4P_5P_6$.

Since the vectors $(P_k - Q)$, $k = 1,\dots n+1$, all lie in the plane of the polygon *P*, the cross products $(P_k - Q) \times (P_{k+1} - Q)$, $k = 1,\dots n$, all point in the same direction normal to the plane of the polygon *P*. Moreover by Equation (4.10),

$$Area(\Delta P_k P_{k+1} Q) = \frac{|(P_{k+1} - Q) \times (P_k - Q)|}{2}.$$

Therefore

$$N(P) \perp P$$

$$|N(P)| = \frac{1}{2}\left|\sum_{k=1}^{n} (P_k - Q) \times (P_{k+1} - Q)\right| = \frac{1}{2}\sum_{k=1}^{n} |(P_k - Q) \times (P_{k+1} - Q)| = Area(P). \qquad (4.12)$$

The expression on the right hand side of Equation (4.11) for the vector $N(P)$ is independent of the choice of the point *Q* in the polygon *P,* since the direction and length of $N(P)$ in Equation (4.12) are independent of the choice of *Q*. In fact, the point *Q* need need not lie inside the polygon *P* or even in the plane of the polygon *P;* the expression on the right hand side of Equation (4.11) represents the same vector $N(P)$ for any point *R*. Indeed, let Q, R be two arbitrary points. Then

$$\sum_{k=1}^{n} (P_k - R) \times (P_{k+1} - R) = \sum_{k=1}^{n} \big((P_k - Q) + (Q - R)\big) \times \big((P_{k+1} - Q) + (Q - R)\big).$$

Expanding the right hand side by the distributive property and using the identity $(Q - R) \times (Q - R) = 0$ yields

$$\sum_{k=1}^{n} (P_k - R) \times (P_{k+1} - R) = \sum_{k=1}^{n} (P_k - Q) \times (P_{k+1} - Q)$$

$$+ \sum_{k=1}^{n} (P_k - Q) \times (Q - R) + \sum_{k=1}^{n} (Q - R) \times (P_{k+1} - Q).$$

Now observe that

$$\sum_{k=1}^{n} (P_k - Q) \times (Q - R) = (P_1 - Q) \times (Q - R) + (P_2 - Q) \times (Q - R) + \cdots + (P_n - Q) \times (Q - R)$$

$$\sum_{k=1}^{n} (Q - R) \times (P_{k+1} - Q) = (Q - R) \times (P_2 - Q) + \cdots + (Q - R) \times (P_{n+1} - Q).$$

Since the cross product is anticommutative and since $P_{n+1} = P_1$, these sums cancel, and we are left with

$$\sum_{k=1}^{n} (P_k - R) \times (P_{k+1} - R) = \sum_{k=1}^{n} (P_k - Q) \times (P_{k+1} - Q).$$

Choosing *R* to be the origin, the expression for computing the vector $N(P)$ simplifies to Newell's formula

$$N(P) = \frac{1}{2}\sum_{k=1}^{n} P_k \times P_{k+1} \qquad (4.13)$$

- $N(P) \perp P$
- $|N(P)| = Area(P)$

We derived Equation (4.13) for convex polygons, but this formula remains valid even for non-convex polygons. For if P is not convex, we can split P into a collection of convex polygons by inserting some extra edges P_jP_k (see Figure 11, right). Applying Equation (4.13) to each of the convex polygons and summing, we find that the terms involving the new edges P_jP_k appear twice with opposite orientations. Thus the terms $P_j \times P_k$ and $P_k \times P_j$ cancel and we arrive again at Equation (4.13).

4.3 Volume. We are interested here mainly in two formulas: one for the volume of tetrahedra and one for the volume of parallelepipeds. A formula for the volume of arbitrary polyhedra is provided in Exercise 5. Since the determinant is defined in terms of volume, formulas for volume are typically expressed in terms of determinants.

Consider a parallelepiped determined by three vectors u, v, w (see Figure 12, left). By the definition of the determinant function

$$Vol(u,v,w) = |\, Det(u,v,w) \,| \,. \qquad (4.14)$$

Now consider a tetrahedron with vertices P, Q, R, S, and let $u = Q - P$, $v = R - P$, $w = S - P$. A tetrahedron is a triangular pyramid (see Figure 12, right). Since

$$\textit{volume of a pyramid} = \frac{1}{3}\,\textit{area of the base of the pyramid} \times \textit{height of the pyramid},$$

the volume of a tetrahedron with vertices P, Q, R, S is

$$Vol(\Delta PQRS) = \frac{1}{3} base \times height = \frac{1}{3} Area(\Delta PQR) \times height(\Delta PQRS)$$

$$= \frac{1}{6} Area(u,v) \times height(u,v,w) = \frac{1}{6} Vol(u,v,w) = \frac{1}{6}\left|Det(u,v,w)\right| .$$

Therefore

$$Vol(\Delta PQRS) = \frac{1}{6}\left|Det(Q-P, R-P, S-P)\right| . \qquad (4.15)$$

$$Vol(u,v,w) = |\,Det(u,v,w)\,| \qquad\qquad Vol(\Delta PQRS) = \frac{\left|Det\left(Q-P,\,R-P,\,S-P\right)\right|}{6}$$

Figure 12: Volume formulas for the parallelepiped and the tetrahedron.

5. Intersection Formulas for Lines and Planes

Formulas for intersecting lines and planes are important in Computer Graphics both for hidden surface procedures and for ray casting algorithms. We shall study hidden surface procedures and ray casting algorithms later in this course. Here we prepare the way by deriving closed formulas for the intersections of lines and planes.

