Fractals from Iterated Functions Systems

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Part I: Fractals as Fixed Points of IFS
Compact Sets

Compact Set

- Bounded -- lies within a circle of finite radius
- Closed -- contains its boundary

Examples

- Polygons
- Circles
**Haussdorf Metric**

*Definition -- Distance Between Compact Sets*

- \( \text{Dist}(x, S) = \text{Min}_{y \in S} \{ \text{Dist}(x, y) \} \)
- \( \text{Dist}(R, S) = \text{Max}_{x \in R} \{ \text{Dist}(x, S) \} \)
  - -- \( \text{Dist}(R, S) \neq \text{Dist}(S, R) \)
  - -- Examples -- Uneven Parallel Lines, Subsets
- \( H(R, S) = \text{Max}\{ \text{Dist}(R, S), \text{Dist}(S, R) \} \)

*Properties of Haussdorf Metric (Barnsley -- Page 33)*

- \( H(R, S) \geq 0 \)
- \( H(R, S) = 0 \iff R = S \)
- \( H(R, S) = H(S, R) \)
- \( H(R, T) \leq H(R, S) + H(S, T) \)
Dist(R,S) ≠ Dist(S,R)

Parallel Lines

\[
\begin{align*}
D_{\text{dist}}(L_1, L_2) &= d \\
D_{\text{dist}}(L_2, L_1) &= D_{\text{dist}}(A, B) > d \\
\end{align*}
\]

Subsets

\[
\begin{align*}
D_{\text{dist}}(R, S) &= 0 \\
D_{\text{dist}}(S, R) &> 0 \\
\end{align*}
\]
Cauchy Sequences and Convergence

Cauchy Sequences of Sets

- \( H(S_{n+m}, S_n) < \varepsilon \) for all \( n > N \) and all \( m > 0 \)

Completeness Theorem  (Barnsley -- Pages 35–37)

- Every Cauchy Sequence (of compact sets) Converges.
- \( x \in S = \lim_{n \to \infty} S_n \iff x = \lim_{n \to \infty} x_n \) \( x_n \in S_n \)
Iterated Function System

Maps on Sets

- \( w(S) = \{w(x) \mid x \in S\} \)

Iterated Function System (IFS)

- \( W = \{w_1, \ldots, w_l\} \) -- Collection of Contractive Maps
- \( W(S) = w_1(S) \cup \cdots \cup w_l(S) \)

Theorem (Barnsley -- Pages 79–81)

- \( \{w_1, \ldots, w_l\} \) Contractive \( \Rightarrow \) \( W \) Contractive
Setup

Space
- Objects = Compact Subsets of the Plane
- Distance = Haussdorf Metric
- Completeness -- Every Cauchy Sequence Converges

Maps
- Iterated Function Systems (IFS)

Fractals
- Fixed Points of IFS (Sets)
  \[ x \in S = \lim_{n \to \infty} S_n \iff x = \lim_{n \to \infty} x_n \quad x_n \in S_n \]
### Spaces and Functions

**Euclidian Space**
- Points
- Standard Distance
- Cauchy Sequences of Points Converge
- Contractive Maps
- Fixed Points
- Trivial Fixed Point Theorem
- Fixed Point Algorithm
  - Start with Any Point and Iterate

**Fractal Space**
- Compact Sets
- Haussdorf Metric
- Cauchy Sequences of Sets Converge
- Iterated Function Systems
- Fractal Sets
- Trivial Fixed Point Theorem
- Fractal Algorithm
  - Start with Any Set and Iterate
Part II: Algorithms and Examples
Fractal Algorithms

Fractal Theorem (Trivial Fixed Point Theorem)
• \(W = \{w_1, \ldots, w_l\}\) contractive IFS \(\Rightarrow W\) has a unique fixed point \(A\)
• \(A = \text{Lim}_{n \to \infty} \left(\underbrace{W \circ \cdots \circ W}_n\right)(B)\) for any set \(B\)

Fractal Algorithm (Deterministic Algorithm)
• \(A_0 = B\) (pick any set \(B\))
• \(A_{n+1} = W(A_n) = w_1(A_n) \cup \cdots \cup w_l(A_n)\)
• \(A_n\) converges to the fractal (fixed point) \(A\)

Fractal Tennis (Random Algorithm)
• \(P_0 =\) Any Point in \(A\)
  -- Pick any point \(Q\) and iterate \(w_i^\prime\)'s on \(Q\) to get a point \(P_0 \in A\).
• \(P_{n+1} \in \{w_1(P_n), \ldots, w_l(P_n)\}\) -- Assign Probabilities to \(w_i^\prime\)'s
• \(A = \{P_0, P_1, \ldots\}\)
How to Generate Fractals

Fractal Strategy

• Given a Fractal $A$
  • Find an IFS $W = \{w_1, \ldots, w_l\}$ that Maps the Fractal $A$ Onto Itself
    -- $W(A) = A$.
  • Then $A$ is the fractal generated by the iterated function system $W$ starting from any compact set $S$.

Examples

• Koch Snowflake
• C-Curve
• Sierpinski Triangle
• Barnsley Figures
• Bezier Curves  {Later}
Condensation Sets

Setup

- $B = \text{Base Case}$
- $C = \text{Condensation Set}$
- $W = \{w_1, \ldots, w_l\} = \text{Contractive Maps}$

Fractal Algorithm with Condensation Set

- $A_0 = B$ (pick any set $B$)
- $A_{n+1} = W(A_n) = w_1(A_n) \cup \cdots \cup w_l(A_n) \cup C$
- $A_n$ converges to the fractal (fixed point) $A$

Examples

- Fractal Staircase
- Fractal Trees
Summary

Fractals

Recursion made visible.

Fixed point of an iterated function system.