Understanding Quaternions

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The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science. -- Clerk Maxwell, 1869.

Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell. – Lord Kelvin, 1892

<u>Quaternion Multiplication</u>

Definition

- $q = a + b\,i + c\,j + dk$
- Sum of a scalar and a vector

Multiplication (Basis Vectors)

- $i^2 = j^2 = k^2 = -1$
- $ij = k \qquad jk = i \qquad ki = j$
- $ji = -k \qquad kj = -i \qquad ik = -j$

Multiplication (Arbitrary Quaternion)

- $(a + \boldsymbol{v})(c + \boldsymbol{w}) = (a\,c - \boldsymbol{v}\cdot\boldsymbol{w}) + (c\,\boldsymbol{v} + a\,\boldsymbol{w} + \boldsymbol{v}\times\boldsymbol{w})$

Properties of Quaternion Multiplication

- Associative
- Not Commutative
- Distributes Through Addition
- Identity and Inverses

Rotations with Quaternions

Conjugate

- $q = a + bi + cj + dk$
- $q^* = a - bi - cj - dk$

Unit Quaternion

- $q(N, \theta) = \cos(\theta) + \sin(\theta) N$
- N = Unit Vector

Rotation of Vectors from Quaternion Multiplication (Sandwiching)

- $S_{q(N,\theta/2)}(v) = q(N, \theta/2)\; v\; q^*(N, \theta/2)$

  -- θ = Angle of Rotation

  -- N = Axis of Rotation

Applications of Quaternions in Computer Graphics

Compact Representation for Rotations of Vectors in 3-Dimensions

- 3×3 Matrices -- 9 Entries
- Unit Quaternions -- 4 Coefficients

Avoids Distortions

- After several matrix multiplications, rotation matrices may no longer be orthogonal due to floating point inaccuracies.
- Non-Orthogonal matrices are difficult to renormalize -- leads to distortions.
- Quaternions are easily renormalized -- avoids distortions.

Key Frame Animation

- Linear Interpolation between two rotation matrices R_1 and R_2 (key frames) fails to generate another rotation matrix.

  $Lerp(R_1, R_2, t) = (1-t)R_1 + tR_2$ -- not necessarily orthogonal matrices.

- Spherical Linear Interpolation between two unit quaternions always generates a unit quaternion.

  $Slerp(q_1, q_2, t) = \dfrac{\sin((1-t)\phi)}{\sin(\phi)} q_1 + \dfrac{\sin(t\phi)}{\sin(\phi)} q_2$ -- always a unit quaternion.

Goals and Motivation

- To provide a *geometric interpretation for quaternions*, appropriate for contemporary Computer Graphics.

- To present better ways to *visualize quaternions*, and the effect of quaternion multiplication on points and vectors in 3-dimensions.

- To derive the formula for *quaternion multiplication from first principles*.

- To develop *simple, intuitive proofs of the sandwiching formulas* for rotation and reflection.

- To show how to apply *sandwiching to compute perspective projections*.

Complex Multiplication

and

Rotation of Vectors in the Plane

<u>**Complex Numbers**</u>

Definition

- $z = a + b\,i \;\leftrightarrow\; v = (a, b)$
- Complex Number $\leftrightarrow$ Vector in the Plane

Complex Multiplication

- $i^2 = -1$
- $(a + b\,i)(c + d\,i) = (ac - bd) + (ad + bc)i$
  - 1 Complex Multiplication $\Leftrightarrow$ 4 Real Multiplications
  - Associative and Commutative
  - Distributes Through Addition
  - Identity and Inverses

<u>Complex Conjugation</u>

Definition

- $z = a + b\,i$
- $z^* = a - b\,i$

Properties

- $(z_1 z_2)^* = z_1^* \, z_2^*$
- $|\,z\,|^2 = z \cdot z = z\,z^* = a^2 + b^2$
- $|\,z_1\; z_2| = |\,z_1||\,z_2|$

Observations

- Multiplication is a linear transformation
  -- $w(z_1 + z_2) = w z_1 + w z_2$ (distributive property)
- Multiplying by a unit complex numbers preserves length
  -- linear isometry

Complex Conjugation

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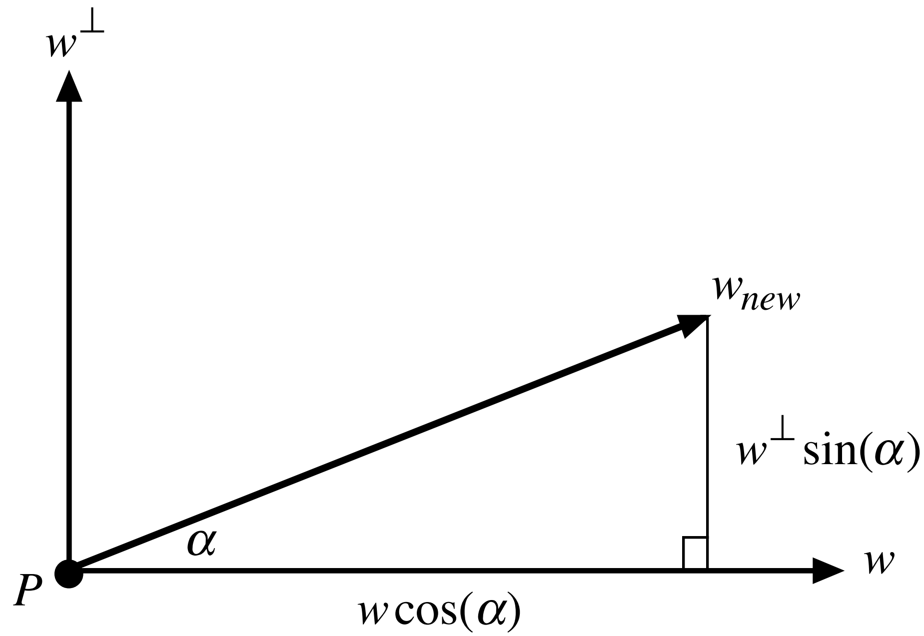
Observations

- Multiplication is a linear transformation
  -- $w(z_1 + z_2) = w z_1 + w z_2$ (distributive property)
- Multiplying by a unit complex numbers preserves length
  -- linear isometry = rotation
- Unit complex numbers are preserved under complex multiplication
  -- composition = complex multiplication

Rotation of Vectors in the Plane



$$w_{new} = w\cos(\alpha) + w^{\perp}\sin(\alpha)$$

$$w = (u, v) \Rightarrow w^{\perp} = (-v, u)$$

$$u_{new} = u\cos(\alpha) - v\sin(\alpha)$$
$$v_{new} = v\cos(\alpha) + u\sin(\alpha)$$

<u>Rotation of Vectors in the Plane Using Matrices</u>

Rotation by Matrix Multiplication

- ϕ = Angle of Rotation
- $R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$
- $v_{new} = v_{old} * R(\phi) = \begin{pmatrix} x_{old} & y_{old} \end{pmatrix} * \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$
  - $x_{new} = x_{old} \cos(\phi) - y_{old} \sin(\phi)$
  - $y_{new} = x_{old} \sin(\phi) + y_{old} \cos(\phi)$

Rotation of Vectors in the Plane Using Complex Numbers

Rotation by Complex Multiplication

- ϕ = Angle of Rotation
- $w = \cos(\phi) + \sin(\phi)\, i$
- $R_w(z) = w\, z$
- $R_w(z) = \left(\cos(\phi) + \sin(\phi)\, i\right)\left(x + y\, i\right) = \left(\underbrace{x\cos(\phi) - y\sin(\phi)}_{x_{new}}\right) + \left(\underbrace{x\sin(\phi) + y\cos(\phi)}_{y_{new}}\right) i$
- $(R_{w_2} \circ R_{w_1})(z) = w_2(w_1 z) = R_{w_2 w_1}(z)$ (Composition $\Leftrightarrow$ Multiplication)

Euler's Formula

Definition

- $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

Observation

- $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \cdots$

  -- $i^2 = -1$

Euler's Formula

- $e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos(\theta) + i\sin(\theta)$

Comparison of Rotation Techniques -- Composition

Matrices

- $R(\phi_1) * R(\phi_2)$
- 8 Real Multiplications

Complex Numbers

- $w_2 w_1$
- 4 Real Multiplications

Questions and Answers

Question: *What is the Geometric Interpretation of a Complex Number?*

Answer: A complex number is a vector in the plane.

Question: *What is the Geometric Interpretation of Complex Multiplication?*

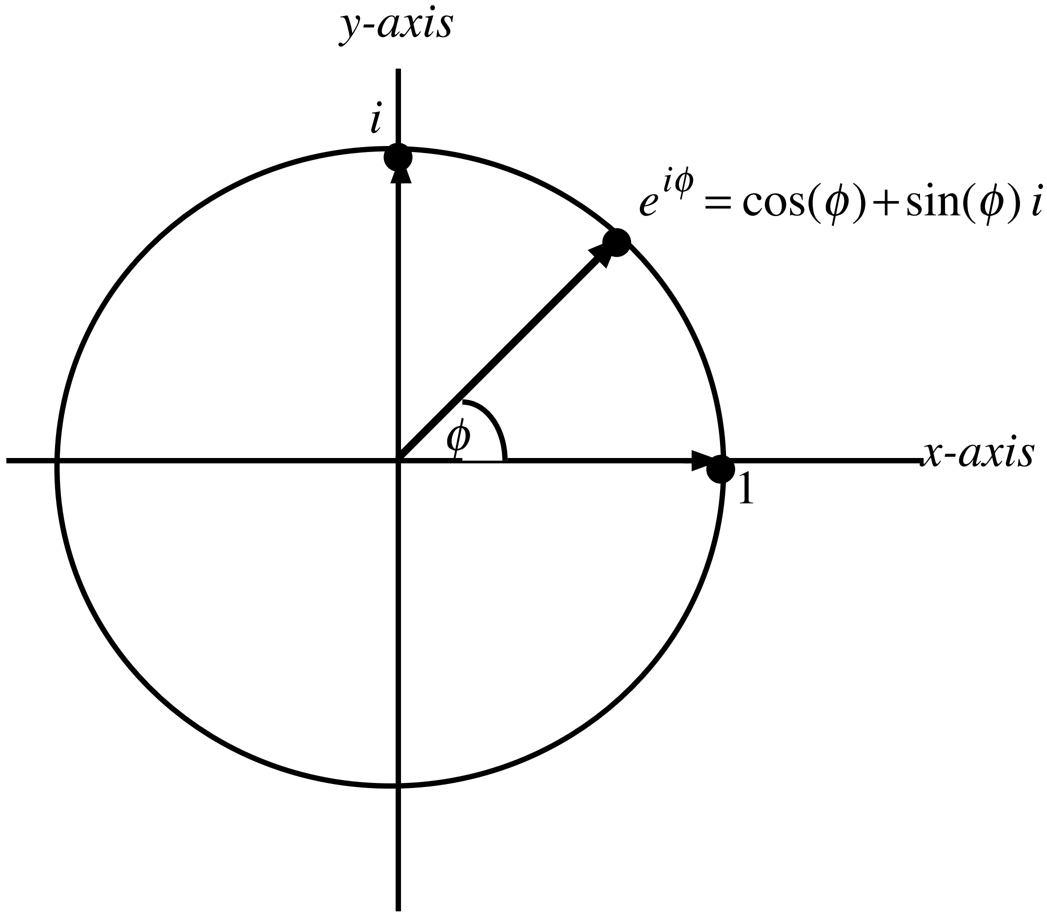
Answer: Multiplying a vector by a complex number $w = r(\cos(\phi) + \sin(\phi)\, i)$ is equivalent to rotating the vector by ϕ and scaling the vector by r.

Question: *Is Complex Multiplication Coordinate Free?*

Answer: No. The identity 1 for complex multiplication represents a special direction which we identify with the x-axis.

Angles are measured with respect to this special direction.

The Unit Circle in the Complex Plane



A Brief History of Multiplication

Natural Numbers

Counting Numbers

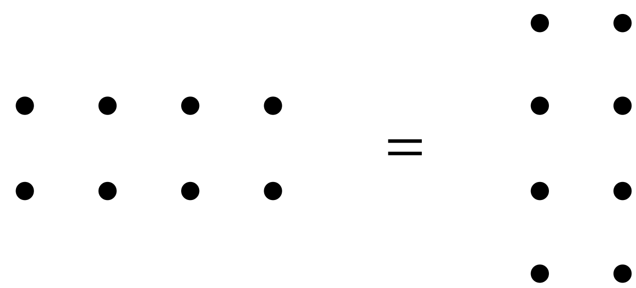
- $0, 1, 2, \cdots$

Multiplication

- Repeated Addition
- $mn = \underbrace{n + \cdots + n}_{m \text{ times}}$

Properties of Multiplication

Commutative (Rotation in 2–D)



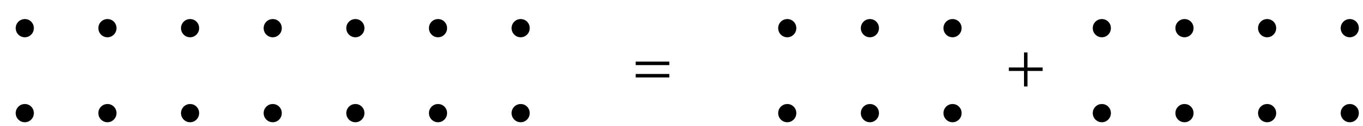
$$2 \times 4 \quad = \quad 4 \times 2$$

Associative (Rotation in 3–D)

$$2 \times (3 \times 4) = (2 \times 3) \times 4$$

Stack dots in 3-D

Distributes through Addition (Splitting and Concatenating)



$$2 \times (3 + 4) \quad = \quad 2 \times 3 \quad + \quad 2 \times 4$$

Extending the Natural Numbers by Solving Polynomial Equations

Rational Numbers

- $n\,x = m$

Negative Numbers

- $x + n = 0$

Real Numbers

- $x^2 - 2 = 0$

Complex Numbers

- $x^2 + 1 = 0$

Multiplication for Rational Number System

Multiplication by Natural Numbers

- $m\left(\dfrac{p}{q}\right)=\underbrace{\dfrac{p}{q}+\cdots+\dfrac{p}{q}}_{m\ \ times}$ (Repeated Addition)

Multiplication of Rationals by Rationals

- $\dfrac{p}{q}\dfrac{r}{s}=\dfrac{pr}{qs}$ (Multiply both side by qs)

Rules

- Associative and Commutative
- Distributes through Addition
- Identity and Inverses

Multiplication for Negative Numbers

Multiplication by Natural Numbers

- $m(-n) = \underbrace{-n + \cdots + -n}_{m\ times}$ (Repeated Addition)

Minus $\times$ Minus = Plus

- $(-1)(-1) = 1$
- $0 = (-1)(1-1) = -1(1) + (-1)(-1) \Rightarrow (-1)(-1) = 1$

Rules

- Associative and Commutative
- Identity and Inverses

$\Rightarrow$ Distributes through Addition

$\Rightarrow$ Preserves Length -- $|m\,n| = |m|\,|n|$

Multiplication for Real Numbers

Multiplication by Natural Numbers

- $mr = \underbrace{r + \cdots + r}_{m\ times}$ (Repeated Addition)

Real Numbers

- Limits of Rational Numbers

Rules

- Associative and Commutative
- Distributes through Addition
- Identity and Inverses

$\Rightarrow$ Preserves Length -- $|rs| = |r|\,|s|$

Multiplication for Complex Numbers

Multiplication by Natural Numbers

- $mw = \underbrace{w + \cdots + w}_{m \ times}$ (Repeated Addition)

Complex Multiplication

- $i^2 = -1$
- $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

Rules

- Associative and Commutative
- Identity and Inverses

$\Rightarrow$ Distributes through Addition

$\Rightarrow$ Preserves Length -- $|w\,z| = |w|\,|z|$

Multiplication in Higher Dimensions

Theorems

- Every polynomial of degree n has n complex roots.
- The complex numbers are a complete metric space.

Consequence

- No new numbers by solving polynomial equations.
- No new numbers as limits of complex numbers.

Question

- What should multiplication mean in dimensions > 2?

Multiplication as Rotation and Scaling

Real Multiplication

- 1 represents the identity
- -1 represents rotation by 180°
- $c > 0$ represents scaling

Complex Multiplication

- 1 represents the identity
- -1 represents rotation by 180°
- i represents rotation by 90°
- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

  represents rotation by θ
- $c > 0$ represents scaling

Multiplication in Higher Dimensions

Multiplication as Rotation and Scaling

- 1 Represents the Identity (Choose a Direction -- Not Coord. Free)
- Unit Vectors Represent Rotations (Similar to Complex Multiplication)
- $c > 0$ Represents Scaling

Rules	*Reasons*
• Associative	• Rotation is Associative
• Distributes through Addition	• Rotation is a Linear Transformation
• Identity and Inverses	• Rotations can be Undone
$\Rightarrow$ NOT Commutative	$\Rightarrow$ Rotation in 3–D not Commutative
$\Rightarrow$ Preserves Length ($\|pq\| = \|p\|\,\|q\|$)	$\Rightarrow$ Rotation is an Isometry

Vector Products in 3–Dimensions

<u>**Motivation**</u>

Observation

- Complex Multiplication Represents Rotating and Scaling Vectors in the Plane.

