Design & Analysis of Algorithms
COMP 482 / ELEC 420

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Flow networks

What if weights in a graph are maximum capacities of some flow of material?

- Pipe network to transport fluid/gas (e.g., water, oil, natural gas, CO_2)
  - Edges – pipes
  - Vertices – junctions of pipes
- Data communication network
  - Edges – network connections of different capacity
  - Vertices – routers (do not produce or consume data just move it)
- Also used in resource planning, economics, ecosystem network analysis

To do:

[CLRS] 26

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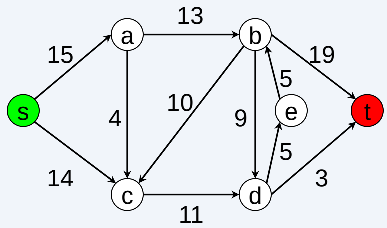
Flow Network Definitions

Directed graph G=(V,E)

One *source* and one *sink*.



Each edge has *capacity* $c(u,v) \geq 0$.

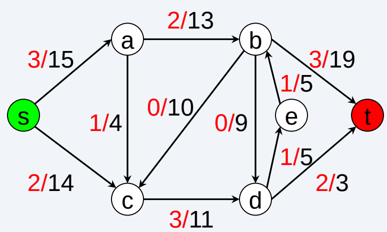
Each vertex is on some path from s to t.

Flow Definitions

How much is currently flowing – $f : V \times V \rightarrow \mathbb{R}$

$$f(V,u) = \sum_{v \in V} f(v,u)$$

$$f(u,V) = \sum_{v \in V} f(u,v)$$



Must satisfy 3 properties:

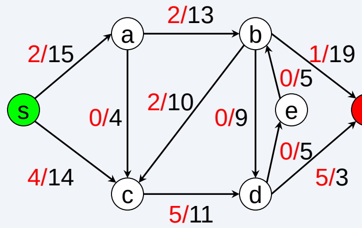
- Capacity constraint:
  $\forall u,v \in V$: $f(u,v) \leq c(u,v)$
- Skew symmetry:
  $\forall u,v \in V$: $f(u,v) = -f(v,u)$
- Flow conservation:
  $\forall u \in V-\{s,t\}$: $f(u,V) = f(V,u) = 0$
  What goes in must go out.

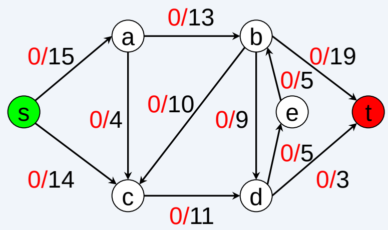
Total value of flow f:

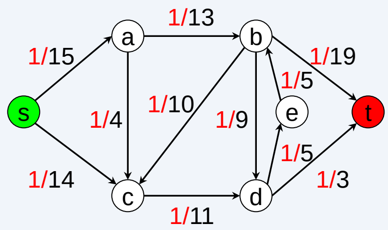
$|f| = f(s,V) = f(V,t)$

Example flows

Valid or invalid? Why?







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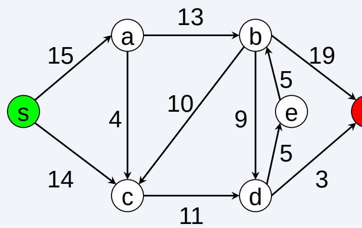
Maximize the flow



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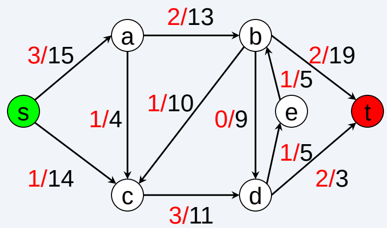
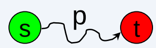
Maximum flow: Algorithm idea

- If we have some flow, …



- …and can find an *augmenting path* 

  can add a constant amount of flow along path:

  ∃ a>0, ∀ (u,v)∈p, f(u,v) + a ≤ c(u,v)

- Then just do it, to get a better flow!