5.1 Intersecting Two Lines. Consider a pair of intersecting lines, L_1, L_2, where L_1 is parallel to the vector v_1 and L_2 is parallel to the vector v_2. Let P_1 be a point on L_1 and let P_2 be a point on L_2. We seek the intersection point P lying on both L_1 and L_2 (see Figure 13).

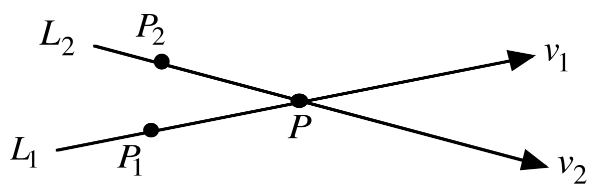


Figure 13: A pair of intersecting lines.

At the intersection point P, there are parameters s, t such that

$$P_1 + sv_1 = P_2 + tv_2, \tag{5.1}$$

or equivalently

$$sv_1 - tv_2 = P_2 - P_1. \tag{5.2}$$

Dotting both sides of Equation (5.2) with both v_1 and v_2 yields two linear equations in the two unknown parameters s, t:

$$\begin{aligned}(v_1 \bullet v_1)s - (v_1 \bullet v_2)t &= v_1 \bullet (P_2 - P_1)\\ (v_1 \bullet v_2)s - (v_2 \bullet v_2)t &= v_2 \bullet (P_2 - P_1)\end{aligned} \tag{5.3}$$

Solving for s, t by Cramer's Rule, we find that at the intersection point

$$s^* = \frac{Det\begin{pmatrix} v_1 \bullet (P_2 - P_1) & -v_1 \bullet v_2 \\ v_2 \bullet (P_2 - P_1) & -v_2 \bullet v_2 \end{pmatrix}}{Det\begin{pmatrix} v_1 \bullet v_1 & -v_1 \bullet v_2 \\ v_1 \bullet v_2 & -v_2 \bullet v_2 \end{pmatrix}} \quad \text{and} \quad t^* = \frac{Det\begin{pmatrix} v_1 \bullet v_1 & v_1 \bullet (P_2 - P_1) \\ v_1 \bullet v_2 & v_2 \bullet (P_2 - P_1) \end{pmatrix}}{Det\begin{pmatrix} v_1 \bullet v_1 & -v_1 \bullet v_2 \\ v_1 \bullet v_2 & -v_2 \bullet v_2 \end{pmatrix}}. \qquad (5.4)$$

Therefore,

$$P = P_1 + s^* v_1 = P_2 + t^* v_2. \qquad (5.5)$$

5.2 Intersecting Three Planes. Consider a plane S determined by a point P on the plane and a vector N normal to the plane. Recall that another point Q lies on the plane S if and only if the vector $Q - P$ is perpendicular to the normal vector N (see Figure 14) -- that is, if and only if

$$N \bullet (Q - P) = 0. \qquad (5.6)$$

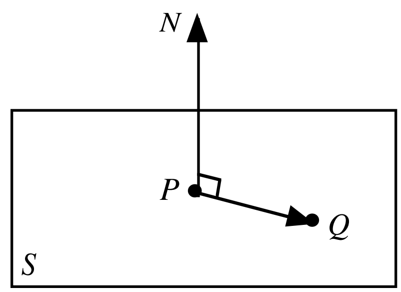


Figure 14: A plane S determined by a point P and a normal vector N. A point Q lies on the plane S if and only if the vector $Q - P$ is perpendicular to the normal vector N.

Now consider three planes S_1, S_2, S_3 intersecting in a common point Q. Let the plane S_k be determined by the point P_k and the normal vector N_k, $k = 1,2,3$. Since the intersection point Q lies on each of the three planes, the point Q must satisfy the equation of each of these planes. Therefore

$$N_k \bullet (Q - P_k) = 0 \qquad k = 1,2,3. \qquad (5.7)$$

Equation (5.7) represents three linear equations in three unknowns, the three coordinates of the point Q. We could solve these equations by linear algebra, using either Cramer's rule or Gaussian elimination. But there is a better way; we can just write down the answer! In fact,

$$Q = R + \frac{\big(N_1 \bullet (P_1 - R)\big) N_2 \times N_3 + \big(N_2 \bullet (P_2 - R)\big) N_3 \times N_1 + \big(N_3 \bullet (P_3 - R)\big) N_1 \times N_2}{Det(N_1\ N_2\ N_3)} \qquad (5.8)$$

where R is any point whatsoever. We can check that this expression for Q is valid by verifying that Q satisfies Equations (5.7). For example, since

$$N_1 \bullet N_1 \times N_2 = N_1 \bullet N_3 \times N_1 = 0 \quad \text{and} \quad N_1 \bullet N_2 \times N_3 = Det(N_1\ N_2\ N_3)$$

it follows that

$$N_1 \bullet (Q - P_1) = N_1 \bullet (R - P_1) + N_1 \bullet (P_1 - R) = 0.$$

Similarly, we can show that

$$N_2 \bullet (Q - P_2) = N_3 \bullet (Q - P_3) = 0 .$$

Therefore Q lies on all three planes. Moreover since R can be any point whatsoever, we can choose R to be the origin. In this case, the expression for computing the intersection point Q simplifies to

$$Q = \frac{(N_1 \bullet P_1) N_2 \times N_3 + (N_2 \bullet P_2) N_3 \times N_1 + (N_3 \bullet P_3) N_1 \times N_2}{Det(N_1, N_2, N_3)} \tag{5.9}$$

An alternative derivation of Equation (5.9) based on solving the linear system in Equation (5.7) by inverting the coefficient matrix is provided in Exercise 13.