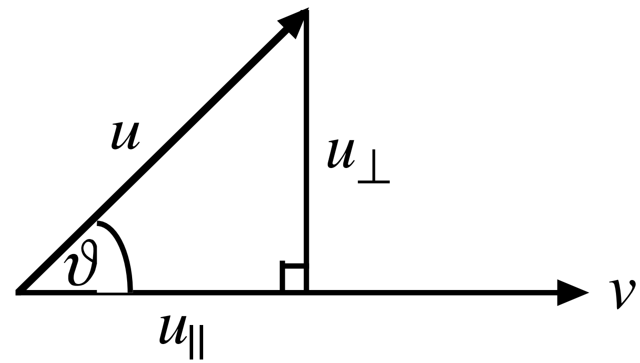
Questions

- Are There Comparable Products for Vectors in 3-Dimensions?
- Can We Apply Vector Products to Represent Rotations in 3-Space?

Vector Product in 3–Dimensions

- Dot Product
- Cross Product

Dot Product



$$u \bullet v = |u|\,|v|\,\cos(\vartheta) \qquad \text{(Coordinate Free)}$$

Properties

-- $u \bullet v = v \bullet u$

-- $u \bullet (v + w) = u \bullet v + u \bullet w$

-- $u_{||} = \left(\dfrac{u \bullet v}{v \bullet v}\right) v$

-- $u \bullet v = 0 \Leftrightarrow u \perp v$

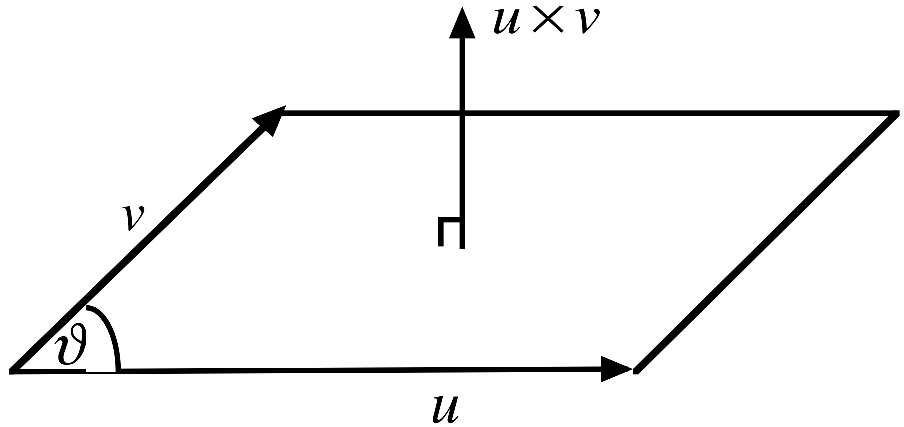
Observations

-- Dot Product is a Scalar

-- Not Associative

-- No Identity, No Inverses

Cross Product



$|u \times v| = |u||v|\sin(\vartheta)$ (Coordinate Free)

Properties	*Observations*
-- $u \times v = -v \times u$	-- Anti-Commutative
-- $u \times (v \times w) = (u \bullet w)v - (u \bullet v)w$ $(u \times v) \times w = (w \bullet u)v - (w \bullet v)u$	-- Not Associative
-- $u \times v \perp u, v$	-- No Identity, No Inverses

Multiplication Modeling Rotation in 2–D

Intuition -- Solving Equations for Vectors in 2–D

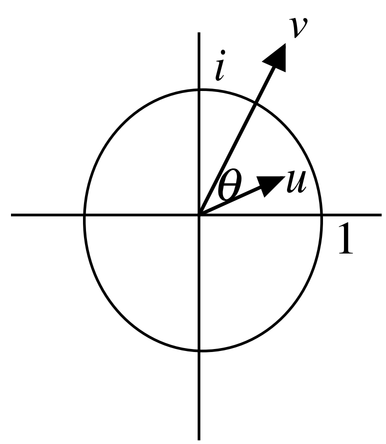
- $ux = v \Leftrightarrow x = u^{-1}v$

- x requires 2 free parameters

  -- scale $|u|$ to $|v|$ -- 1 parameter $\lambda = \dfrac{|v|}{|u|}$

  -- rotate u to v by $\angle(u,v)$ -- 1 parameters $\theta = \angle(u,v)$

- $x = \lambda e^{i\theta}$



No Multiplication Modeling Rotation in 3–D

Theorem

There is No Associative Product with Identity and Inverses for Vectors in 3–D!

Intuition -- Solving Equations for Vectors in 3–D

- $ux = v \Leftrightarrow x = u^{-1}v$
- x requires 4 free parameters

  -- scale $|u|$ to $|v|$ -- 1 parameter

  -- rotate u in the plane $\perp u, v$ by $\angle(u, v)$ -- 3 parameters

  # angle of rotation = 1 parameter

  # unit axis vector = 2 parameters

<u>No Multiplication in 3–D</u>

Theorem

There is No Extension of Complex Multiplication to Vectors in 3–D!

Proof: Consider 3 linearly independent basis vectors: *1, i, j:*

- $i\,j = a1 + b\,i + c\,j$

$$\Rightarrow\ -j = i^2\,j = i(i\,j) = i(a1 + b\,i + c\,j) = a\,i - b + c\,i\,j$$

$$\Rightarrow\ -j == ai - b + c(a + bi + c\,j)$$

$$\Rightarrow\ -j == ai - b + c\,a + b\,c\,i + c^2\,j$$

$$\Rightarrow\ -(1 + c^2)\,j == (ca - b)1 + (a + bc)i \qquad \text{CONTRADICTION}$$

Possible Multiplication Modeling Rotation in 4–D

Intuition -- Solving Equations for Vectors in 4–D

- $ux = v \Leftrightarrow x = u^{-1}v$

  -- Fix 2 mutually orthogonal planes in 4–D

  -- Project u, v into these planes

  # $u = u_1 + u_2$

  # $v = v_1 + v_2$

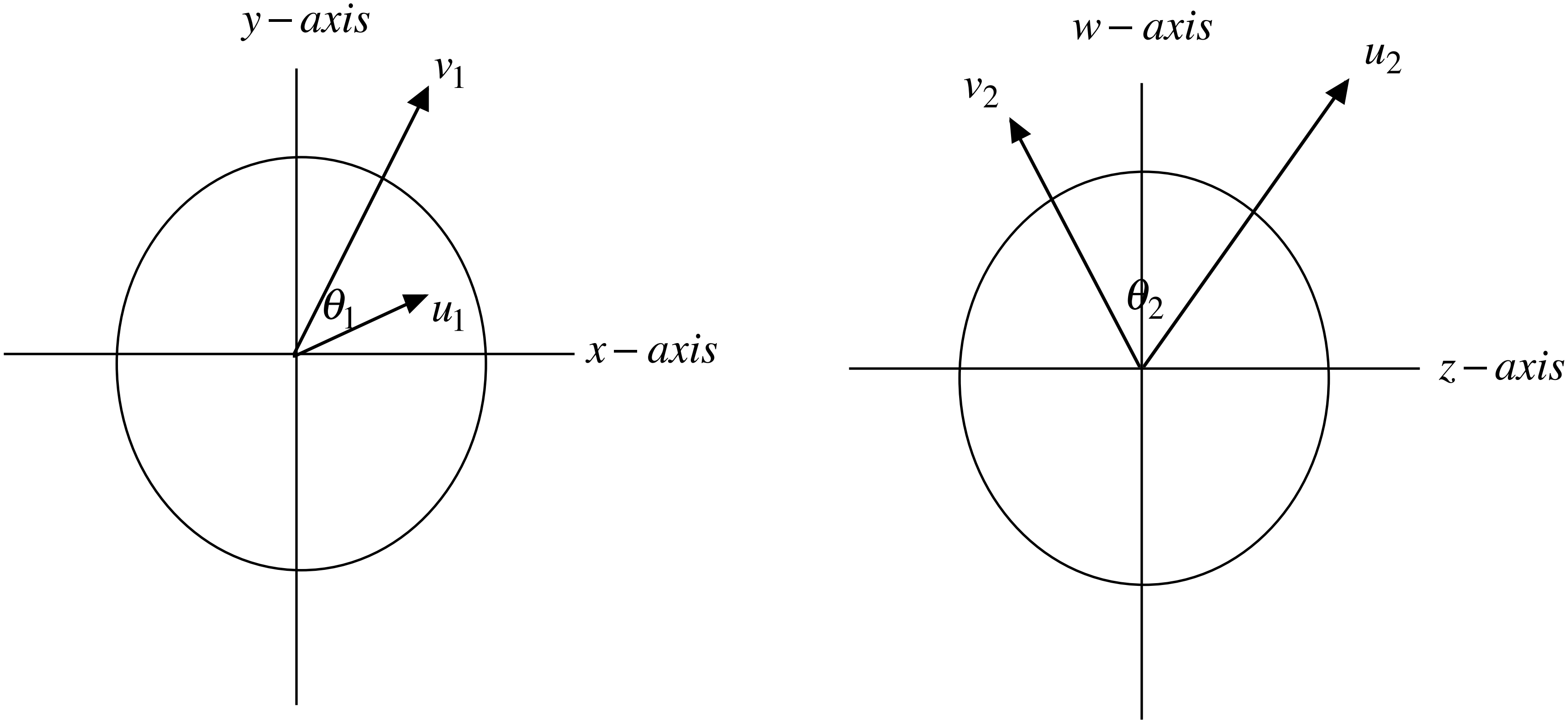
- x requires 4 free parameters

  -- 2 parameters for each plane

  -- scale and rotate in each plane

- $i\,j = k$

Two Mutually Orthogonal Planes in 4–Dimensions



$$u = u_1 + u_2$$

$$v = v_1 + v_2$$

Division Algebras

Operations

- Addition and Subtraction
- Multiplication and Division

Axioms

- Addition – Associative, Commutative, Identity, Inverses (Subtraction)
- Multiplication – Associative, Distributive, Identity, Inverses (Division)

Examples

- Real Numbers -- 1–Dimension
- Complex Numbers -- 2–Dimensions
- Quaternions -- 4–Dimensions

History

Frobenius (1878) and Pierce (1880) proved that the only *associative* real division algebras are real numbers, complex numbers, and quaternions.

The Cayley algebra is the only *nonassociative* division algebra.

Hurwitz (1898) proved that he only *associative* real division algebras with a length preserving product are the real numbers, complex numbers, and quaternions.

Adams (1958, 1960) proved that n-dimensional vectors form an algebra in which division (except by 0) is always possible only for $n = 1, 2, 4, 8$.

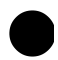
Bott and Milnor (1958) proved that the only finite-dimensional real division algebras occur for dimensions $n = 1, 2, 4, 8$, and these four cases correspond to real numbers, complex numbers, quaternions, and Cayley numbers.

Models for Visualizing 4–Dimensions

<u>Points and Vectors</u>

Points

- Position
- No Direction or Length



Dots

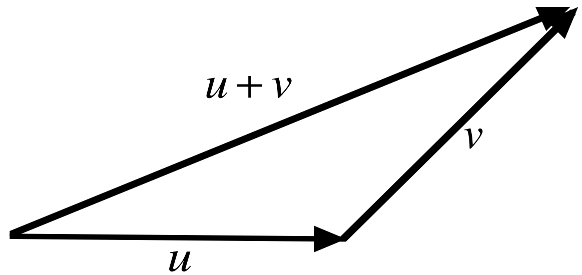
Vectors

- No Position
- Direction and Length

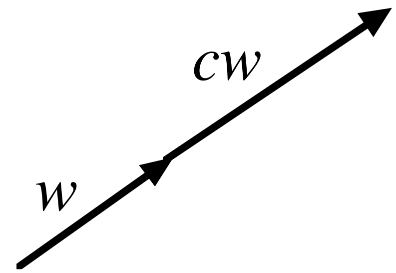
Arrows

Algebra for Points and Vectors in 3–Dimensions

Vectors

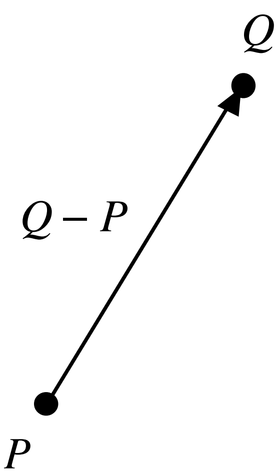


Addition

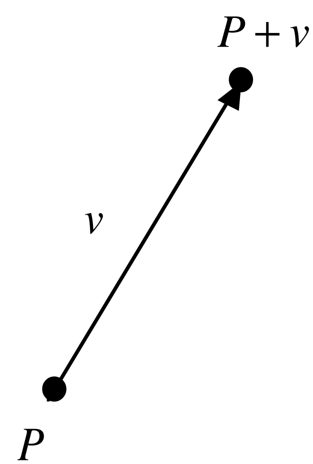


Scalar Multiplication

Points



Subtraction



Addition (Point+Vector)

Linear and Affine Combinations

Vectors -- Linear Combinations

- $\sum_{k=0}^{n} c_k v_k$ -- always defined

Points -- Affine Combinations

- $\sum_{k=0}^{n} c_k P_k = (\sum_{k=0}^{n} c_k) P_0 + \sum_{k=1}^{n} c_k (P_k - P_0)$

$$= P_0 + \sum_{k=1}^{n} c_k (P_k - P_0) \qquad \text{if} \quad \sum_{k=1}^{n} c_k \equiv 1$$

$$= \sum_{k=1}^{n} c_k (P_k - P_0) \qquad \text{if} \quad \sum_{k=1}^{n} c_k \equiv 0$$

$$= \textit{undefined} \qquad \textit{otherwise}$$

Mass–Points

Geometry -- Mass–Points

- (mP, m)

  -- $m = mass$

  -- $P = mP/m = point$

- $(v, 0)$ -- $v = vector$

Algebra -- 4-Dimensional Vector Space

- $(m_1P_1, m_1) + (m_2P_2, m_2) = (m_1P_1 + m_2P_2, m_1 + m_2)$ (Archimedes' Law)