Ford-Fulkerson method

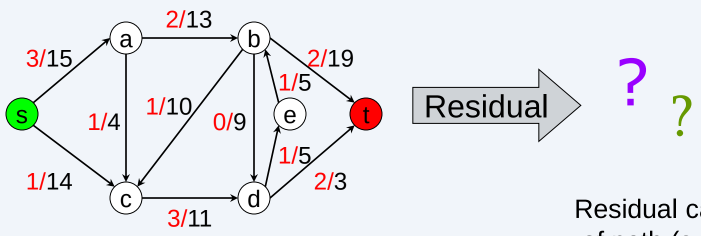
```
Ford-Fulkerson(G,s,t)
initialize flow f to 0 everywhere
while there is an augmenting path p do
   augment flow f along p
return f
```

- How do we find/choose an augmenting path?
- How much additional flow can we send through that path?
- Does the algorithm always find the maximum flow?

Augmenting

Augmenting path – any path in the *residual network*:

- Residual network: $G_f=(V,E_f)$
  $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$
- Residual capacities: $c_f(u,v) = c(u,v) - f(u,v)$
- Residual capacity of path p: $c_f(p) = \min_{(u,v)\in p} c_f(u,v)$



Residual capacity of path (s,c,d,t)?

Observe – Edges in E_f are either edges in E or their reversals: $|E_f| \le 2|E|$.

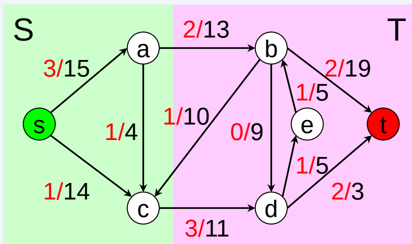
Ford-Fulkerson method, with details

```
Ford-Fulkerson(G,s,t)
  for each edge (u,v)∈G.E do
     f(u,v) = f(v,u) = 0
  while ∃ path p from s to t in residual network G_f do
     c_f = min{c_f(u,v): (u,v)∈p}
     for each edge (u,v) in p do
        f(u,v) = f(u,v) + c_f
        f(v,u) = -f(u,v)
  return f
```

Algorithms based on this method differ in how they choose p.

Does it always find a maximum flow?

Cut – a partition of V into S,T such that $s \in S$, $t \in T$



$$f(S,T) = \sum_{u \in S, v \in T} f(u,v)$$

$$|f| = f(S,T)$$

$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$

Minimum cut – a cut with minimum capacity

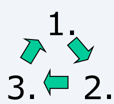
Does it always find a maximum flow?

Max-flow min-cut theorem:

The following are equivalent statements:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S,T)$, for some cut (S,T) of G.

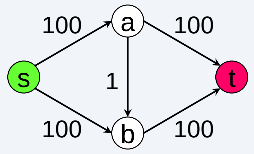
We will prove three parts:



From this we have 2.→1., which means that the Ford-Fulkerson method always correctly finds a maximum flow.

What is the worst-case running time?

- Augmentation = ? O(E)
- How many augmentations?
  - Let's assume integer flows.
  - Each increases the value of the flow by some integer.
  - O(|f*|), where f* is the max-flow.
- Total *worst-case* = O(E×|f*|).



  - How an augmenting path is chosen is very important!

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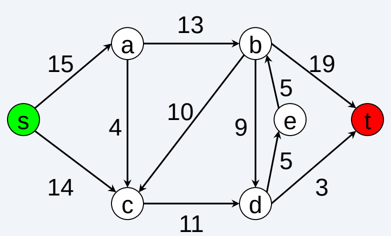
Edmonds-Karp Algorithm

Use **shortest** augmenting path (in #edges).

Run algorithm on our example:



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Edmonds-Karp algorithm analysis: 1

Augmentation = O(E) – Breadth-first search

Will prove: #augmentations = O(VE).

Let d(v) be distance from s to v in residual network.

Will prove: Every |E| iterations, d(t) increases by ≥1.

