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The Bridge Test for Sampling Narrow Passages with Probabilistic Roadmap Planners

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Abstract

Probabilistic roadmap (PRM) planners have been successful in path planning of robots with many degrees of freedom, but narrow passages in a robot's configuration space create significant difficulty for PRM planners. This paper presents a new sampling strategy in the PRM framework for finding paths through narrow passages. A key ingredient of our strategy is the bridge test, *which boosts the sampling density inside narrow passages. The bridge test relies on simple tests of local geometry and can be implemented efficiently in high-dimensional configuration spaces. The strengths of the bridge test and uniform sampling complement each other naturally. We combine them to obtain the final hybrid sampling strategy. Our planner was tested on point robots and articulated robots in planar workspaces. Preliminary experiments show that the hybrid sampling strategy enables relatively small roadmaps to reliably capture the connectivity of configuration spaces with difficult narrow passages.*

1 Introduction

During the past decade, probabilistic roadmap (PRM) planners [ABD+98, BK00, BOvdS99, HLM99, KŠLO96, NSL99, LK01] have emerged as a powerful framework for path planning of robots with many degrees of freedom (dofs). A classic PRM planner [KŠLO96] samples at random a robot's configuration space to construct a network, called a *roadmap*, that approximates the connectivity of the free space. It then searches the roadmap for a collision-free path between given initial and goal configurations. PRM planners are simple to implement and efficient in practice. As a result, they have found many important applications, including robotics, virtual prototyping, computer animation, and computational biology (see, *e.g.*, [ABG+02, ADS02, HLM99, KL00, LK01, SLvGC01, SLB99]).

Despite the success of PRM planners, path planning for many-dof robots is difficult. Several instances of the problem have been proven to be PSPACE-hard [HJW84, Rei79, SS83]. It is unlikely that random sampling, the key idea behind PRM planners, can overcome such difficulty entirely. Indeed, narrow passages in a robot's configuration space pose significant difficulty for PRM planners. Intuitively a narrow passage is a small region whose removal changes the connectivity of the free space. We can also give formal characterizations [BKL+97, HLM99] using the notion of *visibility sets*. To capture the connectivity of the free space accurately, a PRM planner must sample configurations in the narrow passages. This is difficult, because narrow passages have small volumes, and the probability of drawing random samples from small sets is low.

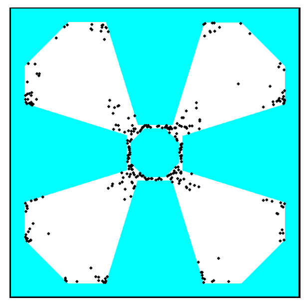


Figure 1. An example of samples generated with the bridge test. In this and all later figures, black dots indicate sampled milestones, and shaded regions indicate obstacles.

In this paper, we propose a new sampling strategy in the PRM framework in order to find paths through narrow passages efficiently. Key to our new strategy is the *bridge test*, which boosts the sampling density inside narrow passages and thus improves the connectivity of roadmaps. In a bridge test, we check for collision at three sampled configurations: the two endpoints and the midpoint of a short line segment s. We accept the midpoint as a new node in the roadmap graph being constructed, if the two endpoints are in collision and the midpoint is collision-free. We call this a bridge test, because the line segment s resembles a bridge: the endpoints of s, located inside obstacles, act as piers, and the midpoint hovers over the free space. For a configuration inside a narrow passage, building *short* bridges through it is easy, due to the geometry of narrow passages; for a configuration in the middle of wide-open free space, doing the same is much more difficult. By favoring short bridges, we increase the chance of accepting configurations inside narrow passages (Figure 1).

The bridge test uses only collision checking as a primitive operation and does not require complex geometric processing in the configuration space. So it can be easily generalized to high-dimensional configuration spaces. It is also simple to implement and runs efficiently.