- $(-mP_1, -m) + (mP_2, m) = \left(m(P_2 - P_1), 0\right)$

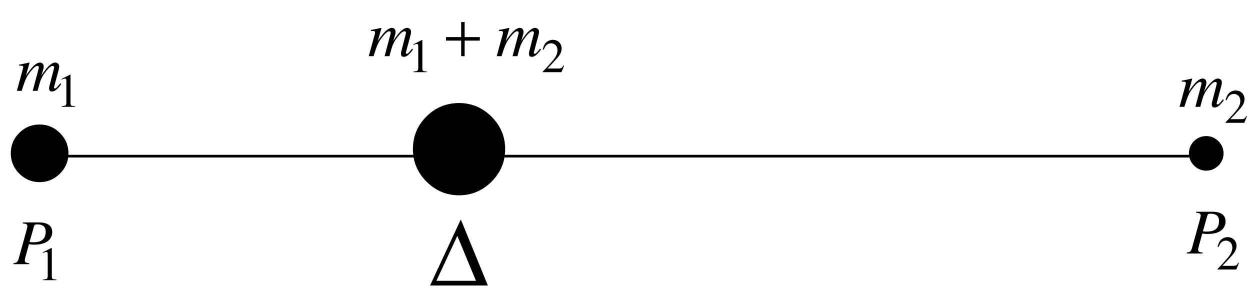
- $c(mP, m) = (cmP, cm)$

- $(v, 0) + (w, 0) = (v + w, 0)$

- $c(v, 0) = (cv, 0)$

- $(mP, m) + (v, 0) = (mP + v, m)$

Archimedes' Law of the Lever



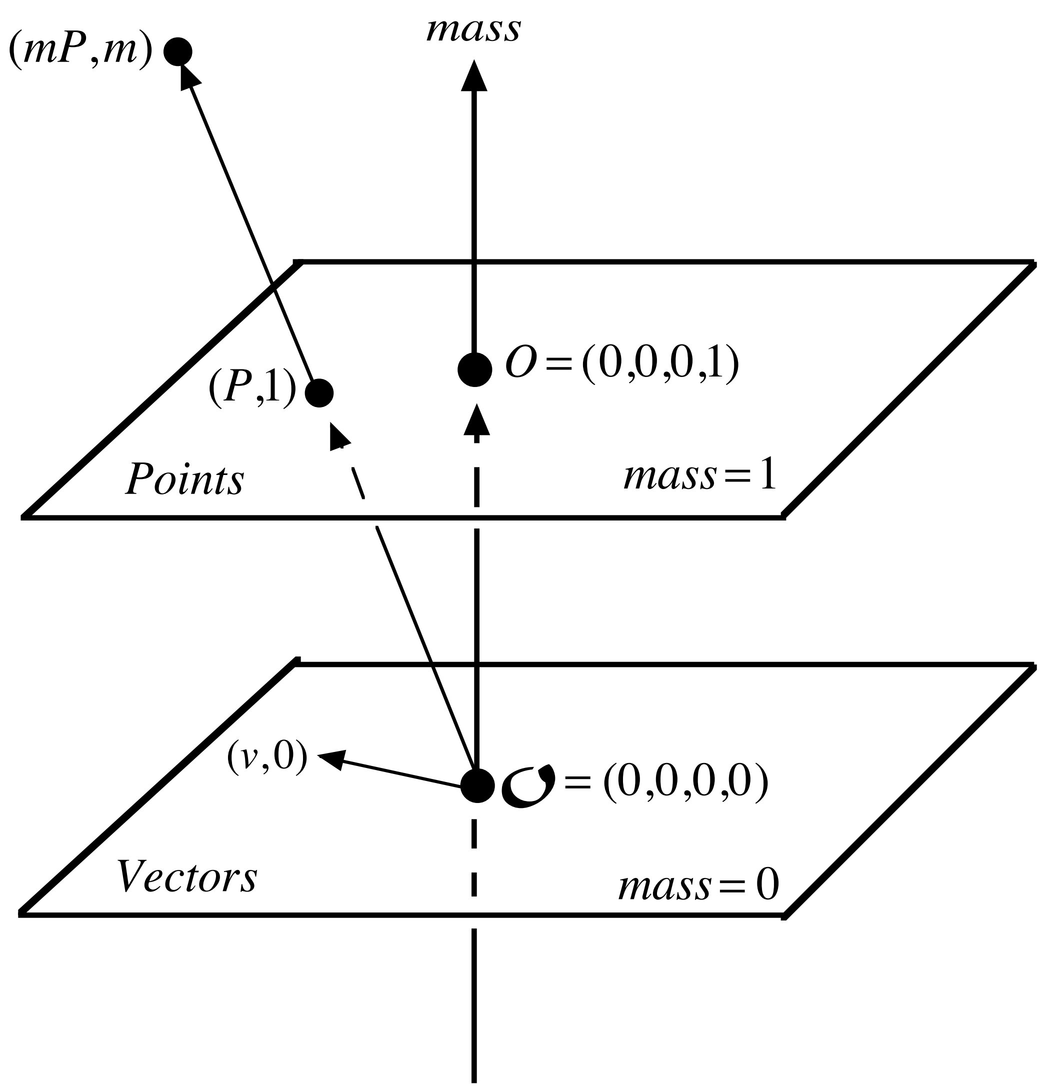
$$(m_1 P_1, m_1) + (m_2 P_2, m_2) = (m_1 P_1 + m_2 P_2, m_1 + m_2)$$

$$d_1 = dist\left(\frac{m_1 P_1 + m_2 P_2}{m_1 + m_2}, P_1\right) = \frac{m_2 \mid P_2 - P_1 \mid}{m_1 + m_2}$$

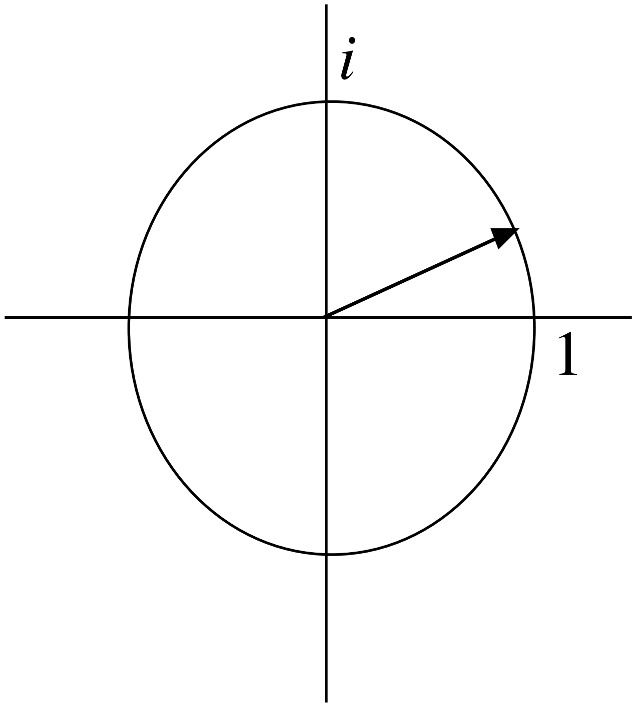
$$\Rightarrow \quad m_1 d_1 = m_2 d_2$$

$$d_2 = dist\left(\frac{m_1 P_1 + m_2 P_2}{m_1 + m_2}, P_2\right) = \frac{m_1 \mid P_1 - P_2 \mid}{m_1 + m_2}$$

The 4–Dimensional Vector Space of Quaternions

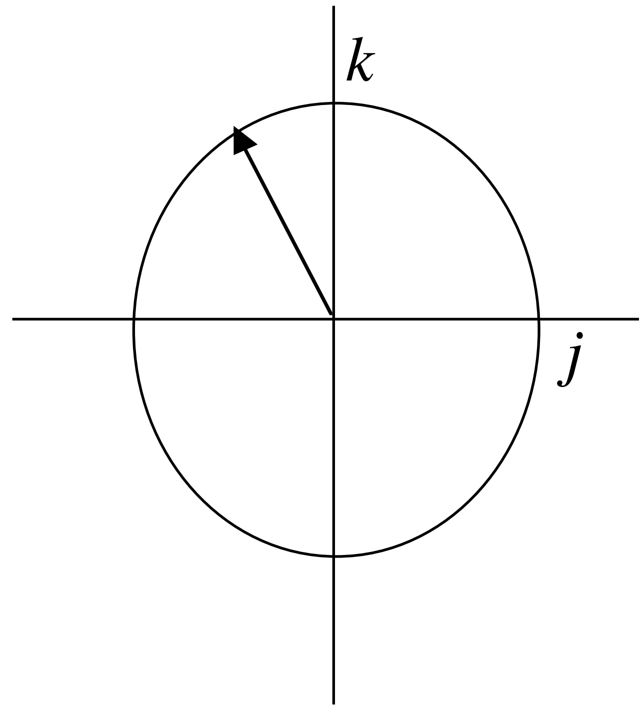


Pairs of Complementary Orthogonal Planes



Complex Plane

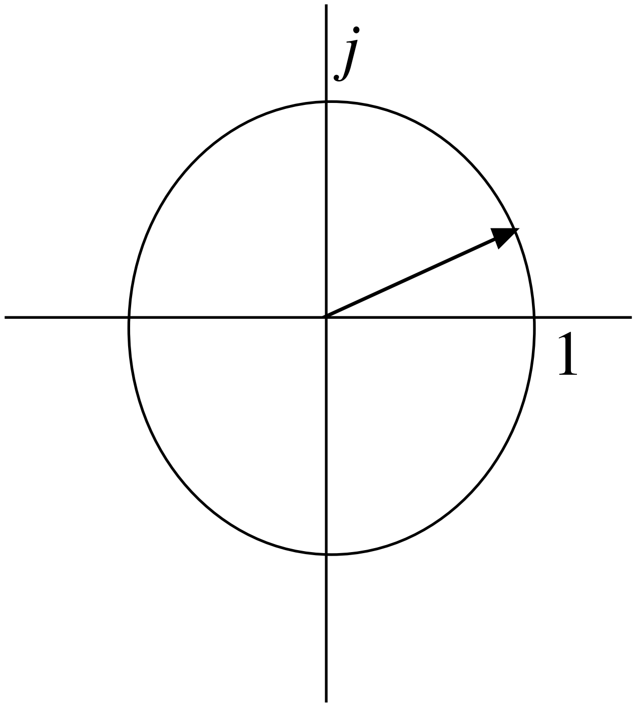
$i^2 = -1$



Orthogonal Plane

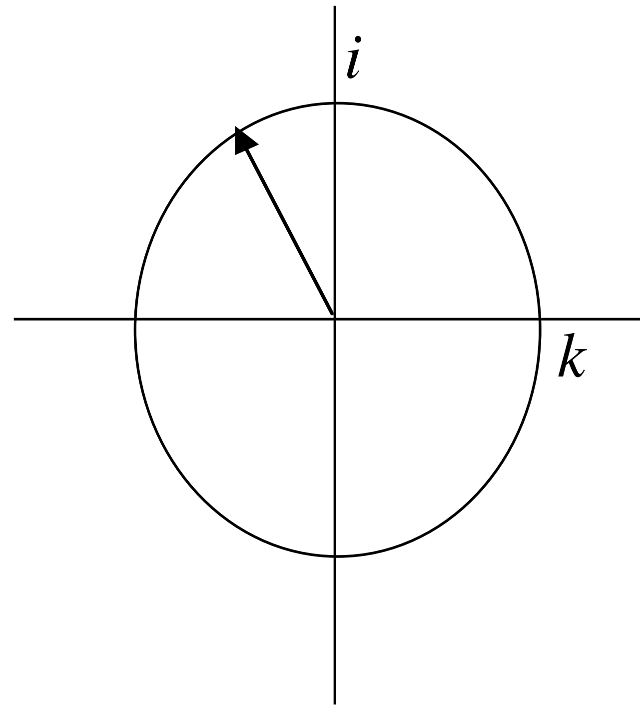
$j, k \perp 1, i$

Pairs of Complementary Orthogonal Planes



Complex Plane

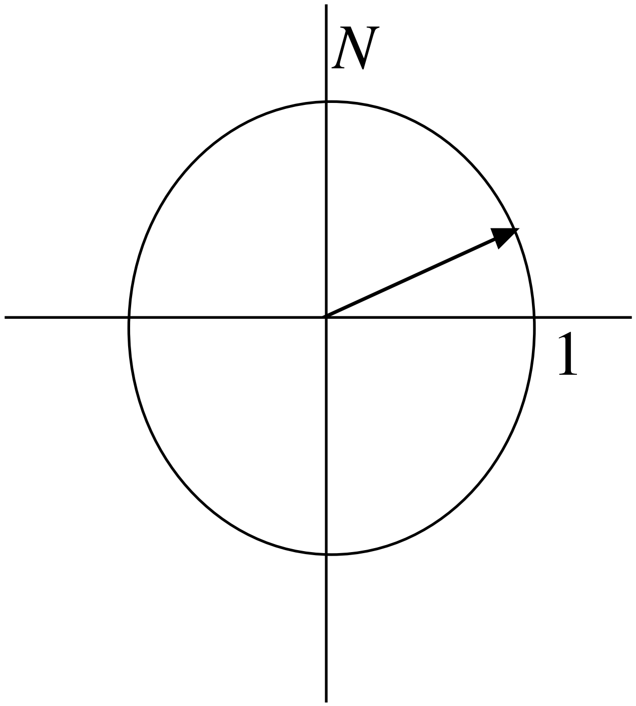
$j^2 = -1$



Orthogonal Plane

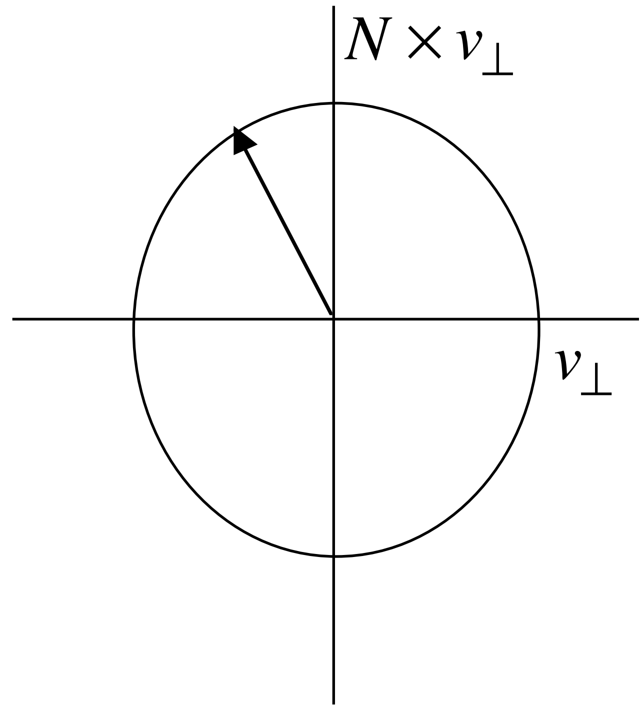
$k, i \perp 1, j$

Pairs of Complementary Orthogonal Planes



Complex Plane

$N^2 = -1$



Orthogonal Plane

$v_\perp, N \times v_\perp \perp 1, N$

Multiplication in 4–Dimensions

Properties of Multiplication

Assumptions

- O is the identity for multiplication

  -- $O\,p = p\,O = p$

- Multiplication is associative and distributes through addition

  -- $p(qr) = (pq)r$

  -- $p(q+r) = pq + pr$ and $(q+r)p = qp + rp$

- Multiplication scales length

  -- $\| pq \| = \| p \| \, \| q \|$

Consequences

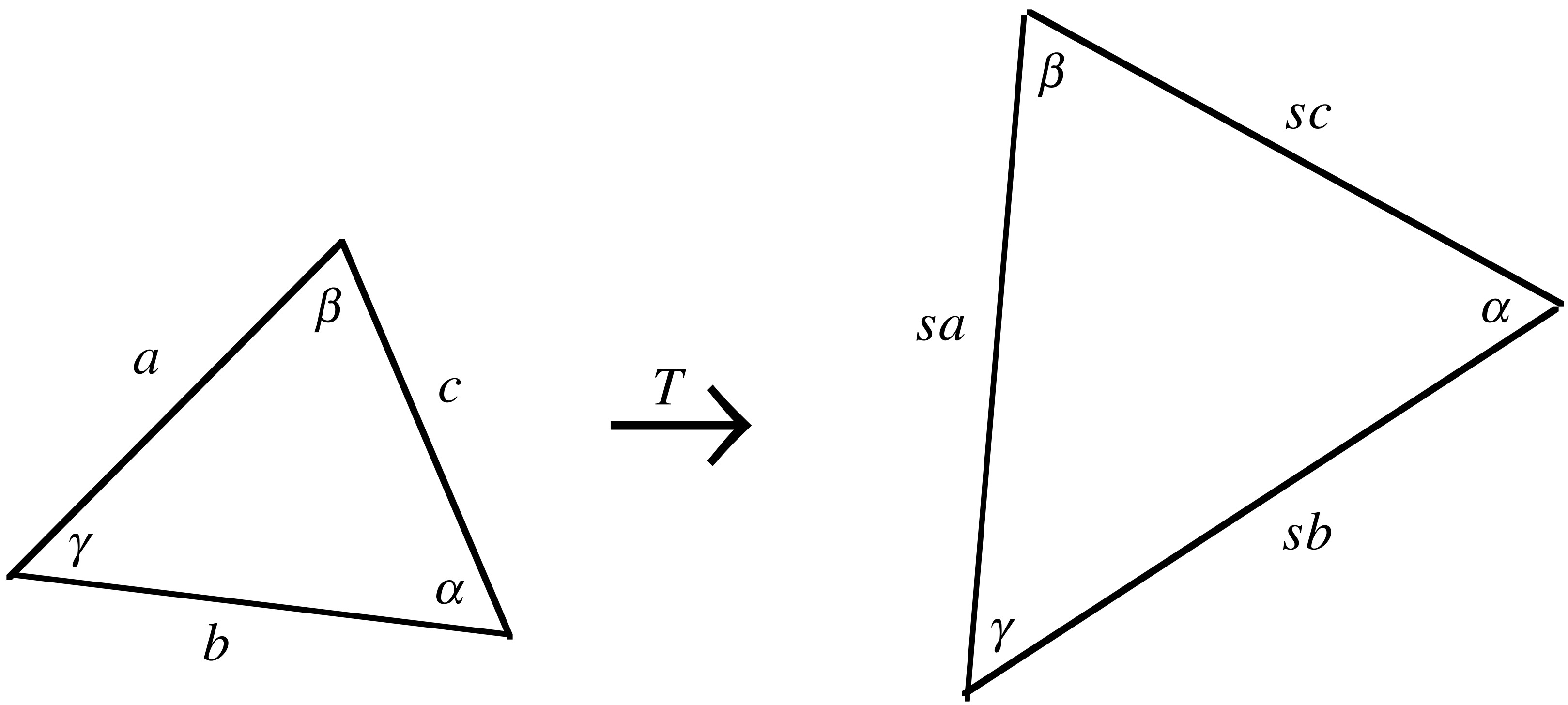
- $(aO + \boldsymbol{v})(bO + \boldsymbol{w}) = (ab)O + a\boldsymbol{w} + b\boldsymbol{v} + \boldsymbol{v}\boldsymbol{w}$

  -- seek formula for multiplication of vectors $\boldsymbol{v}\boldsymbol{w}$

- $\| pq \| = \| p \| \, \| q \| \Rightarrow$ multiplication by p is conformal (preserves angles)

  -- $\| p \| = 1 \Rightarrow$ multiplication by p is a linear isometry (rotation)

Scaling Lengths Preserves Angles



Similar Triangles

Consequences of Conformality

Notation

- O is the identity for multiplication
- v, w, N are unit vectors $(\perp O)$

Consequences of Conformality

- $vw \perp v, w$
- $v^2 = -(v \cdot v)O$
- $v \perp w \Rightarrow vw \perp O$
- $(vw) \cdot O = -v \cdot w$

Lemma 1: $vw \perp v, w$

Proof: By conformality of multiplication:

- $w \perp O \Rightarrow vw \perp vO = v$
- $v \perp O \Rightarrow vw \perp Ow = w$

Lemma 2: $v^2 = -(v \cdot v)O$

Proof: Let $q = \cos(\theta)O + \sin(\theta)v$, with $\theta \neq k(\pi/2)$.