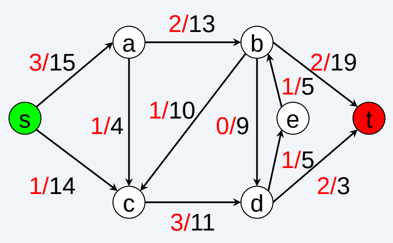
d(t) can increase at most |V| times → O(VE) iterations

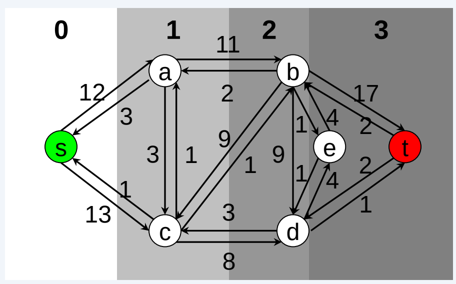
Total = $O(VE^2)$

Edmonds-Karp algorithm analysis: 2

Will prove: Every |E| iterations, d(t) increases by ≥1.

Consider the residual network in levels according to d(v):





As long as d(t) doesn't change, the paths found <u>will only use forward edges</u>.

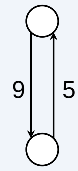
- Each iteration saturates & removes at least 1 forward edge, and adds only backward edges (so no distance ever drops).
- After removing |E| - d(t) +1 forward edges, t will be disconnected from s.

So, within |E| iterations, either

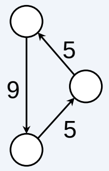
- t is disconnected, & algorithm terminates, or
- A non-forward edge used, & d(t) increased.

Antiparallel edges

Allow:



Disallow:



Flows cancel.

Residual (multi)graph can have parallel edges.

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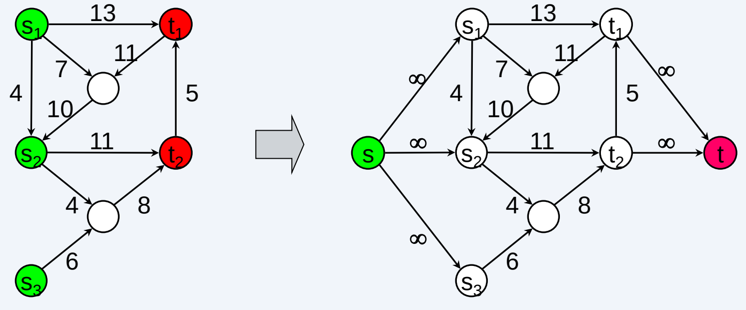
What if we have multiple sources or sinks?



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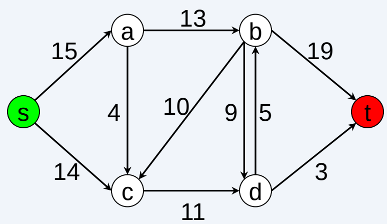
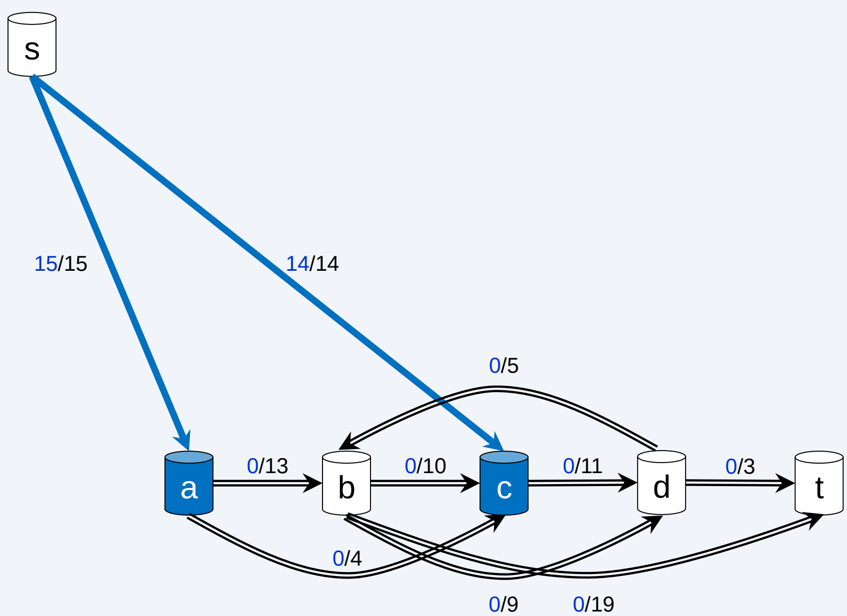
Maximum Flow: Another Algorithm Idea