While being very effective in boosting the sampling density inside narrow passages, the bridge test severely reduces the sampling density in wide-open collision-free regions. This may be undesirable, because an adequate number of nodes are needed in the roadmap to cover the entire free space [BKL+97]. Interestingly the difficulty encountered by the bridge test can be overcome by the uniform sampling strategy, which tends to place many samples in wide-open free space. The strengths of these two strategies complement each other naturally, and we combine them with suitable weights to produce a hybrid sampling strategy to achieve better results. This approach is related to the stratification methods for Monte Carlo integration [KW86].

The difficulty posed by narrow passages and its importance were noted in early work on PRM planners (see, *e.g.*, [KŠLO96]) and were later articulated in [HKL+98]. There are several sophisticated sampling strategies that can alleviate this difficulty, but a satisfactory answer remains elusive. One possibility is to sample more densely near obstacle boundaries [ABD+98, BOvdS99], because configurations inside narrow passages lie close to obstacles. This approach admits a simple, efficient algorithm, the Gaussian sampler [BOvdS99]. However, many configurations near obstacle boundaries lie outside of narrow passages and do not help in improving the connectivity of roadmaps. So despite the improvement, sampling near obstacle boundaries may waste many samples in uninteresting regions. See Figure 3 for a comparison with samples generated with the bridge test. In some special cases, the Gaussian sampler can be extended to reduce the number of wasted samples by paying a higher computational cost [BOvdS99]. Other approaches that can deal with narrow passages include dilating the free space [HKL+98] and retracting to the medial axis of the free space [WAS99]. Both require complex geometric operations that are difficult to implement in high-dimensional configuration spaces. The visibility roadmap [NSL99] is related to the narrow passage problem. It tries to reduce the number of unnecessary samples by checking their visibility.

The rest of the paper is organized as follows. Section 2 gives an overview of our planner. Sections 3 and 4 describe and analyze the bridge test, and show how to combine it with uniform sampling to produce the hybrid sampling strategy. Section 5 reports experiments with our planner on point robots and articulated robots in planar environments. Section 6 discusses alternatives to some choices made in our current planner. Section 7 summarizes the main results and points out direction for future research.

2 Overview of the planner

A classic multi-query PRM planner proceeds in two stages. In the first stage, it tries to construct a roadmap graph G that captures the connectivity of the free space $\mathcal{F}$. The nodes of G are randomly sampled points from $\mathcal{F}$, called *milestones*. There is an edge between two milestones if they can be connected via collision-free canonical paths, typically, straight-line segments. A good roadmap G has two properties. First, the set of milestones in G *covers* the free space well. In other words, for every point $p \in \mathcal{F}$, there is a collision-free straight-line segment between p and a milestone in G with high probability. Second, there is an edge in G between two milestones q and q', if and only if q and q' lie in the same connected component of $\mathcal{F}$. After constructing the roadmap, the planner searches it for a collision-free path between two given query configurations in the second stage. In this paper, we address only the first stage, the roadmap construction. Methods for the second stage are well-known [KŠLO96, ABD+98].

Our goal is to build a good roadmap by sampling a small number of well-placed milestones. The sampling distribution that we use is a weighted mixture of π_{B}, the distribution generated by the bridge test, and π_{U}, the uniform distribution. We describe how to construct π_{B} and combine the two distributions in the next two sections. After generating the milestones, for every pair of milestones close to each other, we check whether a collision-free straight-line segment exists between them. If so, we insert an edge between them into the roadmap.

3 The bridge test

Narrow passages in a free $\mathcal{F}$ are small regions critical in preserving the connectivity of a roadmap built in $\mathcal{F}$. It is difficult to sample in narrow passages because of their small volumes. Any sampling distribution based on the volumes of subsets in $\mathcal{F}$ is likely to fail. In particular, the uniform distribution does not work well. Furthermore, when dealing with a many-dof robot, we do not have an explicit representation of the robot's configuration space $\mathcal{C}$ and cannot locate narrow passages by processing the global geometry of $\mathcal{C}$.