Since multiplication preserves angles,

- $v \perp O \Rightarrow qv \perp qO = q$
- $qv = \big(\cos(\theta)O + \sin(\theta)v\big)v = \cos(\theta)v + \sin(\theta)v^2 \perp \cos(\theta)O + \sin(\theta)v = q$.

But by Lemma 1, $v^2 \perp v$, so

- $v^2 = \alpha O + \beta w \quad w \perp O, v$.

Therefore:

$0 = qv \cdot q = \big(\cos(\theta)v + \sin(\theta)(\alpha O + \beta w)\big) \cdot (\cos(\theta)O + \sin(\theta)v) = (v \cdot v + \alpha)\sin(\theta)\cos(\theta)$

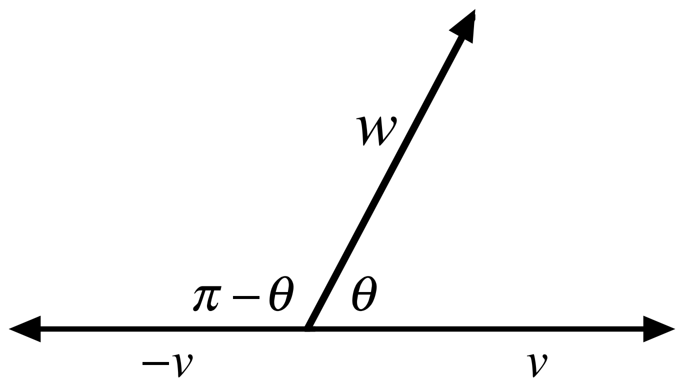
so

- $\alpha = -v \cdot v$
- $\beta = 0$ (because $(v \cdot v) = |v|^2 = |v^2| = \sqrt{\alpha^2 + \beta^2}$)

Corollary 1: *Angle between O and* vw = *Angle between* $-v$ *and* $w = \pi - \theta$.

Proof: By conformality of multiplication:

angle between O and vw = *angle between* $-v$ *and* $(-v)(vw) \,||\, w$



Corollary 2: $v \perp w \Rightarrow vw \perp O$

Proof: *angle between O and* vw = *angle between* $-v$ *and* $w = \pi / 2$.

Corollary 3: $(vw) \cdot O = -v \cdot w$

Proof: $(vw) \cdot O = \| vw \| \, \| O \| \cos(\pi - \theta) = -|v||w|\cos(\theta) = -v \cdot w$

Summary

Notation

- O is the identity for multiplication
- v, w are vectors $(\perp O)$

Consequences of Conformality

- $vw \perp v, w$
- $v^2 = -(v \cdot v)O$
- $v \perp w \Rightarrow vw \perp O$
- $(vw) \cdot O = -v \cdot w$

No Multiplication in R^3

Theorem: *There is no multiplication in* R^3.

Proof: Let O, i, j be an orthonormal basis.

- $ij \perp i, j, O \Rightarrow ij = 0$
- $\| ij \| = \| i \| \| j \| = 1 \neq 0$ CONTRADICTION.

<u>**Quaternion Multiplication**</u>

Recall

- $(aO+v)(bO+w)=(ab)O+aw+bv+vw$

  -- seek formula for multiplication of vectors

  -- multiplication of unit vectors suffice

Vectors

- $vw \perp v,w \Rightarrow vw = aO + b\, v \times w$

  $\Rightarrow \; a=(vw)\cdot O = -v\cdot w$

- $|v|^2\;|w|^2 = \|vw\|^2 = (-v\cdot w)^2 + b^2\,|v\times w|^2 = |v|^2\;|w|^2\left(\cos^2(\theta)+b^2\sin^2(\theta)\right)$

  $\Rightarrow \; b=\pm 1$

Conclusion

- $v\,w = (-v\cdot w)O + v\times w$

<u>Quaternion Multiplication</u>

Quaternion Multiplication

- $(aO+\boldsymbol{v})(bO+\boldsymbol{w}) = (ab)O + a\boldsymbol{w} + b\boldsymbol{v} + \boldsymbol{v}\,\boldsymbol{w}$

  -- $\boldsymbol{v}\,\boldsymbol{w} = (-\boldsymbol{v}\cdot\boldsymbol{w})O + \boldsymbol{v}\times\boldsymbol{w}$

- $(aO+\boldsymbol{v})(bO+\boldsymbol{w}) = (ab-\boldsymbol{v}\cdot\boldsymbol{w})O + a\boldsymbol{w} + b\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{w}$

Basis Vectors

- $i^2 = j^2 = k^2 = -O$
- $ij = k \qquad jk = i \qquad ki = j$
- $ji = -k \qquad kj = -i \qquad ik = -j$

General Cases

$|u| = 1 \Rightarrow u^2 = -O$

$v \perp w \Rightarrow vw = v\times w$

No Multiplication in R^5

Theorem: There is no multiplication in R^5.

Proof: Let O, i, j be orthogonal unit vectors.

- $k = ij \Rightarrow k \perp O, i, j$ (Definition of *k*)
- $l \perp O, i, j, k$ (Definition of *l*)
- $il \perp O, i, j, k, l \Rightarrow il = 0$ (Orthogonality)
- $\| il \| = \| i \| \| l \| = 1 \neq 0$ CONTRADICTION.

No Multiplication in R^d

Theorem: There is no multiplication in R^6 or R^7

Proof: Let O, i, j be orthogonal unit vectors.

- $k = i\,j \Rightarrow k \perp O, i, j$ (Definition of *k*)
- $l \perp O, i, j, k$ (Definition of *l*)
- O, i, j, k and $l, i\,l, j\,l, k\,l$ (8 Mutually Orthogonal Unit Vectors)

CONTRADICTION.

Generalizations

- Similar proofs apply in R^d, $d \neq 2^p$

The Geometry of Quaternion Multiplication

Properties of Quaternion Multiplication

Unit Quaternions

- $\| pq \| = \| p \| \| q \|$
- $\| q \| = 1 \Rightarrow$ multiplication by q (on left or right) is a linear isometry in R^4

  $\Rightarrow$ multiplication by q (on left or right) is rotation in R^4

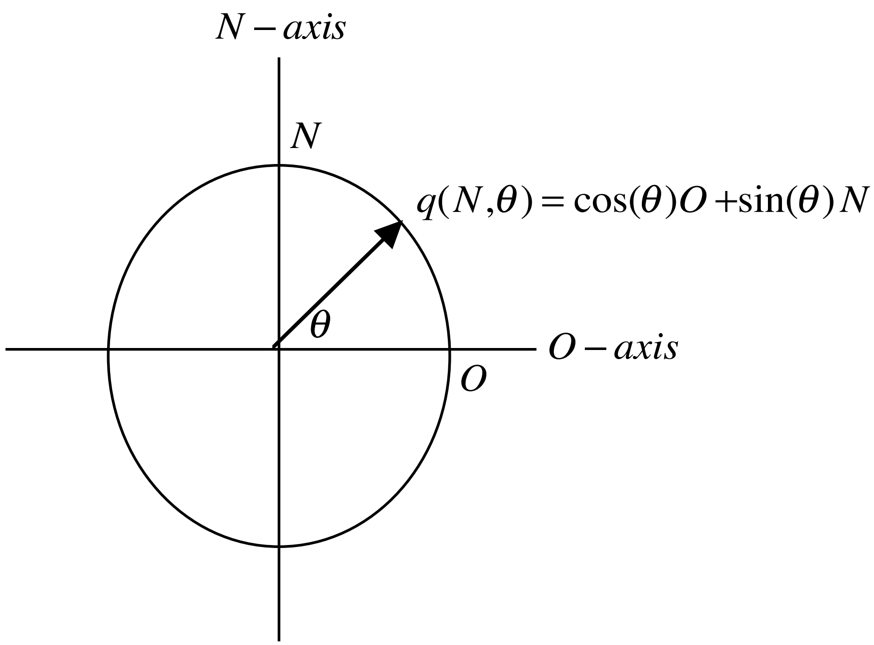
Unit Vectors

- $v\,w = (-v \cdot w)O + v \times w$
- $\| N \| = 1 \Rightarrow N^2 = -O$
- O, N plane is isomorphic to the complex plane

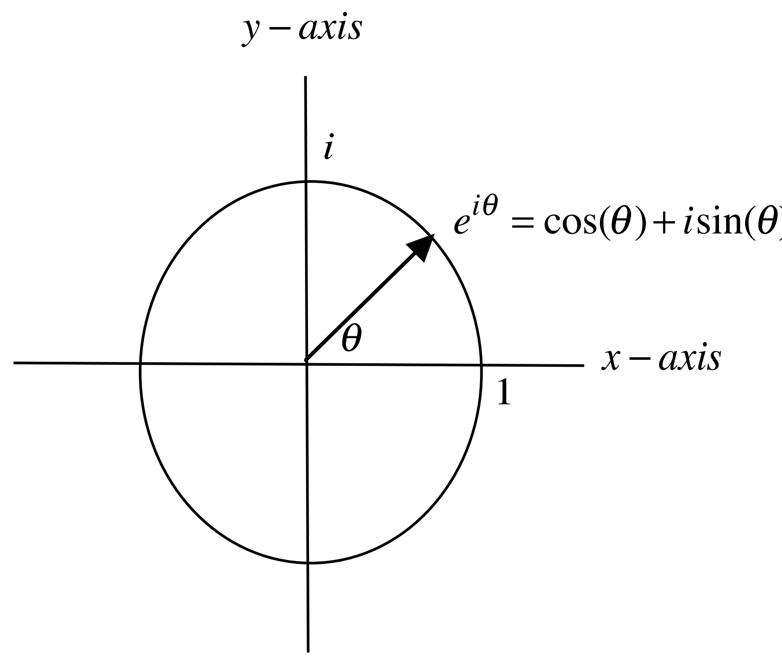
  -- $N^2 = -O$, $\quad O^2 = 1$

  -- $NO = ON = N$

Planes Isomorphic to the Complex Plane



O, N Plane



Complex Plane

$N^2 = -O$, $O^2 = O$ $\qquad$ $i^2 = -1$, $1^2 = 1$

$$q(N,\theta) = \cos(\theta) + sin(\theta)N \leftrightarrow \textit{Rotation} \leftrightarrow e^{i\theta} = \cos(\theta) + sin(\theta)N$$

$\{e^{N\theta}\}$ $\qquad$ *by* θ

Quaternion Multiplication in the Plane of *O, N*

Notation

- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- $q(N, \theta) = \cos(\theta) + \sin(\theta)N$

Multiplication = Rotation

- $e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$
- $q(N, \theta) * q(N, \phi) = q(N, \theta + \phi)$

Products

- $q(N, \theta) * q(N, \phi) = q(N, \theta + \phi)$ (Angles Add)
- $q(N, \theta) * q(N, \phi) = q(N, \phi) * q(N, \theta)$ (Multiplication Commutes)

Conjugation

Definition

- $q = aO + bi + cj + dk$
- $q^* = aO - bi - cj - dk$

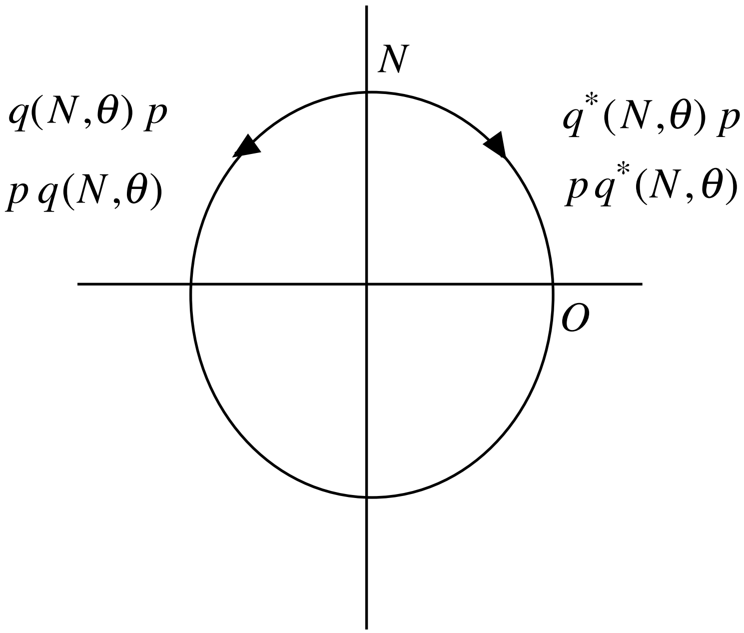
Properties

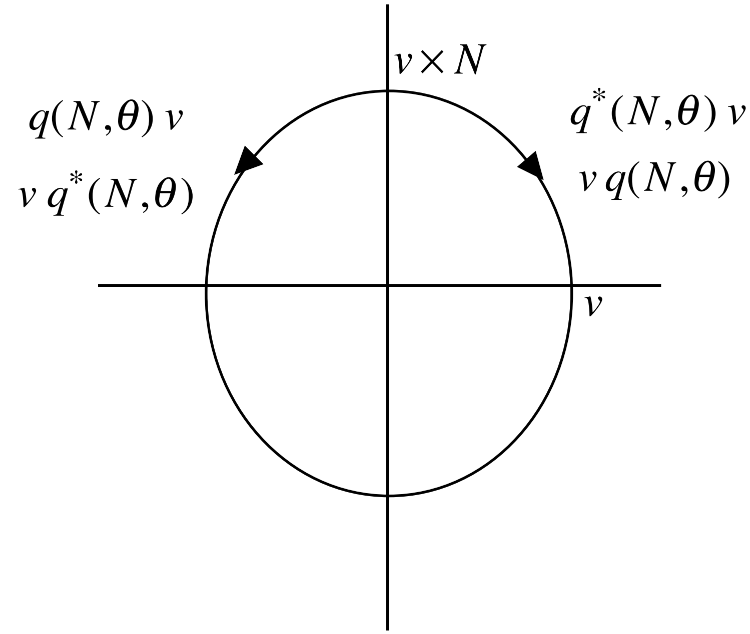
- $q\,q^* = \| q \|^2 \, O$
- $(pq)^* = q^* p^*$
- $q(N,\theta) = \cos(\theta)O + \sin(\theta)N$

  -- $q^*(N,\theta) = \cos(\theta)O - \sin(\theta)N$

  -- $q^*(N,\theta) = q(N,-\theta)$

<u>**Rotation in Complementary Planes**</u>





Plane of ***O***, *N*

$q(N,\theta),\ q^*(N,\theta)$ *Cancel*

Plane Perpendicular to O, N

$q(N,\theta),\ q^*(N,\theta)$ *Reinforce*

Theorem 1: Rotation in the Plane of *O, N*

Let

- $q(N,\theta)=\cos(\theta)+\sin(\theta)N$
- *p = quaternion in the plane of 1, N*

Then

i. $p \to q(N,\theta)\, p$ *rotates p by the angle* θ *in the plane of O, N*

ii. $p \to p\, q(N,\theta)$ *rotates p by the angle* θ *in the plane of O, N*

iii. $p \to p\, q^{*}(N,\theta)$ *rotates p by the angle* $-\theta$ *in the plane of O, N*

iv. $p \to q(N,\theta)\, p\, q^{*}(N,\theta)$ *is the identity on p*

Theorem 2: Rotation in the Plane Perpendicular to *O, N*

Let

- $q(N,\theta) = \cos(\theta) + \sin(\theta)N$
- v = *vector in the plane* $\perp$ *O, N*

Then

$i.\quad v \to q(N,\theta)\,v$ *rotates v by the angle* θ *in the plane* $\perp$ *O, N*

$ii.\quad v \to v\,q(N,\theta)$ *rotates v by the angle* $-\theta$ *in the plane* $\perp$ *O, N*

$iii.\quad v \to v\,q^*(N,\theta)$ *rotates v by the angle* θ *in the plane* $\perp$ *O, N*

$iv.\quad v \to q(N,\theta)\,v\,q^*(N,\theta)$ *rotates v by the angle* 2θ *in the plane* $\perp$ *O, N*

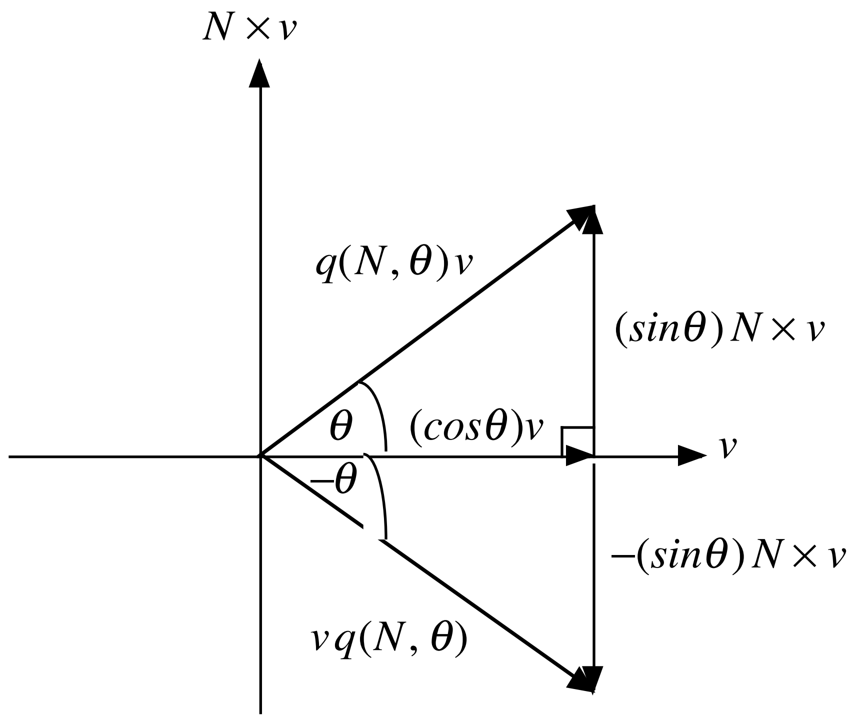
Proof: *i, ii: See diagram*

iii, iv: Follow directly from *i* and *ii*, since $q^*(N,\theta) = q(N,-\theta)$.