5.3 Intersecting Two Planes. Three planes intersect in a point; two planes intersect in a line. Let the plane S_k be determined by the point P_k and the normal vector N_k, $k = 1, 2$. To find the line L in which the planes S_1 and S_2 intersect, we need to find a point Q on L and a direction vector v parallel to L.

The direction vector v is easy. Since the line L lies in both planes, the direction vector v must be perpendicular to the normal to both planes. Therefore we can choose

$$v = N_1 \times N_2 . \tag{5.10}$$

To find a point Q on the line L, we need to solve two linear equations in three unknowns:

$$\begin{aligned} &N_1 \bullet (Q - P_1) = 0 \\ &N_2 \bullet (Q - P_2) = 0 . \end{aligned} \tag{5.11}$$

There are infinitely many solutions to these two equations, since there are infinitely many points Q on the line L. To find a unique solution Q and to make the problem deterministic, we can add the equation of any plane that intersects the line L. Since the vector $N_1 \times N_2$ is parallel to the direction of the line L, any plane with normal vector $N_1 \times N_2$ will certainly intersect the line L. Therefore we can choose any point P_3, and find a unique point Q on the line L by solving the three linear equations in three unknowns

$$\begin{aligned} &N_1 \bullet (Q - P_1) = 0 \\ &N_2 \bullet (Q - P_2) = 0 \\ &(N_1 \times N_2) \bullet (Q - P_3) = 0 \end{aligned} \tag{5.12}$$

for the coordinates of the point Q.

We already know how to solve for the intersection of three planes, since we solved this problem in the previous section. Setting $N_3 = N_1 \times N_2$ in Equation (5.8) and invoking the identities

$$\begin{aligned} &N_2 \times (N_1 \times N_2) = (N_2 \bullet N_2) N_1 - (N_1 \bullet N_2) N_2 \\ &(N_1 \times N_2) \times N_2 = (N_1 \bullet N_2) N_2 - (N_2 \bullet N_2) N_1 \\ &Det(N_1, N_2, N_1 \times N_2) = |N_1 \times N_2|^2 \end{aligned}$$

yields

$$Q = R + \frac{\big(N_1 \bullet (P_1 - R)\big)\big((N_2 \bullet N_2)N_1 - (N_1 \bullet N_2)N_2\big)}{|N_1 \times N_2|^2}$$
$$+ \frac{\big(N_2 \bullet (P_2 - R)\big)\big((N_1 \bullet N_2)N_2 - (N_2 \bullet N_2)N_1\big) + \big(N_1 \times N_2 \bullet (P_3 - R)\big)N_1 \times N_2}{|N_1 \times N_2|^2} \tag{5.13}$$

where R and P_3 are any points whatsoever. Since R and P_3 can be any points whatsoever, we can choose both R and P_3 to be the origin. In this case, the expression for the point Q on the intersection line L simplifies to

$$Q = \frac{(N_1 \bullet P_1)\big((N_2 \bullet N_2)N_1 - (N_1 \bullet N_2)N_2\big) + (N_2 \bullet P_2)\big((N_1 \bullet N_1)N_2 - (N_1 \bullet N_2)N_1\big)}{|N_1 \times N_2|^2}. \tag{5.14}$$

Finally notice that if N_1 and N_2 are unit vectors, then

$$N_1 \bullet N_1 = N_2 \bullet N_2 = 1$$
$$|N_1 \times N_2|^2 = 1 - (N_1 \bullet N_2)^2,$$

so in this case

$$Q = \frac{(N_1 \bullet P_1)\big(N_1 - (N_1 \bullet N_2)N_2\big) + (N_2 \bullet P_2)\big(N_2 - (N_1 \bullet N_2)N_1\big)}{1 - (N_1 \bullet N_2)^2}. \tag{5.15}$$

6. Spherical Linear Interpolation (SLERP)

Interpolation plays a key role in many algorithms in Computer Graphics. The simplest kind of interpolation is *linear interpolation*. Let P_0, P_1 are two points in affine space. Then $v = P_1 - P_0$ is the vector from P_0 to P_1. Therefore the equation of the straight line L joining the points P_0, P_1 is

$$L(t) = P_0 + t\,v = P_0 + t(P_1 - P_0)$$

or equivalently

$$L(t) = (1-t)P_0 + t\,P_1. \tag{6.1}$$

Notice that

$$|L(t) - P_0| = t\,|P_1 - P_0|.$$

Thus, if d is the distance from P_0 to P_1, then $t\,d$ is the distance from P_0 to $L(t)$. Equation (6.1) is called *linear interpolation* because the line $L(t)$ interpolates the points P_0 and P_1, and distance varies linearly along the line $L(t)$.

Suppose, however, that instead of starting with two points along a straight line, we start with two vectors along a circular arc (see Figure 15) and we want to interpolate these vectors uniformly by vectors along this arc. That is, given two vectors v_0, v_1 of the same length, we seek an equation for the vectors $v(t)$ along the arc joining the vectors v_0, v_1 so that if ϕ is the angle from v_0 to v_1, then $t\phi$ is the angle from v_0 to $v(t)$ The following theorem gives a formula for the vectors $v(t)$

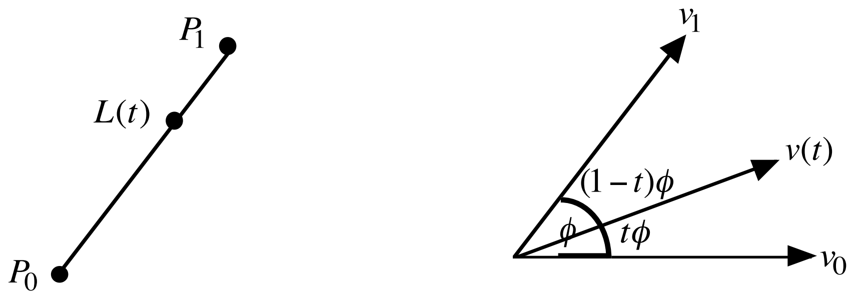


Figure 15: Linear interpolation between two points along a straight line (left), and spherical linear interpolation between two vectors along a circular arc (right).