Our bridge test is designed to boost the sampling density within narrow passages using only simple tests of local geometry. It is based on the following observation. A narrow passage in an n-dimensional configuration space has at least one direction v, in which the robot's motion is very restricted. Small perturbation of the robot's configuration along v results in collision with obstacles. The robot is free to move only in those directions perpendicular to v. There-

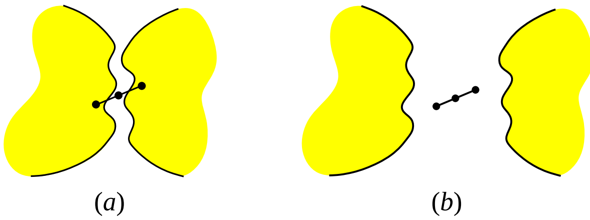


Figure 2. Building short bridges is much easier in narrow passages than in wide-open free space.

fore, for a collision-free point p in a narrow passage, it is easy to sample at random a short line segment s through p such that the endpoints of s lie in obstacles in $\mathcal{C}$ (Figure 2*a*). The line segment s is called a *bridge*, because it resembles a bridge across the narrow passage, with the endpoints of s acting as piers and the point p hovering over the free space. We say that a point $p \in \mathcal{F}$ passes the bridge test, if we succeed in building a bridge through p. A sampled point in $\mathcal{F}$ is accepted as a milestone in our roadmap only if it passes the bridge test.

Clearly building *short* bridges is much easier in narrow passages than in wide-open free space (Figure 2). By favoring short bridges over longer ones, we increase the number of points accepted from the narrow passages.

Sampling milestones. To sample a new milestone using the bridge test, we first pick a line segment s from $\mathcal{C}$ at random by choosing its two endpoints and then determine whether s passes the bridge test. If so, we insert the midpoint of s into the roadmap G as a new milestone. The details are shown below in Algorithm 1, which is called Randomized Bridge Builder (RBB). RBB uses the function CLEARANCE to determine whether a point is collision-free.

Algorithm 1 Randomized Bridge Builder (RBB).

repeat
  Pick a point x from $\mathcal{C}$ uniformly at random.
  if CLEARANCE(x) returns FALSE **then**
    Pick a point x' in the neighborhood of x according to a suitable probability density λ_x.
    if CLEARANCE(x') returns FALSE **then**
      Set p to be the midpoint of line segment $\overline{xx'}$.
      if CLEARANCE(p) returns TRUE **then**
        Insert p into G as a new milestone.

To perform the bridge test, RBB uses only one geometric primitive, CLEARANCE, which can be implemented very efficiently using a collision detection algorithm (see, *e.g.*, [Qui94, GLM96]). The bridge test is purely local and does not require processing the global geometry of $\mathcal{C}$.

Choosing the probability density λ. The density λ determines how frequently a bridge of a particular length and orientation is chosen. We prefer short bridges over longer ones in order to increase the probability of sampling in narrow passages. Let us assume, for now, that there is no reason to prefer one orientation over another. Based on these considerations, we choose λ_x to be a radially symmetric Gaussian with its center at x and a small standard deviation σ. If we have *a priori* information on the narrow passages, then there may be other distributions more advantageous than the radially symmetric Gaussian. See Section 6 for further discussion.