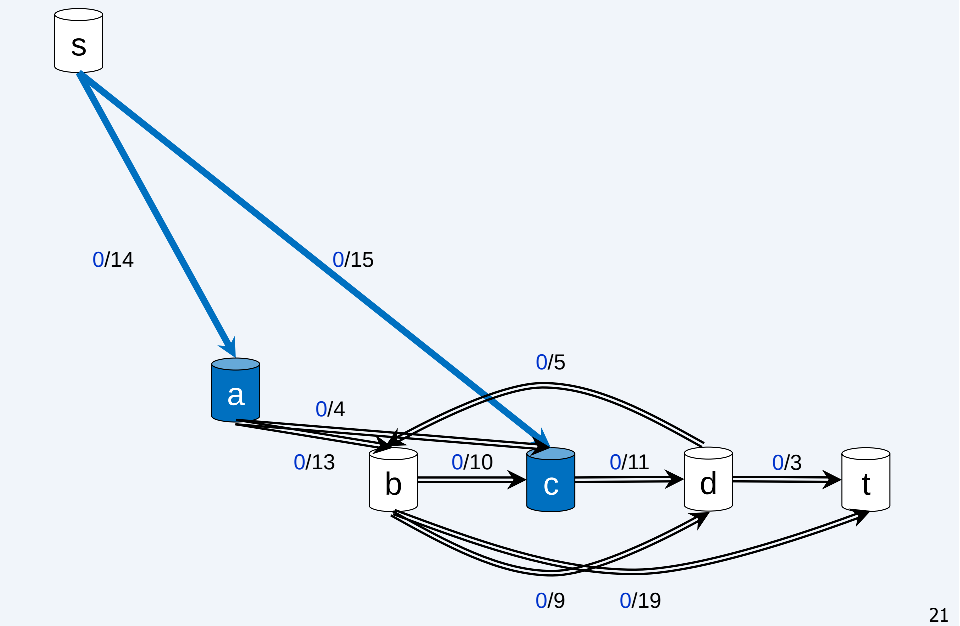
Greedily fill outgoing edges to capacity. Later edges' capacity constraints can lead us to reduce those flows.



Push-Relabel Example – Goldberg & Tarjan (1986)



Relabel a



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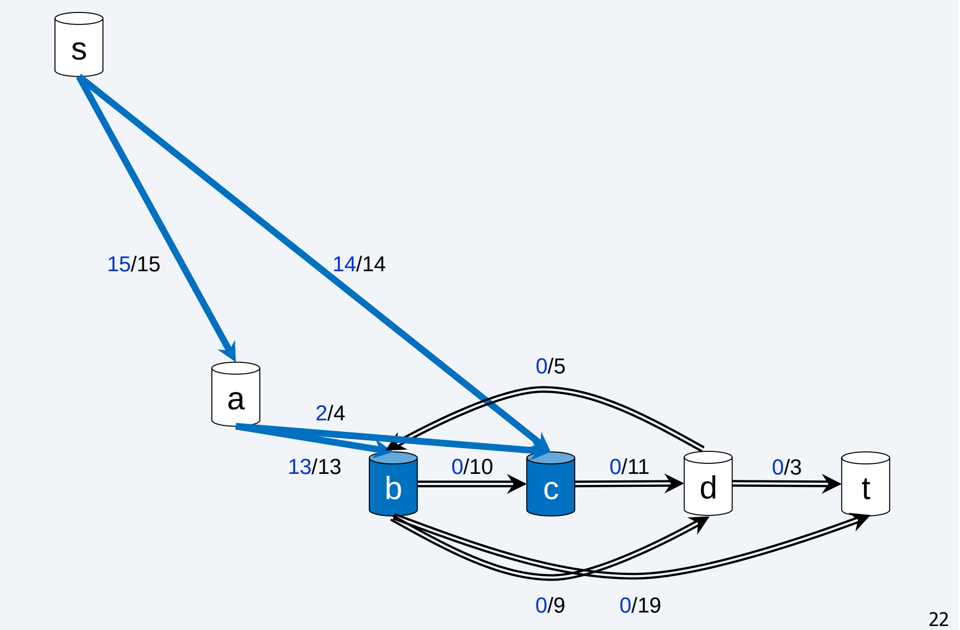
Push Preflow from a to b & c