Proofs of i and ii:

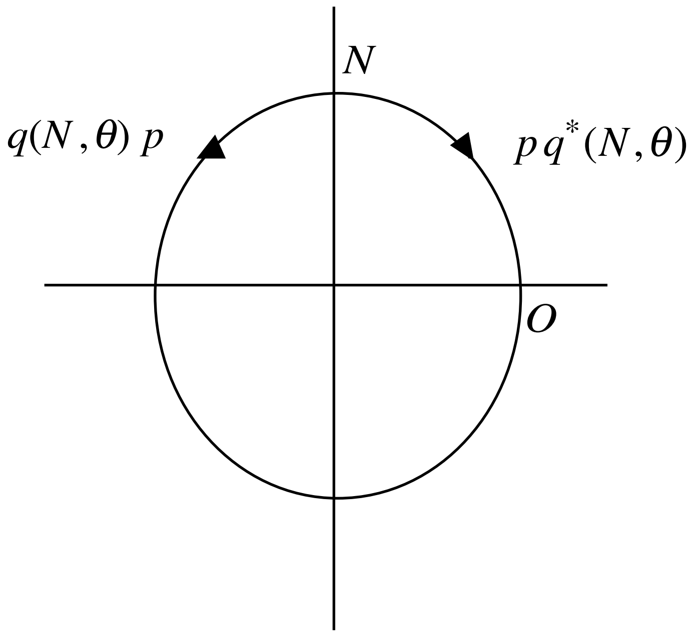
$$i. \quad q(N,\theta)\,v = \big(\cos(\theta)O + \sin(\theta)N\big)\,v = \cos(\theta)\,v + \sin(\theta)\,N\times v$$

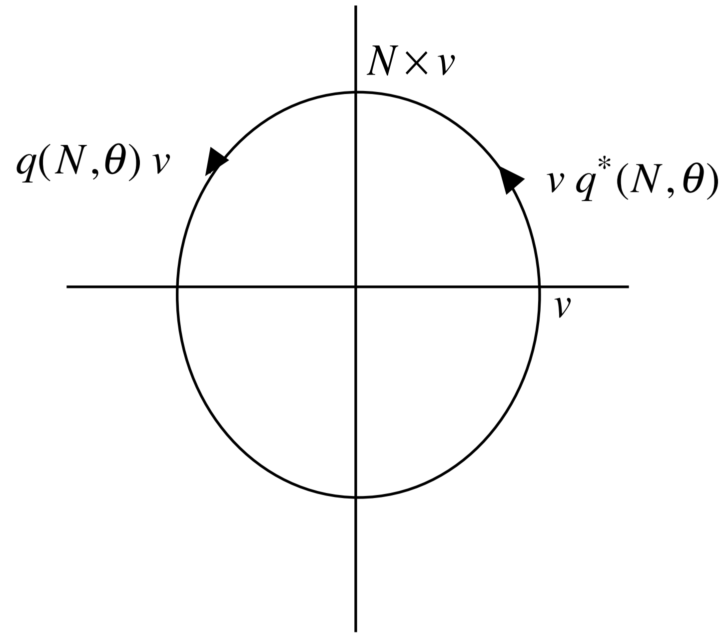
$$ii. \quad v\,q(N,\theta) = v\big(\cos(\theta)O + \sin(\theta)N\big) = \cos(\theta)\,v + \sin(\theta)\,v\times N = \cos(\theta)v - \sin(\theta)N\times v$$



Rotation in the Plane $\perp$ O, N

Sandwiching with Conjugates in Complementary Planes





Plane of O, N

Sandwiching $q(N,\theta)\,p\,q^*(N,\theta)$

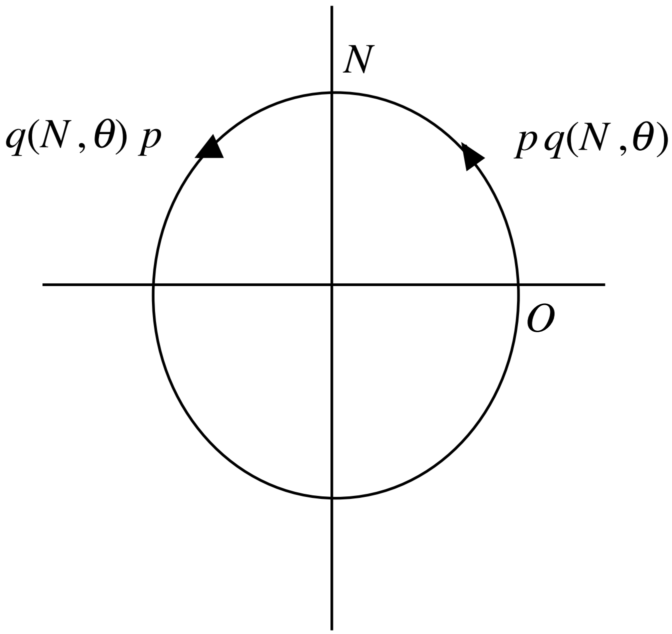
$q(N,\theta)$, $q^*(N,\theta)$ *Cancel*

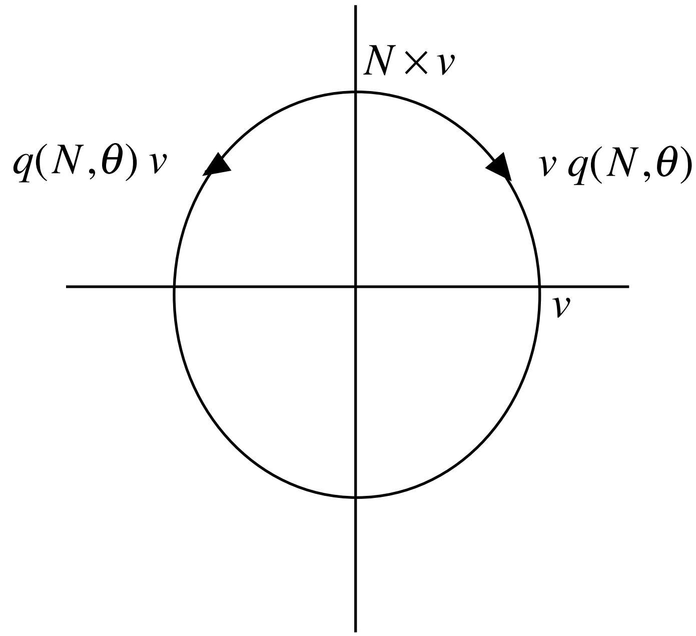
Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\,v\,q^*(N,\theta)$

$q(N,\theta)$, $q^*(N,\theta)$ *Reinforce*

Sandwiching in Complementary Planes





Plane of O, N

Sandwiching $q(N,\theta)\,p\,q(N,\theta)$

$q(N,\theta)$ *on Left and Right Reinforce*

Plane Perpendicular to O, N

Sandwiching $q(N,\theta)\,v\,q(N,\theta)$

$q(N,\theta)$ *on Left and Right Cancel*

3–Dimensional Interpretations of Mutually Orthogonal Planes in 4–Dimensions

Plane $\perp$ *O, N*

- Plane of Vectors $\perp N$
  - -- All Vectors v in 3–Dimensions are $\perp O$ in 4–Dimensions
  - -- $v \perp O, N$ in 4–Dimensions $\Rightarrow$ $v \perp N$ in 3–Dimensions

Plane of O, N

- Points on the Line Through the Point O in the Direction of the Vector N
  - -- $p = cO + sN \rightarrow p \equiv O + \frac{s}{c} N$
  - -- $P(t) = O + tN$ = Line in 3–Dimensions (Second Dimension is Mass)

Sandwiching Summary

$p \rightarrow q(N,\theta)\, p\, q^*(N,\theta)$

- Plane of O, N = Line through O parallel to N

  -- Identity $\rightarrow$ FIXED AXIS LINE

- Plane $\perp$ O, N = Plane $\perp$ N

  -- Rotation by Angle 2θ -- ROTATION

$p \rightarrow q(N,\theta)\, p\, q(N,\theta)$

- Plane $\perp$ O, N = Plane $\perp$ N

  -- Identity $\rightarrow$ FIXED PLANE

- Plane of O, N = Line through O parallel to N

  -- $N \rightarrow -N$ -- MIRROR IMAGE

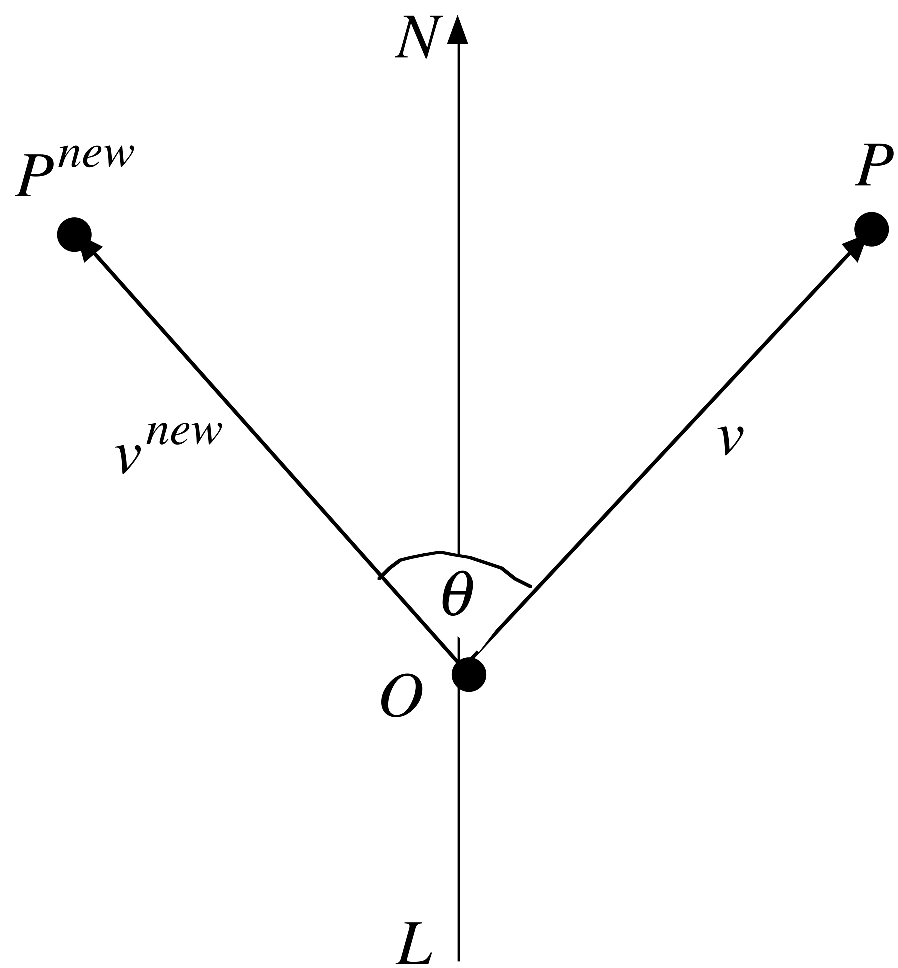
  -- N $\rightarrow$ Mass–Point -- PERSPECTIVE PROJECTION