Analysis of the sampling distribution. One may be curious: what does π_{B}, the probability density created by RBB, look like? To calculate π_{B}, let us first define X and X' to be two random variables, representing respectively the two endpoints of a chosen bridge. By construction, the first endpoint X is distributed uniformly over $\mathcal{B}$, where $\mathcal{B} = \mathcal{C} \backslash \mathcal{F}$. So the density $f_X(x)$ is non-zero if and only if x lies in $\mathcal{B}$. Assume, without loss of generality, that $\mathcal{B}$ has volume 1. Then $f_X(x)$ is 1 if $x \in \mathcal{B}$ and 0 otherwise. Given $X = x$, we choose the other endpoint X' according to the density λ_x. The point X' is accepted only if it lies in $\mathcal{B}$. Let I be a binary function such that for any point $p \in \mathcal{C}$, $I(p) = 1$ if $p \in \mathcal{B}$ and 0 otherwise. The conditional density of X' given X is given by

$$f_{X'|X}(x' \mid x) = \lambda_x(x')I(x')/Z_x,$$

where $Z_x = \int_{\mathcal{C}} \lambda_x(x')I(x')\,dx'$ is a normalizing constant. To calculate the density π_{B} at a point $p \in \mathcal{F}$, we condition on X:

$$\pi_{\text{B}}(p) = \int_{\mathcal{C}} f_{X'|X}(x' \mid x) f_X(x)\,dx. \qquad (1)$$

Note that p is the midpoint of the line segment $\overline{xx'}$ and so $x' = 2p - x$. Substituting the expressions for f_X, $f_{X'|X}$, and x' into (1), we have

$$\pi_{\text{B}}(p) = \int_{\mathcal{B}} \lambda_x(2p - x)I(2p - x)/Z_x\,dx. \qquad (2)$$

Recall that λ_x is chosen to be a Gaussian with its center at x and a small standard deviation. The density λ_x is large if $x' = 2p - x$ lies close to x. Furthermore, the integrand in (1) is non-zero only if $I(2p - x) = 1$, *i.e.*, $x' \in \mathcal{B}$. For a point p in a narrow passage, both conditions are easily satisfied, resulting in a large value for π_{B} at p.

Comparison with sampling near obstacle boundaries. RBB is related to the Gaussian sampler [BOvdS99]. Both use one simple geometric primitive CLEARANCE to create favorable distributions. Their objectives, however, are quite different. RBB increases the sampling density inside narrow passages; the Gaussian sampler increases the sampling density near obstacle boundaries. RBB is slightly more expensive: it makes one more call to CLEARANCE per sample than the Gaussian sampler. However, by focusing on narrow passages, RBB gains efficiency by avoiding sampling

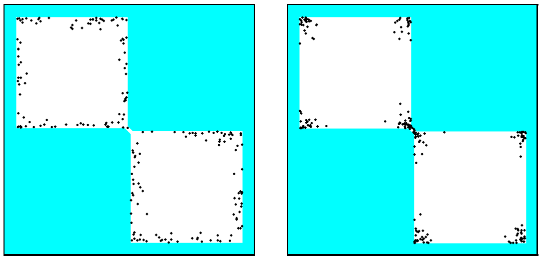


Figure 3. The samples generated by (*a*) the Gaussian sampler and (*b*) RBB.

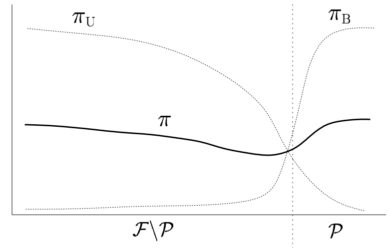


Figure 4. The hybrid sampling distribution. The distributions π_B and π_U perform well on $\mathcal{P}$ and $\mathcal{F}\backslash\mathcal{P}$, respectively. Combining them with suitable weights leads to good performance over the entire sampling domain.

uninteresting obstacle boundaries that do not contribute to improving the connectivity of roadmaps. See Figure 3 for a comparison between these two sampling strategies. In some special cases, an extension of the Gaussian sampler can reduce the number of wasted samples by picking a triple instead of a pair of points and checking that two of the three picked points lie in *different* obstacles [BOvdS99]. It is unclear how well this extension works in general, *e.g.*, in a narrow passage formed by one single non-convex obstacle.