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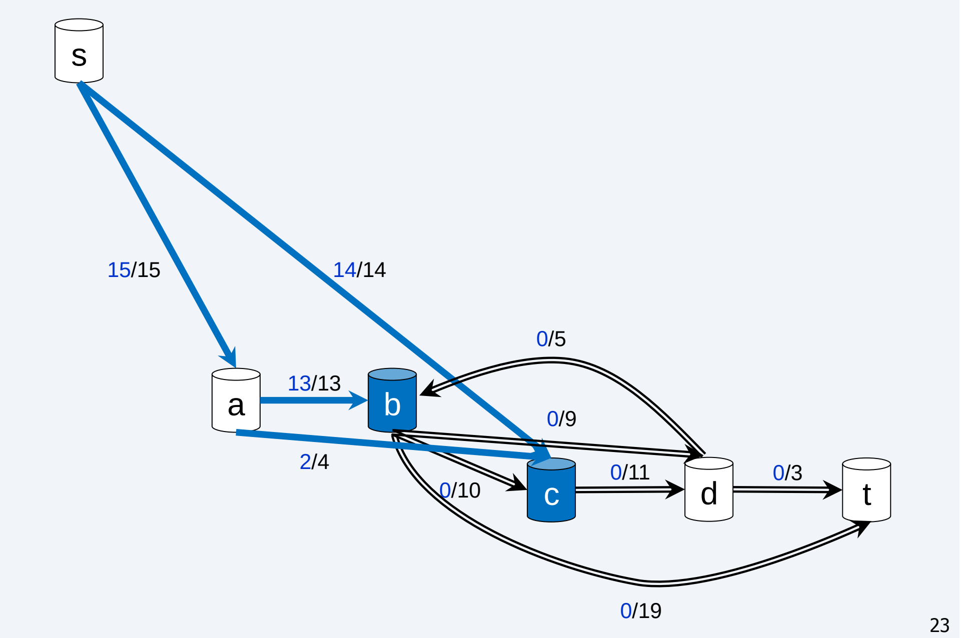
Relabel b



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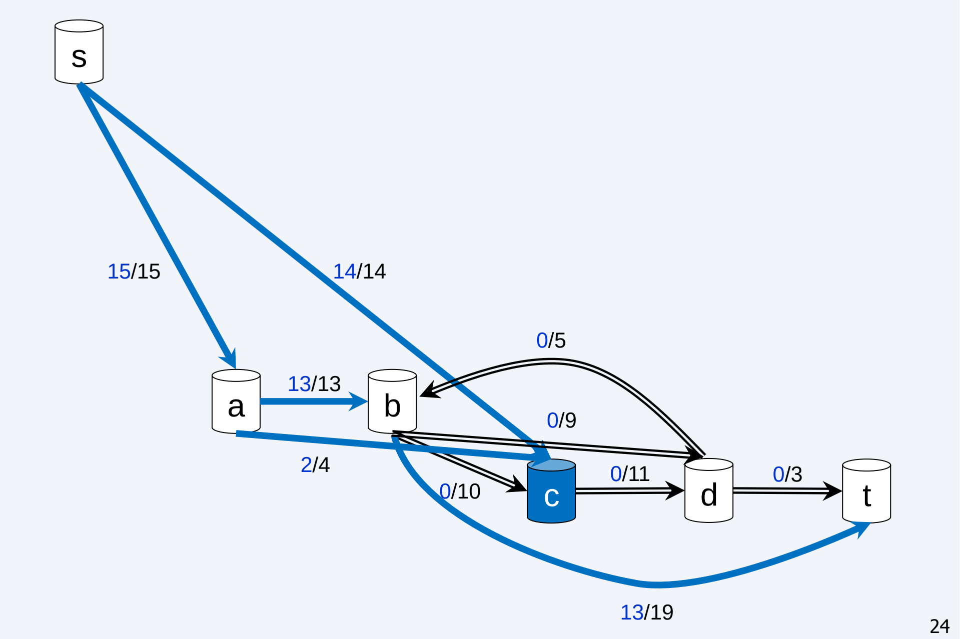
Push Preflow from b



