Design & Analysis of Algorithms
COMP 482 / ELEC 420

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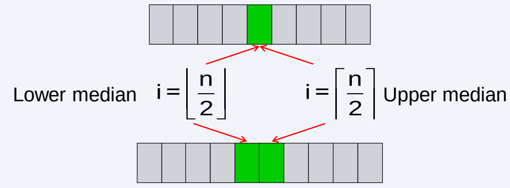
Order Statistics: Select i^{th}-ranked item

Least: $i = 1$
Greatest: $i = n$

<u>To do:</u>
[CLRS] 9 #3



Assume collection is unordered, otherwise trivial.

2

Simple Algorithms

Could sort first: $O(n \lg n)$, but can do better: $O(n)$.

What are algorithms for i=1, 2, 3?

How do these generalize?

3

Modify Quicksort

First, to get <u>average-case</u> O(n).

Then, to get <u>worst-case</u> O(n).
With a very unobvious detail!

4

Blum, Floyd, Pratt, Rivest, Tarjan (1973)

Select(A,n,i):	<u>T(n)</u>	
Divide input into $\lceil n/5 \rceil$ groups of size 5.	O(n)	All this to find a good split.
/* Partition on median-of-medians */		
medians = array of each group's median.	O(n)	
pivot = Select(medians, $\lceil n/5 \rceil$, $\lceil n/10 \rceil$)	$T(\lceil n/5 \rceil)$	
L,G = partition(A, pivot)	O(n)	
/* Find i^{th} element in L, pivot, or G */		
k = # of lesser elements + 1	O(1)	
If i=k, return pivot	O(1)	Only one done.
If i<k, return Select(L, k-1, i)	T(k-1)	
If i>k, return Select(G, n-k, i-k)	T(n-k)	

5

Analysis

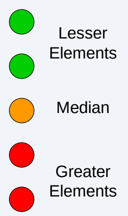
#less #greater

$$T(n) = T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T(\max(k-1, n-k)) + O(n)$$

How to simplify?

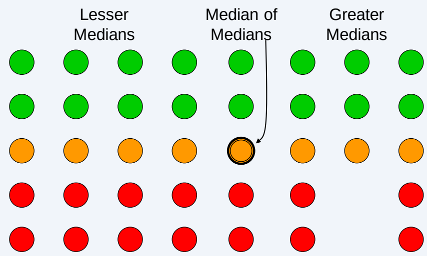
6

Groups of 5



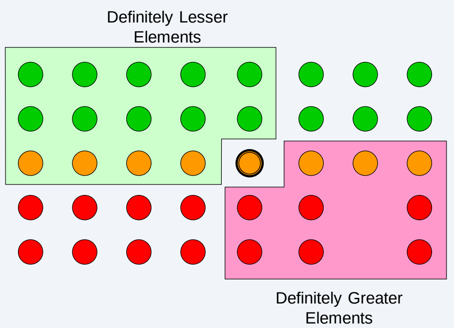
One group of 5 elements.

Groups of Groups of 5

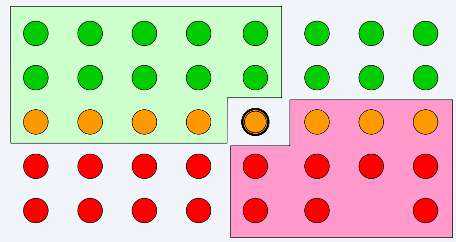


All groups of 5 elements.
(And at most one smaller group.)

Elements Not Needed

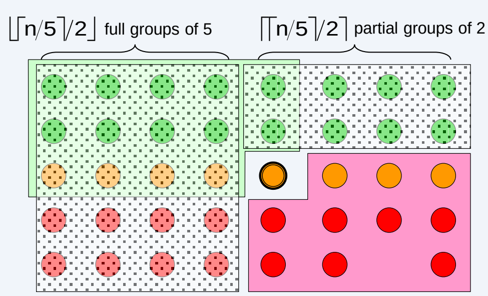


1st Way to Count



Must recur on all elements <u>outside</u> one of these boxes.
How many?

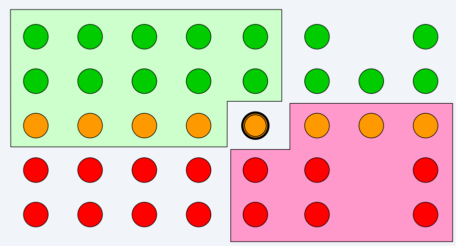
1st Way to Count



Count elements outside smaller box.

At most $5\left\lfloor \left\lceil \frac{n}{5} \right\rceil \Big/ 2 \right\rfloor + 2\left\lceil \left\lceil \frac{n}{5} \right\rceil \Big/ 2 \right\rceil \le \frac{7n}{10} + 7$

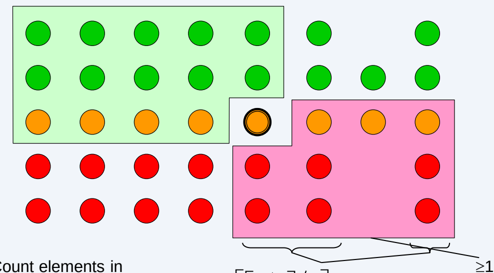
2nd Way to Count



Equivalently, must recur on all elements <u>not inside</u> one of these boxes.
How many?

2nd Way to Count

At most $n - \left(3\left(\left\lceil \frac{\lceil \frac{n}{5} \rceil}{2} \right\rceil - 1\right) + 1\right) \le \frac{7n}{10} + 2$



Count elements in smaller box & pivot.

$\lceil \lceil n/5 \rceil / 2 \rceil - 1$ groups of 3

Plugging Count Back into Recurrence

$$T(n) = T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 2\right) + O(n)$$

Using Substitution Method

Prove $\exists c, n_0 > 0, T(n) \le cn, \forall n \ge n_0$

$$T(n) \le c\left\lceil \frac{n}{5} \right\rceil + c\left(\frac{7n}{10} + 2\right) + kn$$

$$\le c\left(\frac{n}{5} + 1\right) + c\left(\frac{7n}{10} + 2\right) + kn$$ Overestimate ceiling

$$= \frac{9}{10}cn + 3c + kn$$ Algebra

$$\le cn$$ when $0 \le \frac{1}{10}cn - 3c - kn$

$\forall$ c,k, can find a n_0 such that this holds $\forall n \ge n_0$.

Why Groups of 5?

$$T(n) = T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 2\right) + O(n)$$