The idea of sampling near obstacle boundaries fails when the boundaries are uninteresting. The bridge test may fail, too, though much less often. This happens when $\mathcal{F}$ contains sharp corners, because close to the tips of corners, it is easy to build short bridges. In Figure 3(*b*), RBB generated a number of milestones in the six corners of $\mathcal{F}$. These samples are largely unhelpful. Nonetheless, our experiments suggest that the benefits gained by sampling in narrow passages outweigh the computation time wasted on sampling near the sharp corners (see Section 5).

4 Combining complementary sampling distributions

We have seen in Section 3 that the bridge test is effective in boosting the sampling density in $\mathcal{P}$, the subset of $\mathcal{F}$ occupied by narrow passages. The density π_B is heavily biased to $\mathcal{P}$. At the same time, π_B penalizes wide-open collision-free regions: few points are sampled in $\mathcal{F}\backslash\mathcal{P}$. This may be undesirable, because a good roadmap must contain enough milestones to cover the whole free space adequately.

Interestingly we can make up the deficiency of π_B with the uniform distribution π_U, which samples $\mathcal{F}$ with probability proportional to the volumes of subsets in $\mathcal{F}$. For π_U, most samples fall into $\mathcal{F}\backslash\mathcal{P}$. The two sampling distributions complement each other: π_U provides good coverage of $\mathcal{F}\backslash\mathcal{P}$, and the distribution π_B samples more densely in $\mathcal{P}$ and thus improves the connectivity of the roadmap. We combine π_B and π_U with suitable weights to produce our final sampling distribution:

$$\pi = (1 - w) \cdot \pi_B + w \cdot \pi_U, \tag{3}$$

where w is a weight, with $0 \leq w \leq 1$. The choice of w depends on the difficulty of sampling in narrow passages and the number of milestones needed to cover $\mathcal{F}$. Currently we do not have an automatic way of selecting w and set it by hand to give some advantage to π_B, because we assume that $\mathcal{F}$ contains at least some difficult narrow passages.

One fruitful way of thinking about this hybrid distribution π is to divide the free space $\mathcal{F}$ into two subsets, the narrow passages $\mathcal{P}$ and its complement $\mathcal{F}\backslash\mathcal{P}$. We use a different sampling strategy tailored to each subset to achieve good performance over the entire sampling domain. See Figure 4 for an illustration. This approach is related to the stratification methods for Monte Carlo integration [KW86] and the multiple-importance sampling for ray-tracing photo-realistic images [VG95].

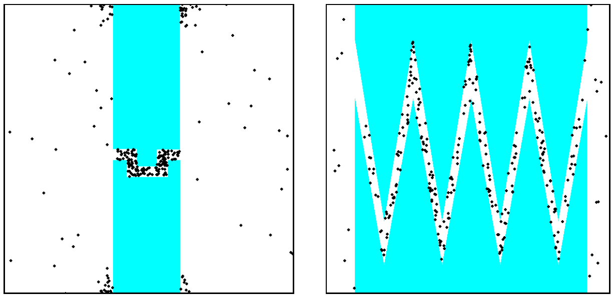
The significance of a hybrid distribution is not about putting together two distributions, but rather about identifying distributions that complement each other's strengths and combining them so that their individual strengths are preserved. Our approach differs from that of the previous work (*e.g.*, [DA01]) in that the two sampling distributions π_B and π_U naturally complement each other. No computation is necessary to explicitly decompose the sampling domain.

To implement the hybrid distribution, we can certainly generate new random points from π_U, but actually we can get at least some of these points "for free" by reusing the points rejected from line 3 of Algorithm 1.