Rotation, Reflection

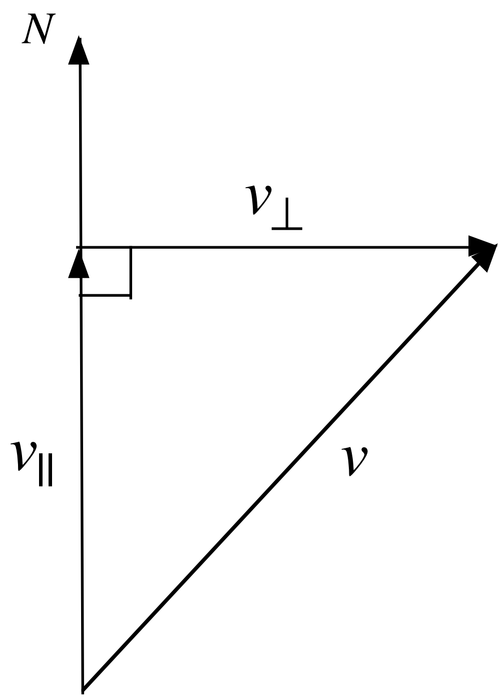
and

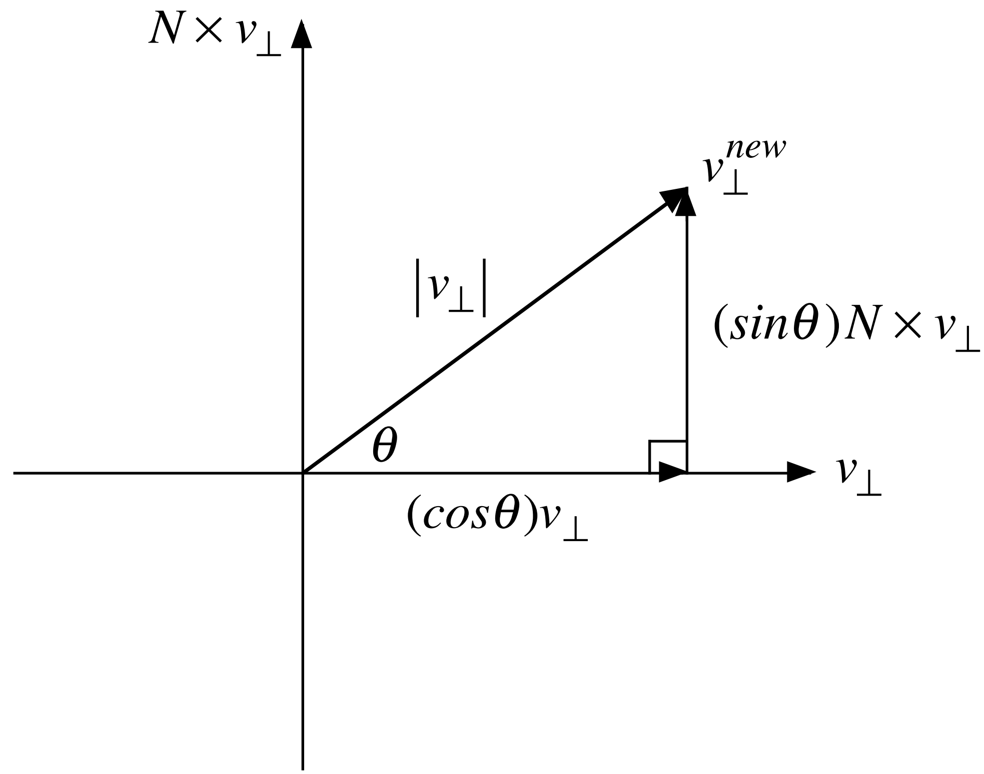
Perspective Projection

Rotation



Decomposing Rotation into Parallel and Perpendicular Components





Decomposition

$v = v_{||} + v_\perp$

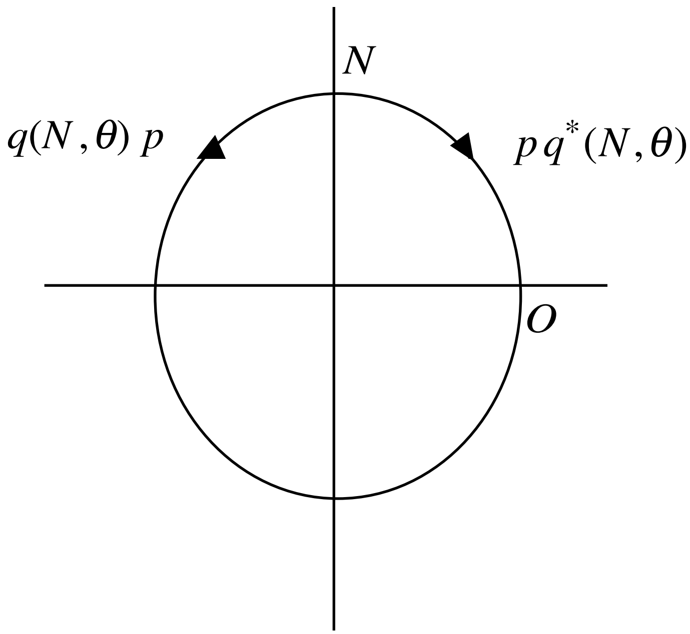
$v_{||}^{new} = v_{||}$

Plane Perpendicular to N

$v_\perp^{new} = \cos(\theta)v_\perp + \sin(\theta)N \times v_\perp$

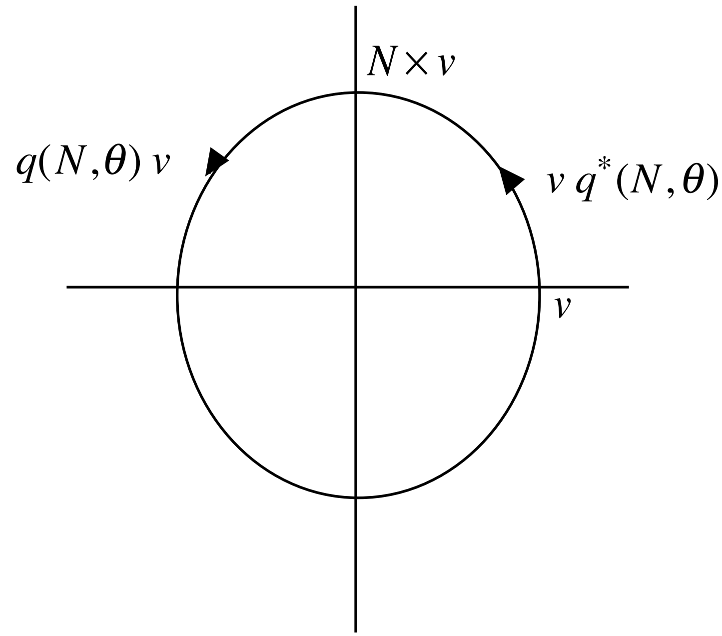
Rotation by θ

Rotation: Sandwiching in Complementary Planes



Plane of O, N

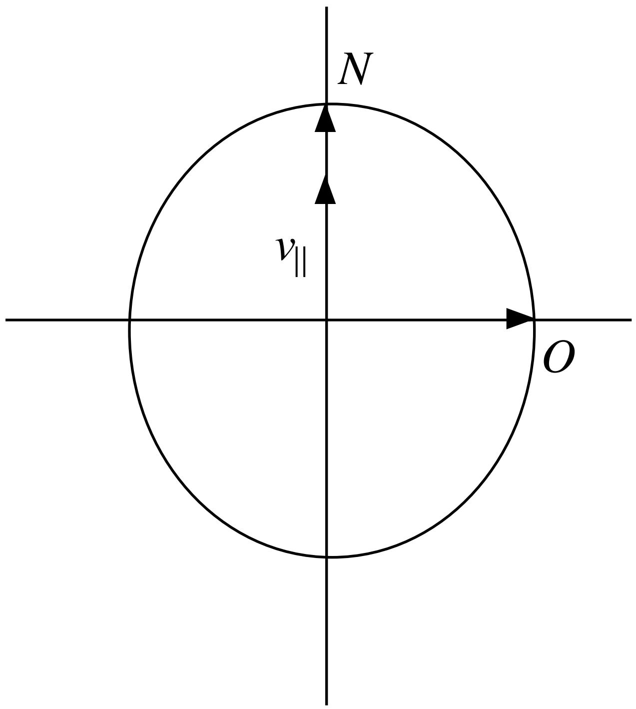
Sandwiching Cancels



Plane Perpendicular to O, N

Sandwiching Adds

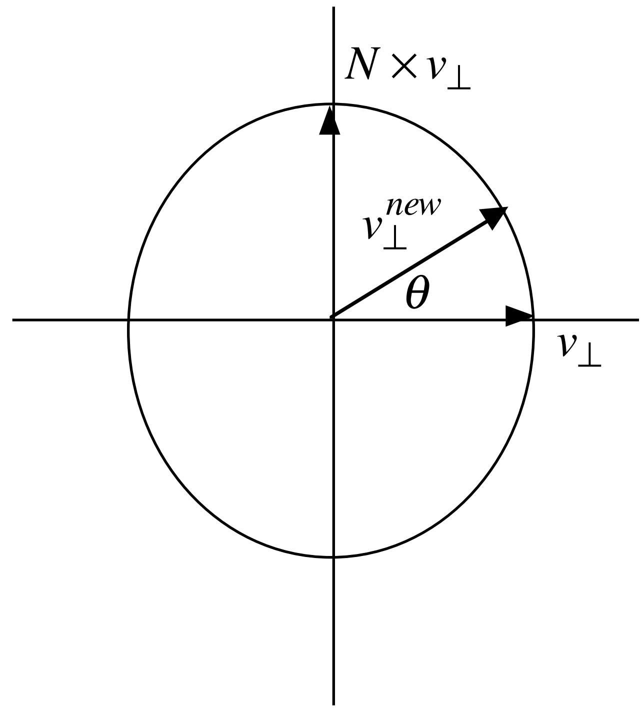
Sandwiching with Conjugates



Plane of O, N

$$q(N,\theta/2)\; v_{||}\; q^{*}(N,\theta/2) = v_{||}$$

Identity



Plane Perpendicular to O, N

$$q(N,\theta/2)\; v_{\perp}^{new}\; q^{*}(N,\theta/2)$$

Rotation by θ

Theorem 1: Sandwiching Rotates Vectors in 3–Dimensions

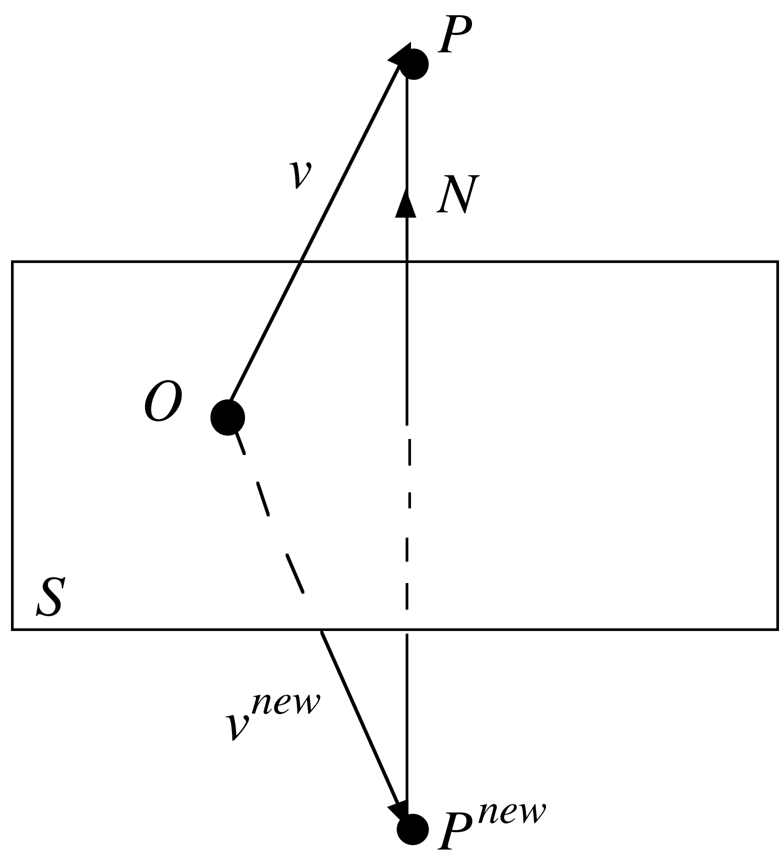
Let

- $q(N, \theta/2) = \cos(\theta/2)O + \sin(\theta/2)N$
- w = *vector in* R^3

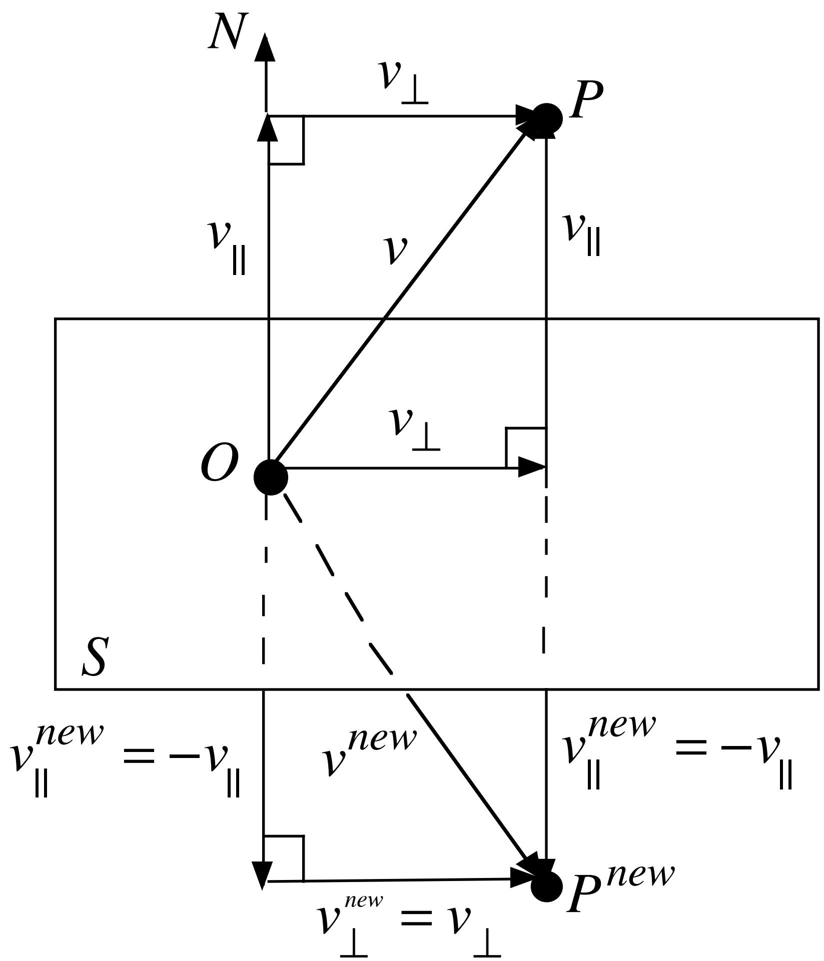
Then

- $q(N, \theta/2)\, w\, q^*(N, \theta/2)$ *rotates* w *by the angle* θ *around the axis* N