Theorem: $v(t) = \dfrac{\sin((1-t)\phi)}{\sin(\phi)} v_0 + \dfrac{\sin(t\phi)}{\sin(\phi)} v_1$, *where ϕ is the angle between v_0 and v_1.*

Proof: The vector $v(t)$ lies in the plane of the vectors v_0, v_1. Therefore there are constants α, β such that

$$v(t) = \alpha v_0 + \beta v_1.$$

Whenever you see sine, think cross product. Crossing both sides of this equation with v_0 and v_1 yields:

$$v_0 \times v(t) = \beta\, v_0 \times v_1$$

$$v(t) \times v_1 = \alpha\, v_0 \times v_1.$$

Taking the lengths of both sides of these two equations, we get

$$|v_0||v(t)| \sin(t\phi) = \beta |v_0||v_1| \sin(\phi)$$

$$|v_1||v(t)| \sin((1-t)\phi) = \alpha |v_0||v_1| \sin(\phi).$$

But $|v_0| = |v_1| = |v(t)|$, so solving these two equations for α, β, we find that

$$\alpha = \frac{\sin((1-t)\phi)}{\sin(\phi)},$$

$$\beta = \frac{\sin(t\phi)}{\sin(\phi)}.$$

Hence

$$v(t) = \frac{\sin((1-t)\phi)}{\sin(\phi)} v_0 + \frac{\sin(t\phi)}{\sin(\phi)} v_1.$$

Notice, in particular, that $v(0) = v_0$ and $v(1) = v_1$.

The formula for $v(t)$ in the preceding theorem is called *spherical linear interpolation* or *SLERP* and we write

$$slerp(v_0,v_1,t) = \frac{\sin\big((1-t)\phi\big)}{\sin(\phi)}v_0 + \frac{\sin\big(t\phi\big)}{\sin(\phi)}v_1, \qquad (6.2)$$

where ϕ is the angle between v_0 and v_1, because $slerp(v_0,v_1,t)$ interpolates the vectors v_0 and v_1, and the angle varies linearly along the arc joining v_0 and $slerp(v_0,v_1,t)$. The term *spherical* refers to the fact that since the vectors v_0 and v_1 have the same length, we can think of these vectors as representing points on a sphere. Equation (6.2) then interpolates two points on a sphere by moving uniformly along a great circle, a geodesic, on the sphere.

Notice that spherical linear interpolation is valid for vectors in any dimension greater than or equal to two, even though our proof using cross product assumed that v_0 and v_1 lie in 3-dimensions. In fact, since v_0 and v_1 lie in a plane, we can always embed these vectors in a 3-dimensional subspace and apply the argument in the preceding theorem. Since the result is coordinate free, spherical linear interpolation is valid in any dimension (see also Exercise 17). We shall have occasion to use spherical linear interpolation in dimensions three and four, when we study shading algorithms and refraction techniques in 3-dimensions and key frame animation with quaternions in 4-dimensions.

7. Inside-Outside Tests

For many algorithms in Computer Graphics such as clipping and shading, it is important to know if a given point lies inside or outside of a fixed polygon. There are two standard ways to solve this problem: ray casting and winding number.

7.1 Ray Casting. In ray casting, we fire a ray from the given point P to the fixed polygon S. If the point P lies outside the polygon S, the number of intersections between the ray and the polygon is even, since the ray must alternate entering and exiting the polygon; if the point P lies inside the polygon S the number of intersections between the ray and the polygon is odd, since the line must alternate exiting and entering the polygon before exiting the polygon one final time (see Figure 16).

Ray casting works fine in most cases, but ray casting can get into trouble if the ray intersects the polygon at or near one of the vertices of the polygon. Should a vertex count as one or two intersections? If we count a vertex as one intersection, we may get the wrong result for points outside the polygon; if we count a vertex as two intersections, we may get the wrong result for points inside the polygon. There is no consistent way to handle vertices that always gives the correct answer. Fortunately, if we fire rays in random directions, we are not likely to hit a vertex, so usually ray casting works well, but there is another approach that avoids this problem altogether.

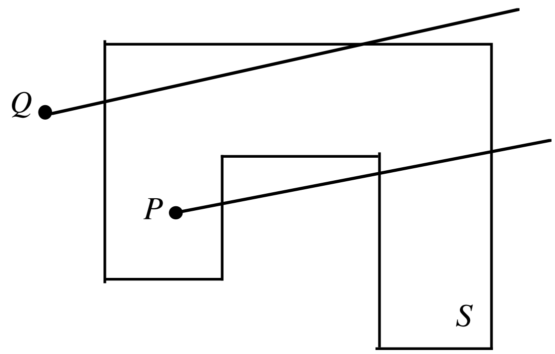


Figure 16: Ray casting. A line emanating from a point P inside the polygon S will intersect the polygon in an odd number of points; a line emanating from a point Q outside the polygon S will intersect the polygon is an even number of points.