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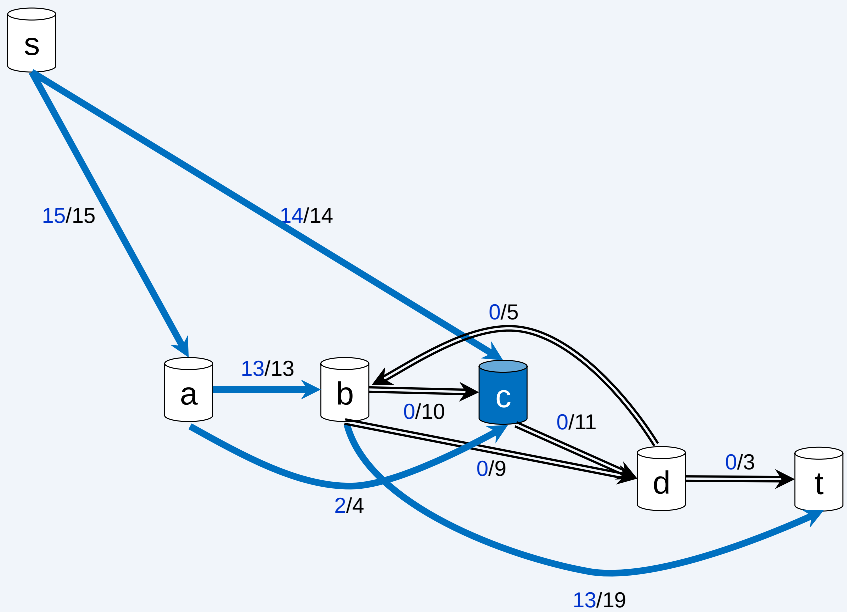
Relabel c



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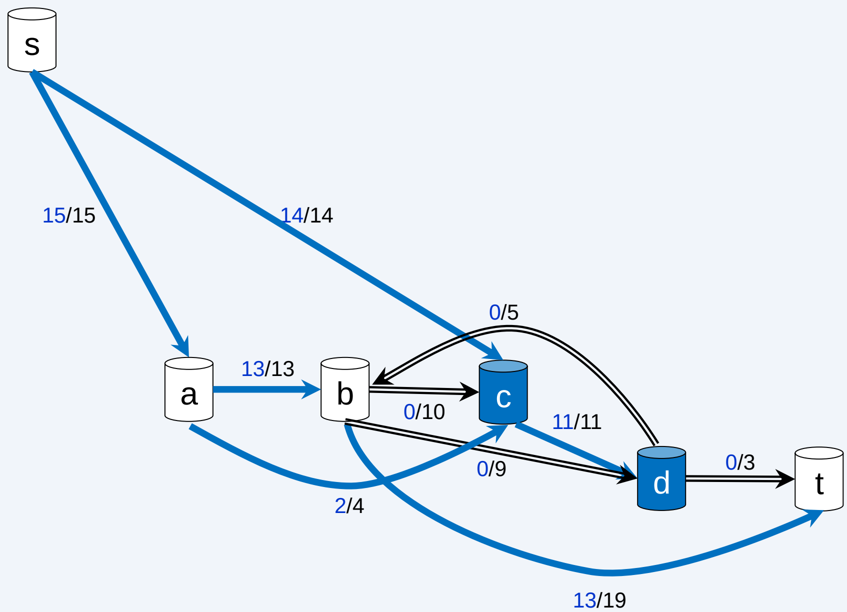
Push Preflow from c to d