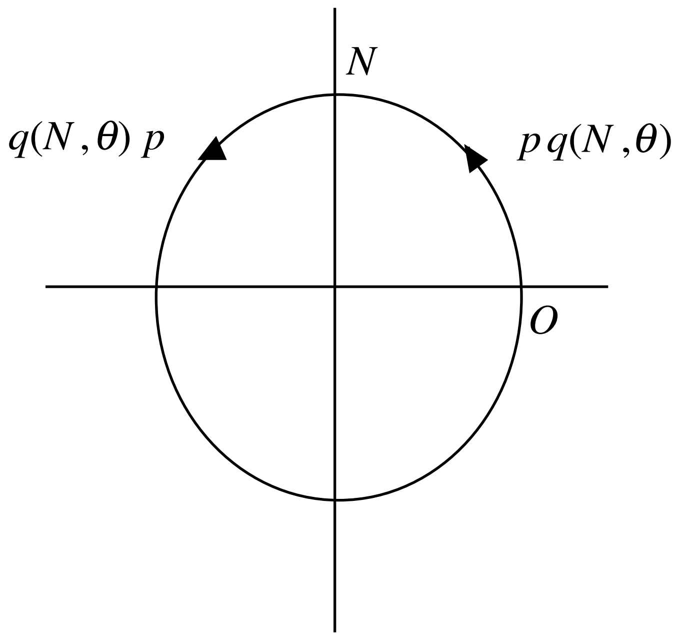
Mirror Image



Mirror Image

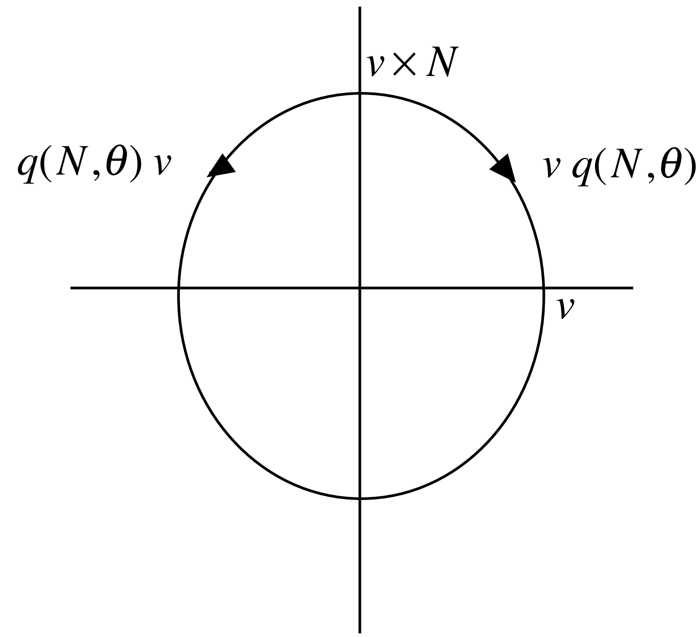


Mirror Image: Sandwiching in Complementary Planes



Plane of O, N

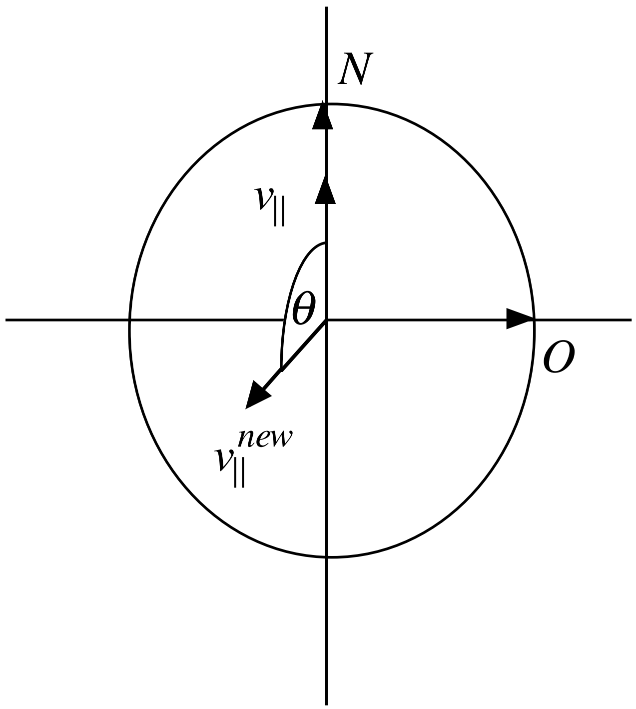
Sandwiching Adds



Plane Perpendicular to O, N

Sandwiching Cancels

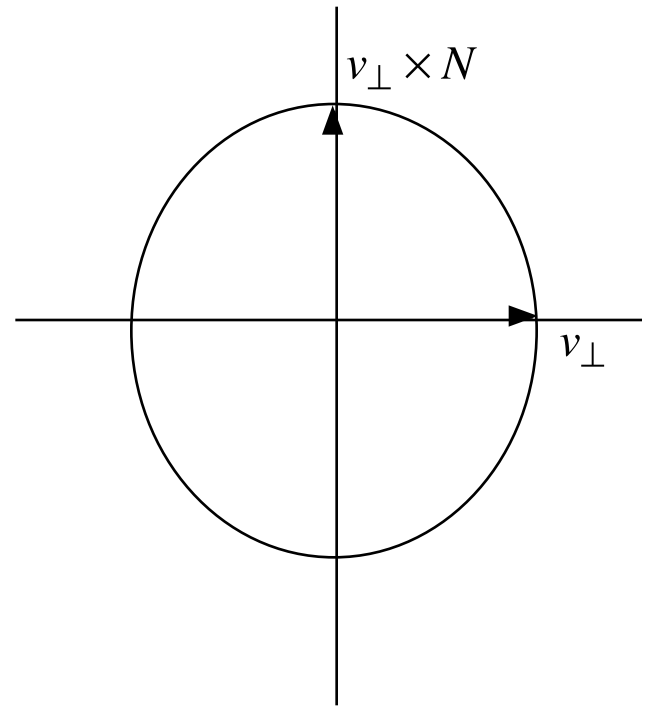
Sandwiching



Plane of O, N

$$q(N,\theta/2)\ v_{||}\ q(N,\theta/2)$$

Rotation by θ



Plane Perpendicular to O, N

$$q(N,\theta/2)\ v_{\perp}\ q(N,\theta/2) = v_{\perp}$$

Identity

Theorem 2: Sandwiching Reflects Vectors in 3–Dimensions

Let

- w *= vector in* R^3

Then

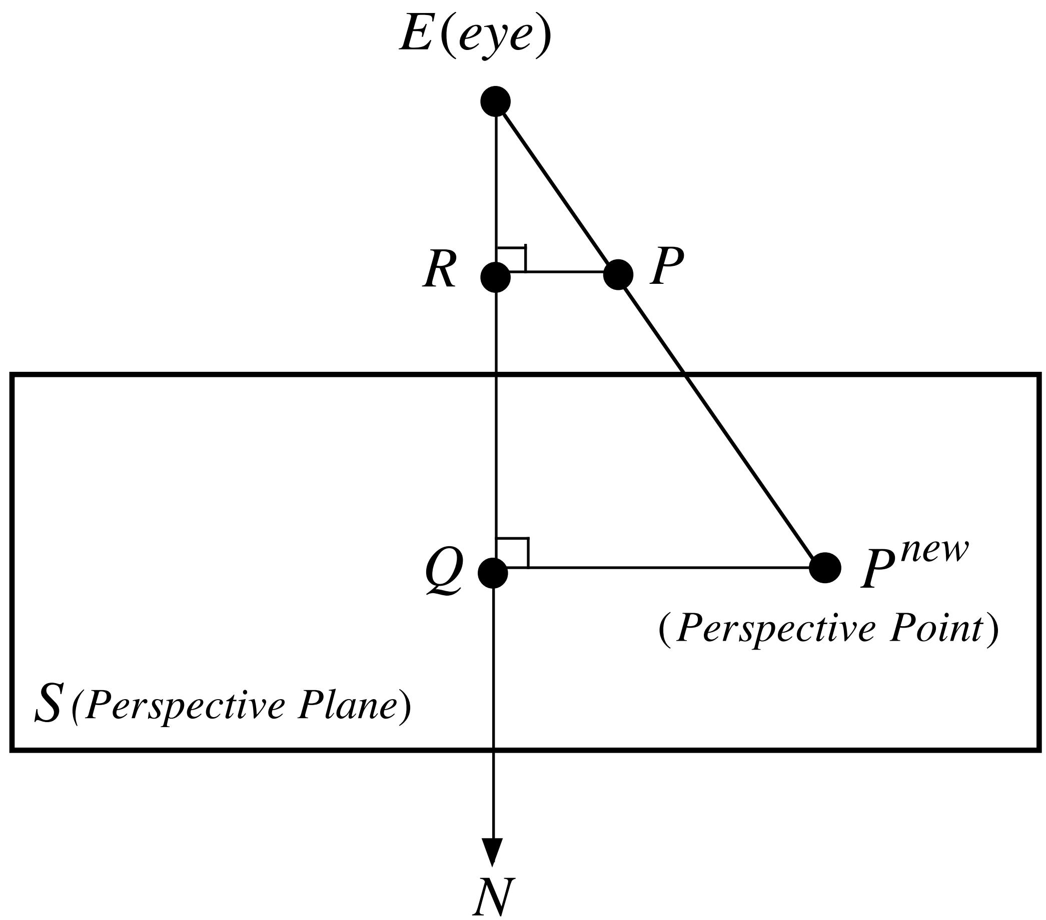
- $N\,w\,N$ *is the mirror image of* w *in the plane* $\perp N$

Proof: Take $\theta = \pi$. Then sandwiching w with:

$$q(N, \pi/2) = \cos(\pi/2) + \sin(\pi/2)N = N$$

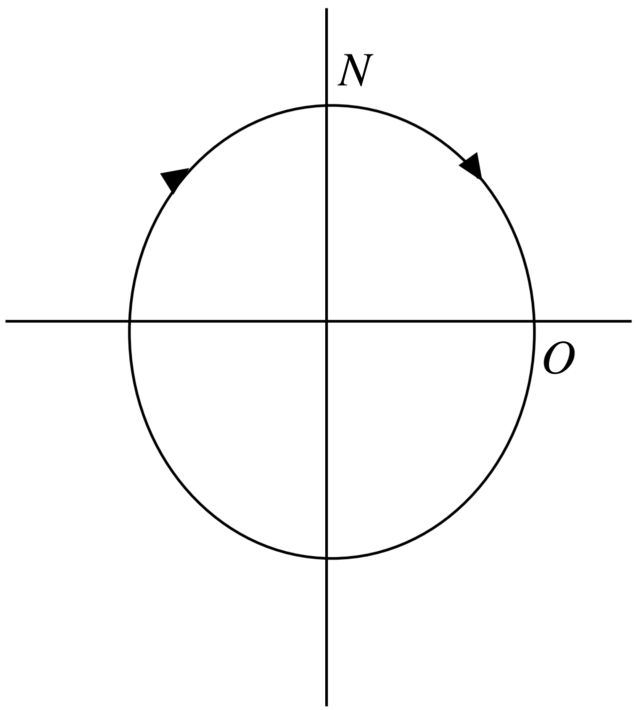
gives the mirror image of w in the plane $\perp N$.

Perspective Projection

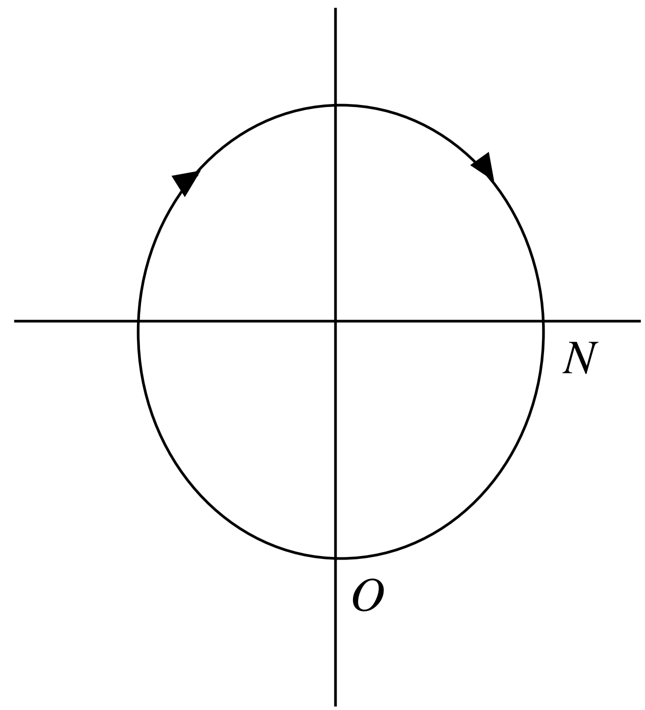


$$\Delta EQP^{new} \approx \Delta ERP$$

<u>Perspective Projection: Sandwiching in the Plane of *O, N*</u>



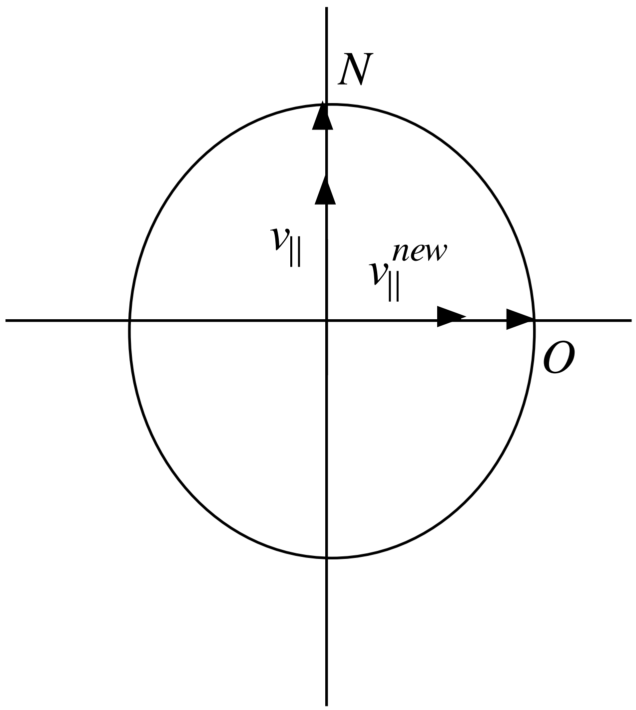
Plane of O, N
Before Sandwiching with $q(N, -\pi/4)$

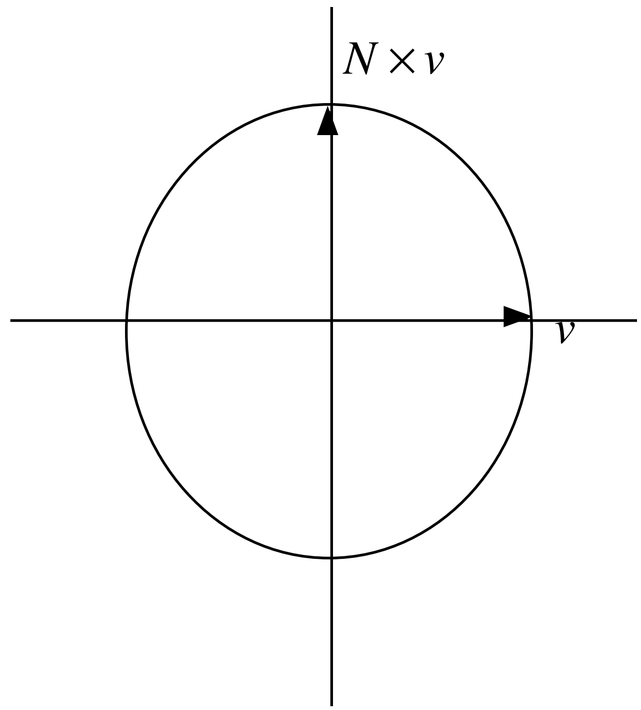


Plane of O, N
After Sandwiching with $q(N, -\pi/4)$

Length along N is mapped to mass at O

Sandwiching





Plane of O, N

$$q(N, -\pi/4)\ d\,N\ q(N, -\pi/4) = d\,O$$

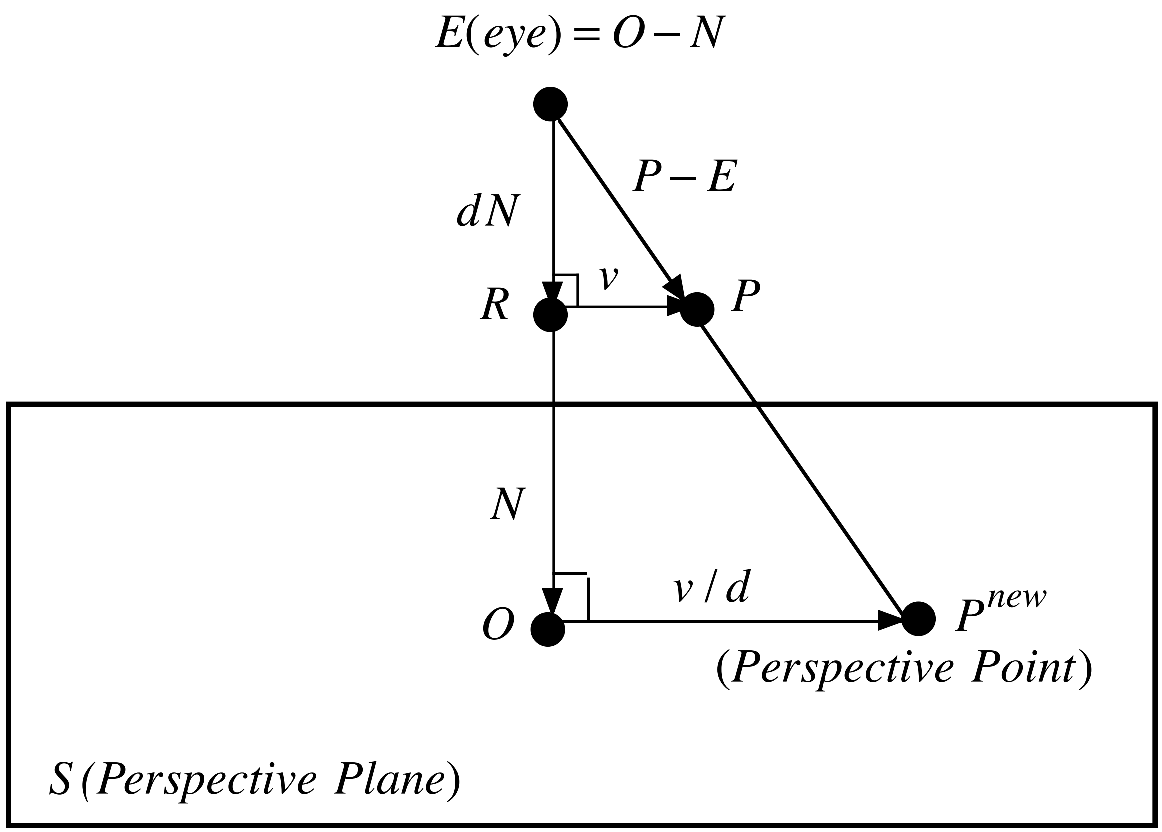
Rotation by $\pi/2$

Plane Perpendicular to O, N

$$q(N, -\pi/4)\ v\ q(N, -\pi/4) = v$$

Identity

Perspective Projection



$$P - E = dN + v \to dO + v \to O + v/d$$

$$\Delta EOP^{new} \approx \Delta ERP$$

Theorem 3: Sandwiching Vectors to the Eye with $q(N,-\pi/4)$ Gives Perspective

Let

- *S = plane through the origin O perpendicular to the unit normal N*
- $E = O - N =$ *eye point*
- *P = point in* R^3

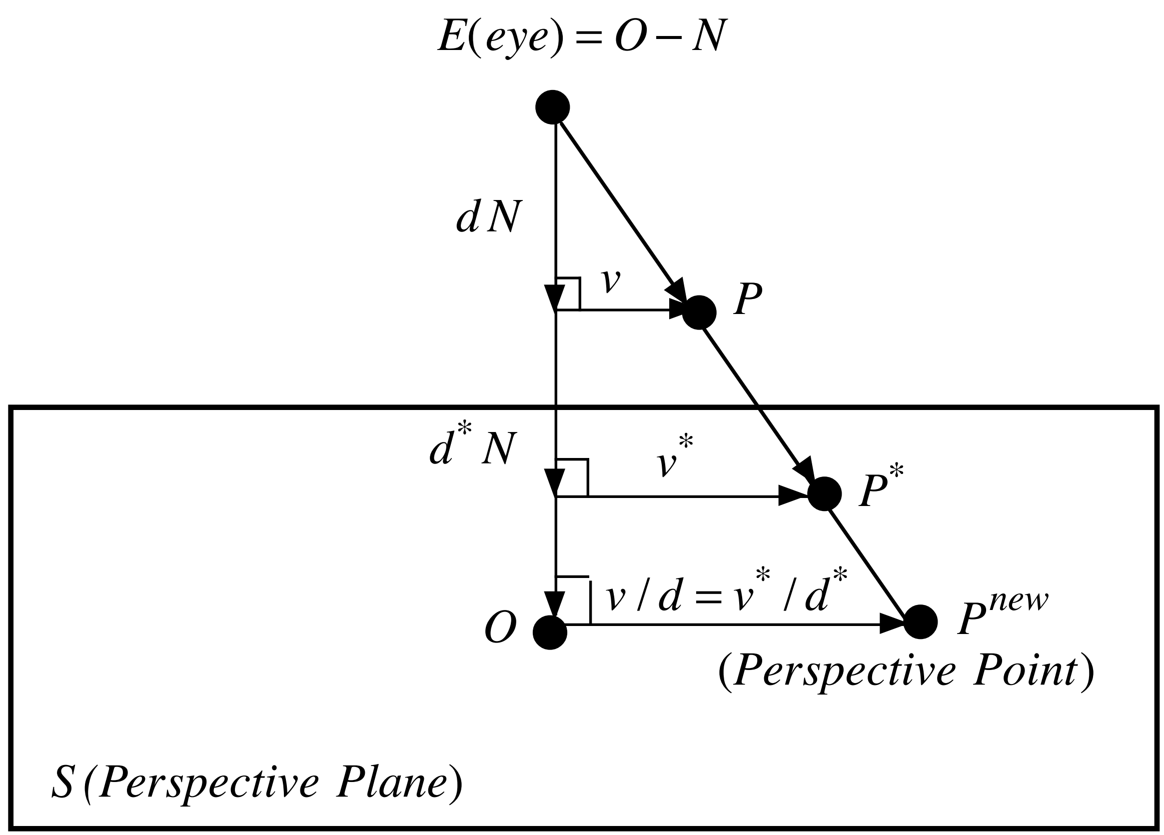
Then

- $q(N,-\pi/4)\,(P-E)\,q(N,-\pi/4)$ *is a mass-point, where:*

  -- *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*

  -- *the mass is equal to the distance d of the point P from the plane through the eye point E perpendicular to the unit normal N.*