7.2 Winding Number. The *winding number* relative to a point P of an oriented closed planar curve C with no self intersection points is the number of times the curve C circles around the point P. Let $Wind(C,P)$ denote the winding number of C relative to P. Then

$$Wind(C,P) = 0 \Leftrightarrow P \text{ lies outside of } C$$

$$Wind(C,P) = \pm 1 \Leftrightarrow P \text{ lies inside of } C.$$

When the point P lies inside the curve C, the orientation of the curve determines the sign of the winding number; counterclockwise is positive orientation, clockwise is negative orientation.

For polygons S there is a closed formula for computing the winding number for any point P. Let $P_1,\ldots,P_n$ be the vertices of the polygon S, and let $P_{n+1} = P_1$. Then

$$Wind(S,P) = \frac{1}{2\pi}\sum_{k=1}^{n} ArcSin\left(\frac{\left|(P_{k+1}-P)\times(P_k-P)\right|}{|P_{k+1}-P|\;|P_k-P|}\right). \qquad (7.1)$$

This formula is valid for any polygon; the polygon need not be regular or even convex.

To understand why this formula works, think about a turtle located at the point P facing initially in the direction of the vertex P_1. To measure how much the polygon winds around P, the turtle measures the angles $A_k = \angle P_k P P_{k+1}$ for $k = 1,\ldots,n$. For each angle A_k, the turtle executes the command TURN A_k. After executing TURN A_1, the turtle is facing the vertex P_2; after executing TURN A_k, the turtle is facing the vertex P_{k+1}. Thus the turtle starts and ends facing the vertex P_1, so the total amount the turtle turns must be an integer multiple of 2π. Some angles may be positive (counterclockwise) and some negative (clockwise), but the total turning must always be an integer multiple of 2π.

To calculate the angles A_k for $k = 1,\ldots,n$, we can use the cross product. The angle $A_k = \angle P_k P P_{k+1}$ is the angle between the vectors $P_k - P$ and $P_{k+1} - P$ (see Figure 17), and from the definition of the cross product

$$Sin\left(\angle P_k P P_{k+1}\right) = \frac{\left|(P_{k+1} - P) \times (P_k - P)\right|}{|P_{k+1} - P|\ |P_k - P|}.$$

Therefore,

$$A_k = \angle P_k P P_{k+1} = ArcSin\left(\frac{\left|(P_{k+1} - P) \times (P_k - P)\right|}{|P_{k+1} - P|\ |P_k - P|}\right)$$

so

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} ArcSin\left(\frac{\left|(P_{k+1} - P) \times (P_k - P)\right|}{|P_{k+1} - P|\ |P_k - P|}\right).$$

Since the total turtle turning must be an integer multiple of 2π, the winding number is given by

$$Wind(S,P) = \frac{1}{2\pi}\sum_{k=1}^{n} A_k = \frac{1}{2\pi}\sum_{k=1}^{n} ArcSin\left(\frac{\left|(P_{k+1} - P) \times (P_k - P)\right|}{|P_{k+1} - P|\ |P_k - P|}\right).$$

When the point *P* lies outside of the polygon *S,* the winding number $Wind(S,P) = 0$. We can verify this result in the following fashion. If the point *P* is far away from the polygon *S*, then the turtle will never face in a direction 180° away from the centroid of the polygon (see Figure 17). Therefore the turtle never turns through more than a total of π radians, so since the total turtle turning is an integer multiple of 2π, the total turtle turning must be zero. Now as long as we do not cross an edge of the polygon, the winding number is a continuous function, so if we move the turtle a little bit, the winding number can only change by a little bit. But the winding number is an integer, so if the winding number can only change by a little bit, then the winding number cannot change at all. Therefore as we move the point *P* outside the polygon *S*, the winding number never changes; hence if *P* lies outside of *S*, then $Wind(S,P) = 0$.

Conversely, when the point *P* lies inside of the polygon *S,* the winding number $Wind(S,P) = \pm 1$. We can verify this result in the following fashion. If *S* is a regular polygon and the point *P* is at the center, then the sum of the central angles is $\pm 2\pi$, so the winding number is ± 1 (see Figure 17). Again as long as we do not cross an edge of the polygon, the winding number is a continuous function, so if we move the turtle a little bit, the winding number can only change by a little bit. Hence once again, since the winding number is an integer, the winding number cannot change at all inside the polygon. Thus if the point *P* lies inside a regular polygon *S*, then $Wind(S,P) = \pm 1$. Moreover, this result remains valid even if *S* is not a regular polygon because we can deform a regular polygon into an arbitrary polygon. Since the winding number is a continuous function, the winding number is unchanged under small deformations of the polygon; therefore no matter how we deform that polygon, if the point *P* lies inside the polygon *S*, then $Wind(S,P) = \pm 1$.

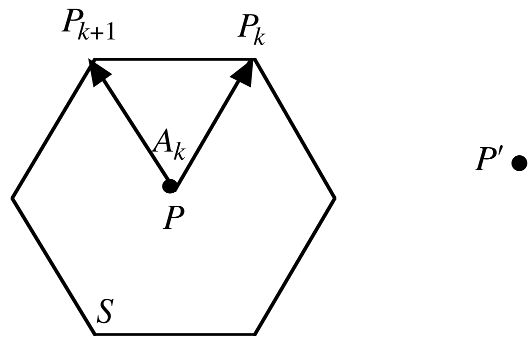


Figure 17: Winding number. If S is a regular polygon and the point P is at the center, then the sum of the central angles is $\pm 2\pi$, so the winding number is ± 1. If the point P' is far away from the polygon S, then the turtle will never face in a direction 180° away from the centroid of the polygon, so the winding number is 0.