5 Implementation and Experiments

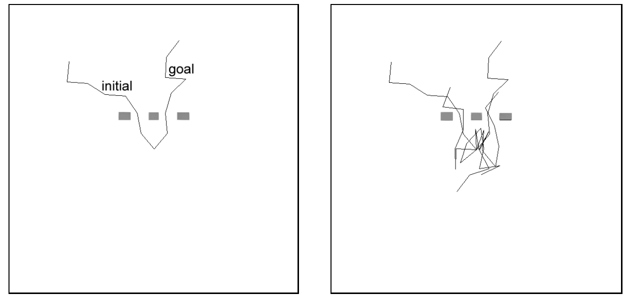
To test the hybrid sampling strategy, we applied it to both a point robot and a seven-dof articulated robot in planar environments. Preliminary experiments indicate that our planner is able to efficiently capture the connectivity of free spaces containing difficult narrow passages.

Implementation details. There are two parameters that we need to choose in order to fix our hybrid sampling strategy. First, recall that in RBB, the density λ is chosen to be a Gaussian with a small standard deviation σ to bias towards sampling short bridges. In our experiments, we set σ to be roughly 10% of the allowable range of motion. Making σ too small may adversely impacts the performance of



(*a*) (*b*)

Figure 5. Environments used for testing our planner.



(*a*) (*b*)

Figure 6. A 7-dof planar articulated robot with a fixed base. (*a*) Initial and goal configurations. (*b*) Some intermediate configurations for a computed path.

the planner. The reason is that if a bridge is too short, the second endpoint of the bridge would be unable to cross the narrow passage and would fall in $\mathcal{F}$, thus causing many potentially useful points to fail the bridge test. The second parameter is the weight for combining π_{B} and π_{U}. We use the ratio 5:1 in favor of π_{B}, meaning that for every five milestones generated from π_{B}, we pick one milestone from π_{U}.

Our program was implemented in Java, and the results reported below were acquired from a PC with a 1.8GHz Pentium 4 processor.

Experimental results. We first tested our planner on a point robot in four planar environments, containing different kinds of narrow passages. Environment A (Figure 1) consists of four chambers connected by multiple narrow corridors. To go from one chamber to the diagonally opposite one, the point robot must pass through two narrow corridors. Environment B (Figure 3) contains a very narrow and short corridor that connects two large square chambers. The corridor in environment C (Figure 5*a*) is longer and has multiple turns. So each milestone in the corridor has low visibility and covers only a small portion of the free space. Environment D (Figure 5*b*) contains a very long narrow corridor. It illustrates an interesting scenario in which RBB and the Gaussian sampler behave similarly, because every point in the narrow passages is also close to obstacle boundaries and *vice versa*.

We also performed preliminary experiments with a planar articulated robot with seven dofs. Figure 6 shows the test environment E as well as a computed path. At both the initial and goal configurations, the robot is trapped in a narrow opening and must execute difficult maneuvers in order to find a path.

For environments A–D, we generated 30 random queries that require the point robot to go through the narrow passages. For environment E, we handpicked the query. We then performed 10 independent runs for each query. We terminated the planner as soon as a path was found between query configurations, and recorded the running times and other statistics. For the purpose of comparison, we performed the same experiments with three sampling strategies: uniform sampling, pure RBB (without mixing with uniform sampling), and hybrid sampling. For pure RBB, we enlarged the standard deviation of the Gaussian to be 20% of the allowable range in order to cover the free space adequately. The results for each environment and sampling strategy are averaged and reported in Table 1.

Table 1. Performance statistics of different sampling strategies. Note that the running times were acquired from a Java implementation. So the relative performance is more important than the absolute values of running times.

Env.	Sampler	N_{mil}	N_{clear}	N_{con}	Time (sec.)
A	uniform	550	1327	2220	0.56
	RBB	125	2797	517	0.06
	hybrid	54	3969	243	0.06
B	uniform	657	1676	2604	0.76
	RBB	45	2770	149	0.03
	hybrid	36	6647	104	0.04
C	uniform	493	832	2004	0.18
	RBB	128	5619	504	0.08
	hybrid	81	8644	396	0.06
D	uniform	205	607	1242	0.04
	RBB	265	4283	1576	0.12
	hybrid	207	4062	1545	0.09
E	uniform	11876	31197	75260	1112
	RBB	4037	91067	21002	203
	hybrid	3574	123701	16301	173