Hidden Surfaces



$$d < d^* \Rightarrow P\ obscures\ P^*$$

Summary: Sandwiching with $q(N,-\pi/4)$

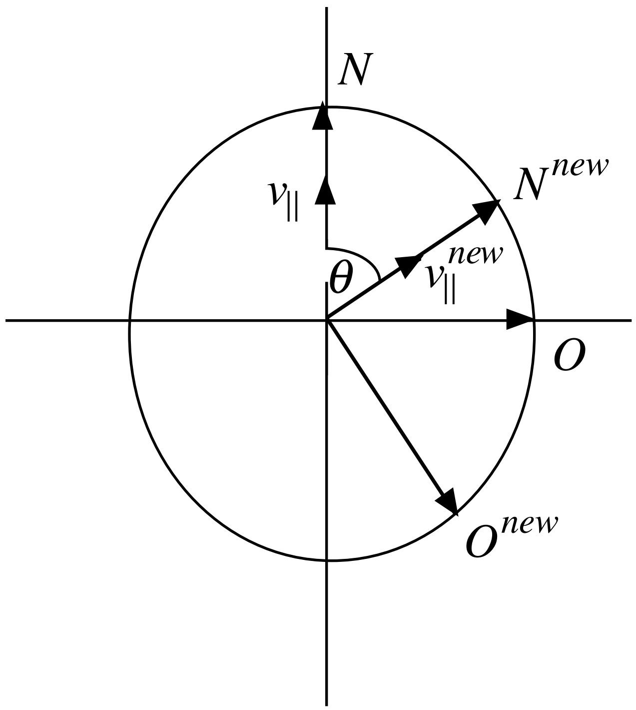
Maps the Vector N to the Point O

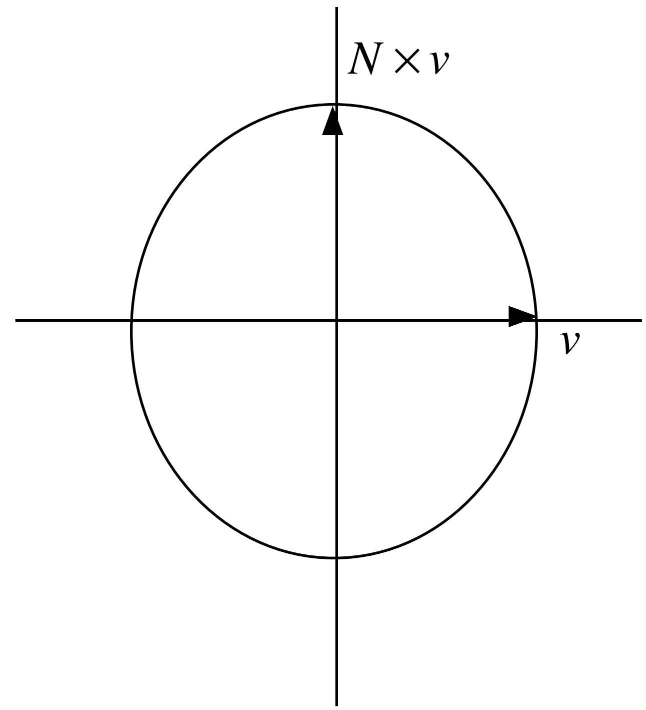
- $q(N,-\pi/4)\ N\ q(N,-\pi/4) = O$
- Projects a Vector to a Point
- Projects Points into a Plane

Converts Distance Along N to Mass at O

- $q(N,-\pi/4)\ dN\ q(N,-\pi/4) = d\,O$
- No Information is Lost
- Hidden Surfaces

Sandwiching





Plane of O, N

$q(N,-\theta/2)\ N\ q(N,-\theta/2) = \sin(\theta)O + \cos(\theta)N$

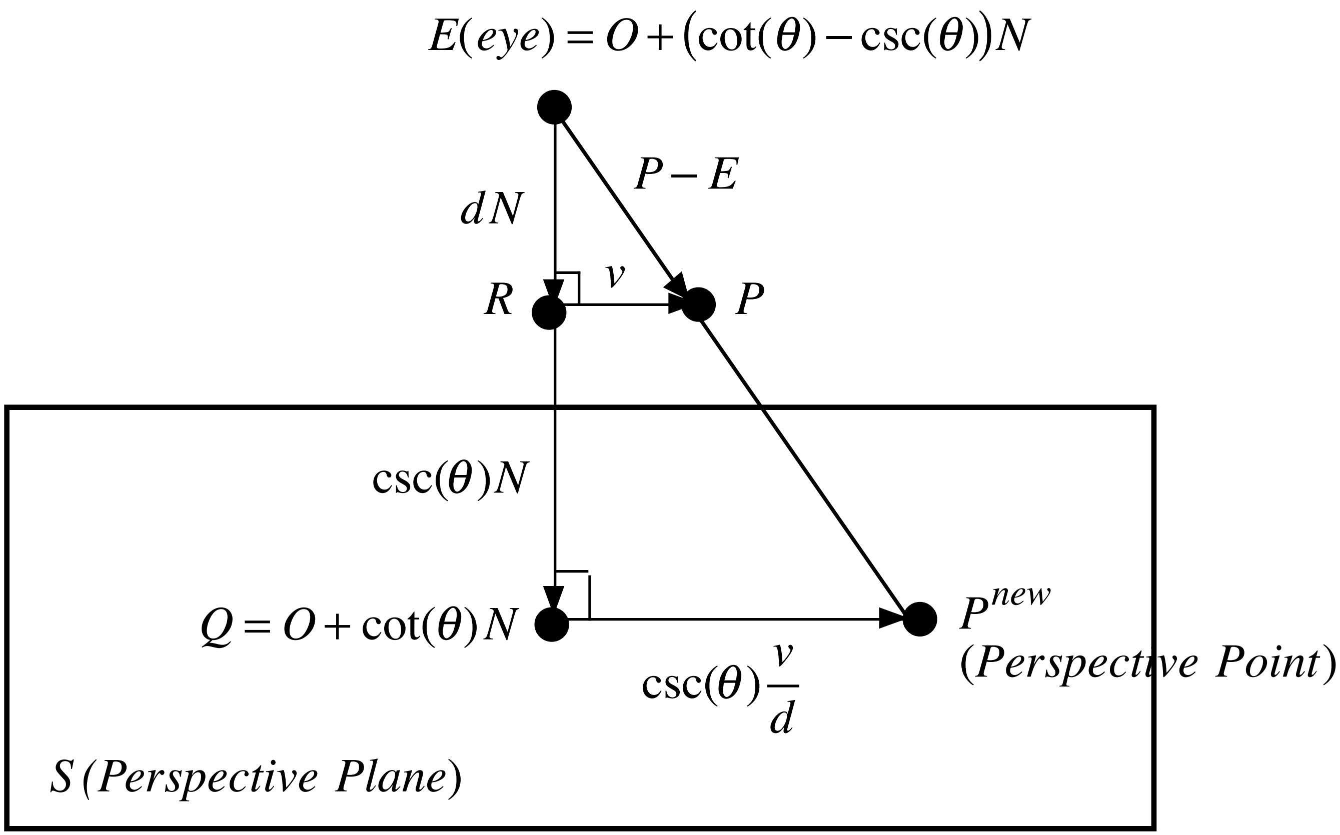
Rotation by $-\theta$

Plane Perpendicular to O, N

$q(N,\theta/2)\ v\ q(N,\theta/2) = v$

Identity

Perspective Projection



$$P - E = dN + v \longrightarrow d\sin(\theta)O + d\cos(\theta)N + v \equiv O + \cot(\theta)N + \csc(\theta)\frac{v}{d}$$

$$\Delta EQP^{new} \approx \Delta ERP$$

Theorem 4: Sandwiching Vectors to the Eye with $q(N,-\theta/2)$ Gives Perspective

Let

- S = *plane through the point* $O+\cot(\theta)N \equiv q(N,-\theta/2)\, N\, q(N,-\theta/2)$ *perpendicular to the unit normal* N

- $E = O + \big(\cot(\theta) - \csc(\theta)\big)N =$ *eye point*

- P = *point in* R^3

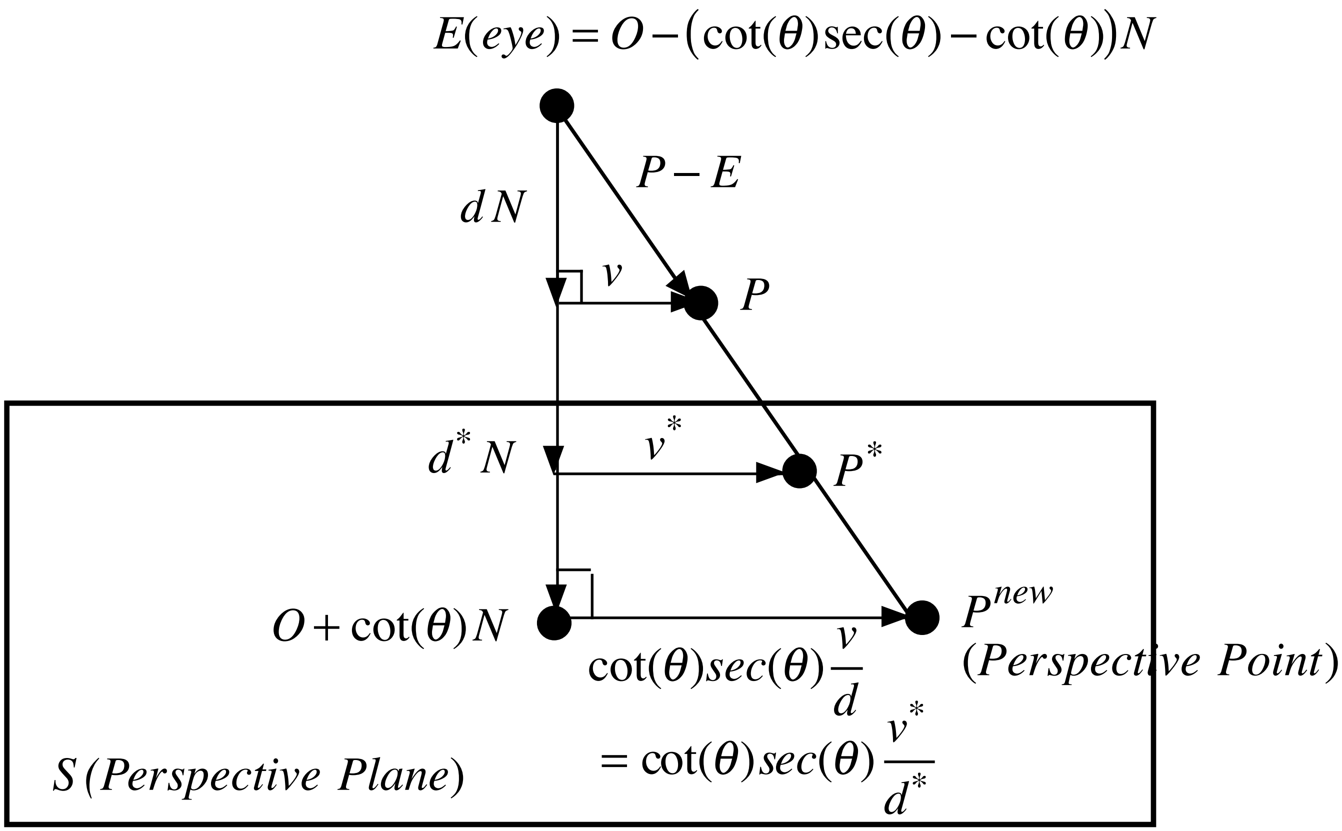
Then

- $q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2)$ *is a mass-point, where:*

  -- *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*

  -- *the mass is equal* $\sin(\theta)$ *times the distance d of the point P from the plane through the eye point E perpendicular to the unit normal N.*

<u>Hidden Surfaces</u>



$$d\sin(\theta) < d^*\sin(\theta) \Leftrightarrow d < d^* \Rightarrow P\ obscures\ P^*$$

Conclusion:

Summary and Insights

<u>**Summary**</u>

Rotations in 4–D

- $p \to q(N,\theta)\,p$ *rotates p by the angle* θ *in the plane of O, N*
- $p \to p\,q(N,\theta)$ *rotates p by the angle* θ *in the plane of O, N*
- $v \to q(N,\theta)\,v$ *rotates v by the angle* θ *in the plane* $\perp$ *O, N*
- $v \to v\,q(N,\theta)$ *rotates v by the angle* $-\theta$ *in the plane* $\perp$ *O, N*

Rigid Motions in 3–D

- Each Rigid Motion in 3–D is the Composite of Two Rotations in 4–D
- $w \to q(N,\theta/2)\,w\,q^{*}(N,\theta/2)$ *rotates w by the angle* θ *around the axis N*
- $w \to N\,w\,N$ *is the mirror image of w in the plane* $\perp$ *N*

<u>Main Insights</u>

Quaternions are Mass-Points

- A quaternion is not merely the sum of a scalar and a vector
- Quaternions have a physical-geometric interpretation compatible with the standard model of space used in contemporary Computer Graphics

Multiplying by a Unit Quaternion Rotates Vectors in 4–Dimensions

The Identity O for Quaternion Multiplication Represents a Fixed Point, the Origin for Points in 3–Dimensions

The Plane of O, N and the Plane Perpendicular to O, N have Special Algebraic and Geometric Properties

Sandwiching is the Fundamental Operation on Quaternions

The Plane of *O*, *N*

The plane of O, N is isomorphic to the complex plane.

- Therefore multiplication by unit quaternions $q(N,\theta)$ rotates quaternions in this plane, and left and right multiplication commute.

The plane of O, N represents a line in 3-dimensions.

- The line through the point *O* parallel to the vector *N*.
- Rotations in 4–dimensions that are the identity on the plane of *O*, *N* are the identity on a line in 3–dimensions.
- Therefore rotations of quaternions that are the identity on the plane of *O*, *N* in 4-dimensions correspond to rotations about a line in 3-dimensions.

Rotations in the plane of O, N correspond to perspective projections in 3-dimensions.

- Rotations of quaternions in the plane of *O*, *N* convert distance into mass.
- Therefore no information is lost, even though these rotations correspond to projections in 3-dimensions. This observation allows us to detect hidden surfaces while performing perspective projection.

The Plane Perpendicular to *O, N*

Multiplication by the unit quaternions $q(N,\theta)$ *on the left represents counterclockwise rotations; multiplication by the unit quaternions* $q(N,\theta)$ *on the right represents clockwise rotations.*

The plane perpendicular to O, N represent a plane of vectors in 3-dimensions

- The plane of vectors perpendicular to *N*.

Rotations in 4-dimensions that are the identity on the plane perpendicular to O, N are the identity on a plane of vectors in 3-dimension.

- Therefore rotations of quaternions that are the identity on the plane perpendicular to *O, N* in 4–dimensions correspond to reflections in and projections onto a plane perpendicular to *N* in 3–dimensions.

Sandwiching

Sandwiching measures the anticommutativity of quaternion multiplication.

We can harness sandwiching to fix points in complementary planes in 4–dimensions.

- Sandwiching a quaternion between a unit quaternion $q(N,\theta)$ and its conjugate $q^*(N,\theta)$ is the identity on quaternions in the plane of *O, N*.

- Sandwiching a quaternion with a unit quaternion $q(N,\theta)$ is the identity on the quaternions in the plane perpendicular to *O, N*.

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