In addition to correctly distinguishing the difference between inside and outside, the winding number has one additional advantage: Equation (6.1) for the winding number is numerically stable. If $\angle P_k P P_{k+1}$ is small, then

$$\angle P_k P P_{k+1} \approx Sin(\angle P_k P P_{k+1}) = \frac{|(P_{k+1} - P) \times (P_k - P)|}{|P_{k+1} - P| \, |P_k - P|},$$

so we do not lose precision for small angles when we calculate the cross product.

8. Summary

Vector algebra is a potent tool for analyzing geometry. Trigonometric laws, metric expressions, intersection formulas, and inside-outside tests have all been derived here using vector techniques. Vector techniques are cleaner and more powerful than coordinate methods. From now on, whenever you encounter a geometric problem, you should eschew coordinate methods in favor of vector techniques.

We have encountered two helpful analytic rules that you should remember forever:

i. *Whenever you see cosine, think dot product.*

ii. *Whenever you see sine, think cross product.*

Similarly, there are three useful geometric heuristics that you should always keep in mind:

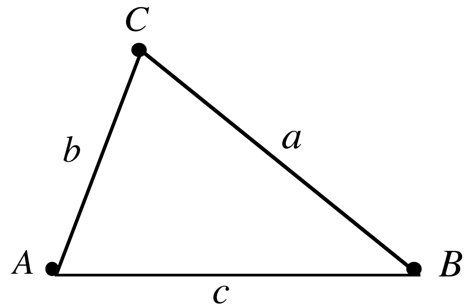
iii. *Whenever you see distance, think dot product.*

iv. *Whenever you see area, think cross product.*

v. *Whenever you see volume, think determinant.*

Below, for easy reference, we list all the key formulas that we have derived in this chapter.

8.1 Trigonometric Laws



Law of Cosines: $c^2 = a^2 + b^2 - 2ab\,Cos(C)$

Law of Sines: $\dfrac{a}{Sin(A)} = \dfrac{b}{Sin(B)} = \dfrac{c}{Sin(C)}$

8.2 Metric Formulas

8.2.1 Distance

Distance Between Two Points

$$Dist^2(P,Q) = |Q-P|^2 = (Q-P)\bullet(Q-P)$$

Distance Between a Point and a Line

$$Dist^2(P,L) = Dist^2(P,Q+tv) = (Q-P)\bullet(Q-P) - \frac{\big((Q-P)\bullet v\big)^2}{v\bullet v}$$

Distance Between a Point and a Plane

$$Dist(Q,S) = Dist(Q,P+su+tv) = \frac{Vol(u,v,Q-P)}{Area(u,v)} = \frac{|Det(u,v,Q-P)|}{|u\times v|}$$

$$Dist(Q,S) = \frac{|(Q-P)\bullet N|}{|N|} \qquad N \parallel u\times v$$

Distance Between Parallel Lines

$$Dist^2(L_1,L_2) = Dist^2(P_1+sv,P_2+tv) = (P_2-P_1)\bullet(P_2-P_1) - \frac{\big((P_2-P_1)\bullet v\big)^2}{v\bullet v}$$

Distance Between Skew Lines

$$Dist(L_1,L_2) = Dist^2(P_1+sv_1,P_2+tv_2) = \frac{\big|Det\big((P_2-P_1),v_1,v_2\big)\big|}{|v_1\times v_2|}$$

8.2.2 Area

Area of a Parallelogram

$$Area(u,v) = |u \times v|$$

Area of a Triangle

$$Area(\Delta PQR) = \frac{|(Q-P) \times (R-P)|}{2}$$

Area of a Planar Polygon

Vertices $P = \{P_1, \ldots, P_n\}$

$$Area(P) = \frac{1}{2}\left|\sum_{k=1}^{n} P_k \times P_{k+1}\right|$$

8.2.3 Volume

Volume of a Parallelepiped

$$Vol(u,v,w) = |Det(u,v,w)|$$

Volume of a Tetrahedron

$$Vol(\Delta PQRS) = \frac{\left|Det\left(Q-P, R-P, S-P\right)\right|}{6}$$

8.3 Intersections

Two Lines: $L_1(s) = P_1 + sv_1 \quad L_2(t) = P_2 + tv_2$

$$s^* = \frac{Det\begin{pmatrix} v_1 \bullet (P_2 - P_1) & -v_1 \bullet v_2 \\ v_2 \bullet (P_2 - P_1) & -v_2 \bullet v_2 \end{pmatrix}}{Det\begin{pmatrix} v_1 \bullet v_1 & -v_1 \bullet v_2 \\ v_1 \bullet v_2 & -v_2 \bullet v_2 \end{pmatrix}} \quad \text{and} \quad t^* = \frac{Det\begin{pmatrix} v_1 \bullet v_1 & v_1 \bullet (P_2 - P_1) \\ v_1 \bullet v_2 & v_2 \bullet (P_2 - P_1) \end{pmatrix}}{Det\begin{pmatrix} v_1 \bullet v_1 & -v_1 \bullet v_2 \\ v_1 \bullet v_2 & -v_2 \bullet v_2 \end{pmatrix}}.$$

$$P = P_1 + s^* v_1 = P_2 + t^* v_2.$$

Three Planes: $S_1 : N_1(Q - P_1) = 0, \quad S_2 : N_2(Q - P_2) = 0, \quad S_3 : N_3(Q - P_3) = 0$

$$Q = \frac{\left(N_1 \bullet P_1\right) N_2 \times N_3 + \left(N_2 \bullet P_2\right) N_3 \times N_1 + \left(N_3 \bullet P_3\right) N_1 \times N_2}{Det(N_1\ N_2\ N_3)}$$

