Design & Analysis of Algorithms
COMP 482 / ELEC 420

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Math Background: Review & Beyond

1. Asymptotic notation
2. Math used in asymptotics
3. Recurrences
4. Probabilistic analysis

To do:
[CLRS] A, B, 3.2 #1

Comparisons

Some useful algebra: a<b, c<d implies…

- -b < -a
- 1/b < 1/a – Unless a,b have different signs!
- a+c < b+c
- ac < bc – If c>0
- a+c < b+d – But not necessarily a-c < b-d or a-c > b-d !
- $a^n < b^n$ – If a,b>0

Floor & Ceilings

$$a-1 < \lfloor a \rfloor \le a \le \lceil a \rceil < a+1$$ ∀ a>0

$$\frac{a-(b-1)}{b} \le \left\lfloor \frac{a}{b} \right\rfloor \le \frac{a}{b} \le \left\lceil \frac{a}{b} \right\rceil \le \frac{a+(b-1)}{b}$$ ∀ integer a>0, ∀ b≥1

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lceil \frac{a}{2} \right\rceil = a$$ ∀ integer a

Logarithm Notation

$\lg n = \log_2 n$ $\ln n = \log_e n$

$\lg^k n = (\lg n)^k$
$\lg \lg n = \lg (\lg n)$
$\lg n + k = (\lg n) + k$

$\lg^{(0)} n = n$ $\lg^{(i+1)} n = \lg \lg^{(i)} n$

$\lg^* n = \min \{i \ge 0 \mid \lg^{(i)} n \le 1\}$
How many times can you take the logarithm before it's ≤ 1?

Exponentials & Logarithms

$a^{\log_b c} = c^{\log_b a}$ —c=b→ $a = b^{\log_b a}$

$\log_a x = (\log_b x)(\log_a b)$ —x=a→ $1 = (\log_b a)(\log_a b)$
$O(\log_a x) = O(\log_b x) = O(\log x)$

$\log_b^c n = o(n^a)$ ∀a>0

Summations

Arithmetic $\sum_{i=0}^{n} i =$?

Geometric $\sum_{i=0}^{n} x^i =$? if $x \neq 1$

$\sum_{i=0}^{\infty} x^i =$? if $|x| < 1$

Harmonic $\sum_{i=1}^{n} \frac{1}{i} =$?

Telescoping $\sum_{i=0}^{n-1} (a_i - a_{i+1}) =$?

Solve as telescoping sum:

$$\sum_{i=1}^{n-1} \frac{1}{i \times (i+1)} = ?$$

$$= \sum_{i=1}^{n-1} \left(\frac{i+1}{i \times (i+1)} - \frac{i}{i \times (i+1)} \right)$$

$$= \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$= 1 - \frac{1}{n}$$

7

Summations – Combining with Calculus

Have: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ if $|x| < 1$

Differentiate: $\sum_{i=0}^{\infty} i \times x^{i-1} = \frac{1}{(1-x)^2}$ if $|x| < 1$

Algebra: $\sum_{i=0}^{\infty} i \times x^{i} = \frac{x}{(1-x)^2}$ if $|x| < 1$

8

Bounding Summations: Upper Bound

$$\sum_{i=1}^{n} \lg i = \quad ?$$

$$\sum_{i=1}^{n} \lg i \leq \sum_{i=1}^{n} \lg n = n \lg n$$

$$\sum_{i=1}^{n} \lg i = O(n \lg n)$$

9

Bounding Summations: Lower Bound

$$\sum_{i=1}^{n} \lg i \geq ?$$

10

Logarithms

More generally,

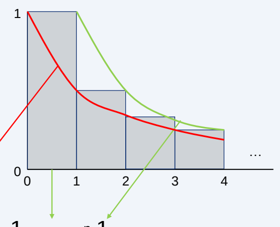
$$\sum_{i=0}^{n} i^c (\log i)^d = \Theta\left(n^{c+1} (\log n)^d\right)$$

$$\forall c, d > 0$$

11

Bounding Summations by Integrals

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1)$$



$$\ln(n+1) = \int_1^{n+1} \frac{1}{x} dx \leq \sum_{i=1}^{n} \frac{1}{i} \leq 1 + \int_1^{n} \frac{1}{x} dx \quad = (\ln n) + 1$$

Recall: ln 1 = 0

12

Factorials

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$n! = \omega(2^n)$

$n! = o(n^n)$

$\log n! = \Theta(n \log n)$

13

Permutations & Combinations

of permutations of an n-set = n!

# of k-combinations of an n-set = $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

14