N_{mil} : number of milestones in the resulting roadmap
N_{clear}: number of calls to CLEARANCE
N_{con} : number of calls to check collision-free connection between two milestones

Table 1 shows consistent results from experiments on different robots and environments. Typically the hybrid sampling is the best performer in terms of running times. To obtains milestones, it rejects more samples and makes more calls to CLEARANCE; however, it produces a small roadmap with well-placed milestones and thus greatly reduces the number of checks that a pair of milestones can be connected via collision-free straight-line segments. In general, such a connection check is much more expensive than a call to CLEARANCE. So hybrid sampling was able to achieve good overall performance. Pure RBB is slightly inferior to hybrid sampling because RBB takes longer to cover the free space adequately. The only exception to our general observations is environment D. As expected, all three sampling strategies generated roadmaps of comparable sizes to answer the queries. However, since hybrid sampling and RBB made more calls to CLEARANCE, they took slightly longer.

6 Discussion

Several issues in the design of the bridge test worth careful thought, in particular, the choice of the density function λ.

So far we have assumed that λ is a radially symmetric Gaussian, which works well if a robot's dofs are all symmetric, *e.g.*, a point robot. The symmetry breaks down on free-flying rigid-bodies and articulated robots, for which each dof must be scaled to reflect its influence on the global movement of the robot. To deal with the problem, we assign a different Gaussian standard deviation for each dof. The resulting density function is still a Gaussian, but no longer radially symmetric. This simple extension has already been implemented for our planar articulated robots.

Furthermore, the Gaussian density function peaks at 0, which means that the narrower a passage is, the more likely it will be sampled. This is reasonable if we have no prior information on narrow passages at all. In practice, we often have some estimates. For instance, if a robot is stuck in a narrow corridor, it has a rough idea of how much room there is to maneuver based on its knowledge of the environment. Let us assume that the width of the narrow passage is roughly c. Then it is useless to sample bridges much shorter than c, because the bridge test is bound to fail, as explained in Section 5. Instead, λ should peak at roughly c and quickly decrease to 0 as the length of the bridge gets shorter.

7 Conclusion and future work

We have presented a new sampling strategy in the PRM framework for finding paths through narrow passages. A key ingredient of the new strategy is the bridge test, which boosts the sampling density inside narrow passages. The bridge test makes use of one geometric primitive, which checks whether a configuration is collision-free. It is purely local and can be implemented efficiently in high-dimensional configuration spaces. The strengths of the bridge test and the uniform sampling are complementary. We combine them to produce a hybrid sampling strategy that generates small roadmaps that cover the free space well and have good connectivity. In our preliminary test on point robots and seven-dof articulated robots in planar environments, our planner was able to reliably capture the connectivity of free spaces with difficult narrow passages.

Several interesting issues regarding the bridge test and the hybrid sampling strategy require further exploration.

First, we are conducting additional experiments with high-dof planar articulated robots to better understand the strength and weakness of our algorithm. We are also implementing the algorithm for free-flying rigid bodies in 3-D workspaces. Based on our experience with PRM planners and that of other researchers, we are confident that our new algorithm will perform well in 3-D workspaces.

Second, the bridge test sometimes produces false positives, as shown by the milestones sampled near sharp corners of the free space. A heuristic that seems to work well in our experiments is to try building an additional bridge orthogonal to the one that already exists. Failing such an orthogonal bridge test indicates that the sampled point is likely near a sharp corner rather than in a narrow passage. We are exploring more rigorous ways to eliminate these false positives.

Finally, we would like to further develop the hybrid sampling strategy. The important issues here are to identify sampling distributions that naturally complement each other and do not require explicit decomposition of the sampling domain and to find a systematic weight assignment method that preserves the strengths of individual distributions.

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