Two Planes: $S_1: N_1(Q-P_1)=0$, $S_2: N_2(Q-P_2)=0$

$$v = N_1 \times N_2$$

$$Q = \frac{(N_1 \bullet P_1)\big((N_2 \bullet N_2)N_1 - (N_1 \bullet N_2)N_2\big) + (N_2 \bullet P_2)\big((N_1 \bullet N_1)N_2 - (N_1 \bullet N_2)N_1\big)}{|N_1 \times N_2|^2}$$

$$L(t) = Q + t\,v$$

8.4 Interpolation

Linear Interpolation

$$L(t) = (1-t)\,P_0 + t\,P_1$$

Spherical Linear Interpolation (SLERP)

$$slerp(v_0, v_1, t) = \frac{\sin\big((1-t)\phi\big)}{\sin(\phi)} v_0 + \frac{\sin(t\phi)}{\sin(\phi)} v_1, \text{ where } \phi \text{ is the angle between } v_0 \text{ and } v_1.$$

8.5 Winding Number

Vertices $S = \{P_1, \ldots, P_n\}$

$$Wind(S,P) = \frac{1}{2\pi} \sum_{k=1}^{n} ArcSin\left(\frac{|(P_{k+1}-P) \times (P_k - P)|}{|P_{k+1}-P|\,|P_k - P|} \right)$$

Exercises:

1. Consider a point Q and a line L determined by a point P and a direction vector v.
   a. Show that
   $$Dist^2(Q,L) = \frac{((Q-P)\times v)^2}{v \bullet v}.$$
   b. Why is Equation (4.2) preferable to the formula in part *a*?

2. Consider a circle inscribed in a triangle (see Figure 18). Let C_{in} denote the center of this circle and let r_{in} denote the radius of this circle. Verify that:
   a. $$C_{in} = \frac{|P_3 - P_2|P_1 + |P_1 - P_3|P_2 + |P_2 - P_1|P_3}{|P_3 - P_2| + |P_1 - P_3| + |P_2 - P_1|}$$
   b. $$r_{in} = \frac{2\,Area(\Delta P_1 P_2 P_3)}{Perimeter(\Delta P_1 P_2 P_3)}$$

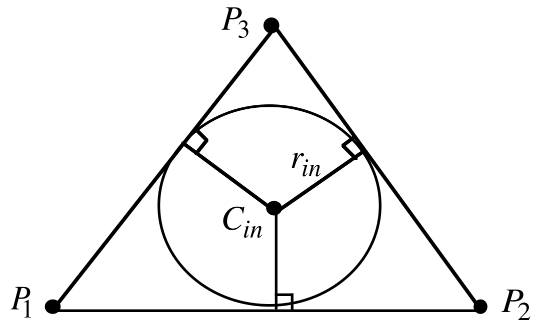


Figure 18: A circle inscribed in a triangle.

3. Show that:

   a. $Area(\Delta PQR) = \frac{1}{2}\left| Det\begin{pmatrix} P & 1 \\ Q & 1 \\ R & 1 \end{pmatrix} \right| = \frac{1}{2}\left| Det\begin{pmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{pmatrix} \right|.$

   b. $Vol(\Delta PQRS) = \frac{1}{6}\left| Det\begin{pmatrix} P & 1 \\ Q & 1 \\ R & 1 \\ S & 1 \end{pmatrix} \right| = \frac{1}{6}\left| Det\begin{pmatrix} p_1 & p_2 & p_3 & 1 \\ q_1 & q_2 & q_3 & 1 \\ r_1 & r_2 & r_3 & 1 \\ s_1 & s_2 & s_3 & 1 \end{pmatrix} \right|.$

4. Consider a pyramid V with apex A and polygonal base P with vertices $P_1,\ldots,P_m$, and let $P_{m+1} = P_1$ Show that

$$Vol(V) = \frac{1}{6}\left| \sum_{j=1}^{m} Det(A - Q, P_j, P_{j+1}) \right|,$$

where Q is any point in the plane of the polygon P.

5. Consider a convex polyhedron V with polygonal faces $S_1,\ldots,S_n$ oriented consistently either clockwise or counterclockwise with respect to the outward pointing normals to the faces. With respect to this orientation, let $P_{k,1},\ldots,P_{k,m(k)}$ be the vertices of S_k, and let $P_{k,m(k)+1} = P_{k,1}$, $k = 1,\ldots,n$. Finally, let A be a point in the interior of V, and let Q_k be any point in the plane of the polygon S_k, $k = 1,\ldots,n$.

   a. Using Exercise 4, show that

$$Vol(V) = \frac{1}{6}\left| \sum_{k=1}^{n} \sum_{j=1}^{m(k)} Det(A - Q_k, P_{k,j}, P_{k,j+1}) \right|.$$

   b. Show that the formula in part *a* is valid for any point A, even for points outside the polyhedron V. That is, show that for any two points A, B

$$\frac{1}{6}\left|\sum_{k=1}^{n}\sum_{j=1}^{m(k)} Det(A-Q_k, P_{k,j}, P_{k,j+1})\right| = Vol(V) = \frac{1}{6}\left|\sum_{k=1}^{n}\sum_{j=1}^{m(k)} Det(B-Q_k, P_{k,j}, P_{k,j+1})\right|.$$

c. Choosing A to be the origin, conclude from parts *a,b* that the expression for computing the volume of a polyhedron V simplifies to

$$Vol(V) = \frac{1}{6}\left|\sum_{k=1}^{n}\sum_{j=1}^{m(k)} Det(Q_k, P_{k,j}, P_{k,j+1})\right|.$$

d. Show that the formula in part *c* is valid even for non-convex polyhedra.

6. Let Q be the point defined in Equation (5.13). Verify that

$$N_1 \bullet (Q - P_1) = 0$$
$$N_2 \bullet (Q - P_2) = 0.$$

7. What does it mean if $Det(N_1\ N_2\ N_3) = 0$ in Equation (5.8) or Equation (5.9)?

8. What does it mean if $|N_1 \times N_2|^2 = 0$ in Equation (5.13) or Equation (5.14)?

9. Consider a line L through a point P in the direction v, and a plane S through a point Q with normal vector N. Show that the line L and the plane S intersect in the point R, where

$$R = P + \frac{N \bullet (Q-P)}{N \bullet v} v.$$

10. Consider a pair of intersecting lines, L_1, L_2, where L_1 is parallel to the vector v_1 and L_2 is parallel to the vector v_2. Let P_1 be a point on L_1 and let P_2 be a point on L_2.

a. Show that the point P on the intersection of the two line L_1 and L_2 is the same as the point on the intersection of the three planes defined by the equations

$$(v_1 \times v_2) \bullet (P - P_1) = 0$$
$$\big((v_1 \times v_2) \times v_1\big) \bullet (P - P_1) = 0$$
$$\big((v_1 \times v_2) \times v_2\big) \bullet (P - P_2) = 0$$

b. Using part *a* and Equation (5.8), find a closed form expression for the intersection point P.

c. Show that the expression you derived in part *b* is equivalent to the result in Equation (5.5).

11. Consider a pair of skew lines, L_1, L_2, where L_1 is parallel to the vector v_1 and L_2 is parallel to the vector v_2. Let P_1 be a point on L_1, and let P_2 be a point on L_2. Show that Equations (5.4) and (5.5) provide the points of closest approach of the two lines.

12. Verify that:

a. $$M = \begin{pmatrix} u^T & v^T & w^T \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \Rightarrow$$

$$M^{-1} = \frac{\begin{pmatrix} v \times w \\ w \times u \\ u \times v \end{pmatrix}}{Det(u\ v\ w)} = \frac{\begin{pmatrix} v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \\ w_2u_3 - w_3u_2 & w_3u_1 - w_1u_3 & w_1u_2 - w_2u_1 \\ u_2v_3 - u_3v_2 & u_3v_1 - u_1v_3 & u_1v_2 - u_2v_1 \end{pmatrix}}{Det(u\ v\ w)}$$

b. $$M = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \Rightarrow$$

$$M^{-1} = \frac{\begin{pmatrix} (v \times w)^T & (w \times u)^T & (u \times v)^T \end{pmatrix}}{Det(u\ v\ w)} = \frac{\begin{pmatrix} v_2w_3 - v_3w_2 & w_2u_3 - w_3u_2 & u_2v_3 - u_3v_2 \\ v_3w_1 - v_1w_3 & w_3u_1 - w_1u_3 & u_3v_1 - u_1v_3 \\ v_1w_2 - v_2w_1 & w_1u_2 - w_2u_1 & u_1v_2 - u_2v_1 \end{pmatrix}}{Det(u\ v\ w)}$$

13. Let S_k be the plane determined by the point P_k and the normal vector N_k, $k = 1,2,3$, and let Q be the point of intersection of the three planes S_1, S_2, S_3.

a. Show that

$$\begin{matrix} N_1 \bullet (Q - P_1) = 0 \\ N_2 \bullet (Q - P_2) = 0 \\ N_3 \bullet (Q - P_3) = 0 \end{matrix} \quad \Leftrightarrow \quad Q * \begin{pmatrix} N_1^T & N_2^T & N_3^T \end{pmatrix} = \begin{pmatrix} N_1 \bullet P_1 & N_2 \bullet P_2 & N_3 \bullet P_3 \end{pmatrix}.$$

b. Using part *a* and Exercise 12*a*, show that

$$Q = \frac{(N_1 \bullet P_1) N_2 \times N_3 + (N_2 \bullet P_2) N_3 \times N_1 + (N_3 \bullet P_3) N_1 \times N_2}{Det(N_1\ \ N_2\ \ N_3)}.$$

14. What is the value of the winding number

$$Wind(S,P) = \frac{1}{2\pi} \sum_{k=1}^{n} ArcSin\left(\frac{|(P_{k+1} - P) \times (P_k - P)|}{|P_{k+1} - P|\ |P_k - P|} \right)$$

when

a. P is a point on an edge of the polygon S?

b. P is a vertex of the polygon S?

15. Prove in two ways that nonsingular affine transformations preserve ratios of volumes:
    a. by using the fact that nonsingular transformations preserve ratios of distances along lines.
    b. by invoking Exercise 3b.

16. Let v_0, v_1 be unit vectors. Show that the unit vector that bisects the angle between v_0 and v_1 is

$$v = \sec(\phi/2)\left(\frac{v_0 + v_1}{2}\right),$$

where ϕ is the angle between v_0 and v_1.

17. In this exercise we provide an alternative derivation of the formula for $slerp(v_0, v_1, t)$ using only the dot product.
    a. Derive a formula for $slerp(v_0, v_1, t)$ by dotting both sides of the equation

    $$v(t) = \alpha v_0 + \beta v_1$$

    with v_0, v_1 and solving the resulting linear equations for α, β.
    b. Using the trigonometric identity

    $$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi),$$

    show that the formula for $slerp(v_0, v_1, t)$ derived in part *a* is equivalent to the formula for $slerp(v_0, v_1, t)$ provided in Equation (6.2).
    c. Conclude that Equation (6.2) for $slerp(v_0, v_1, t)$ is independent of the dimension of the vectors v_0, v_1.