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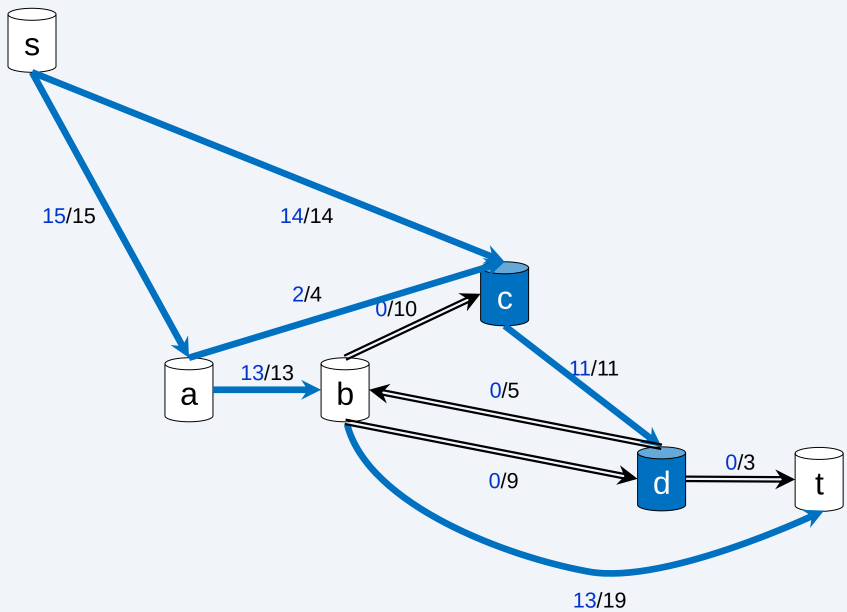
Relabel c



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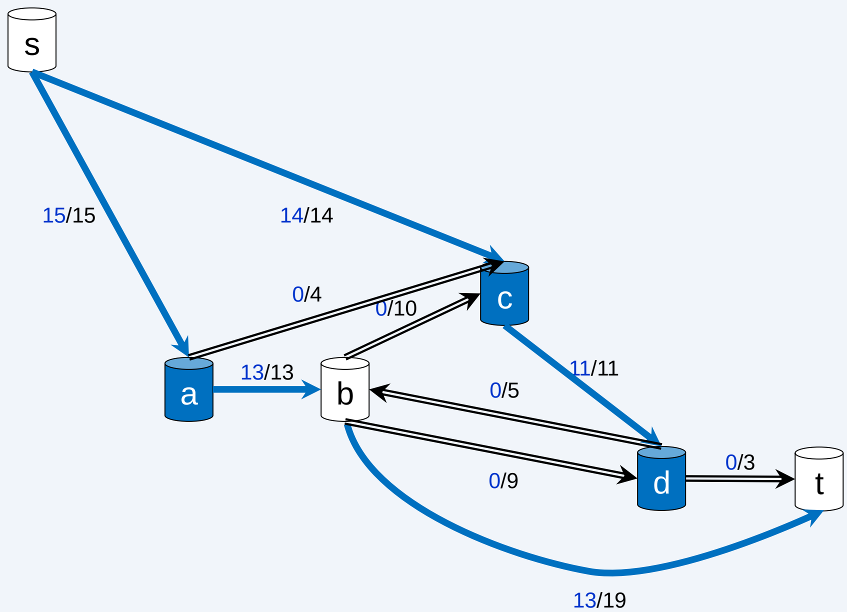
Push Preflow from c to a



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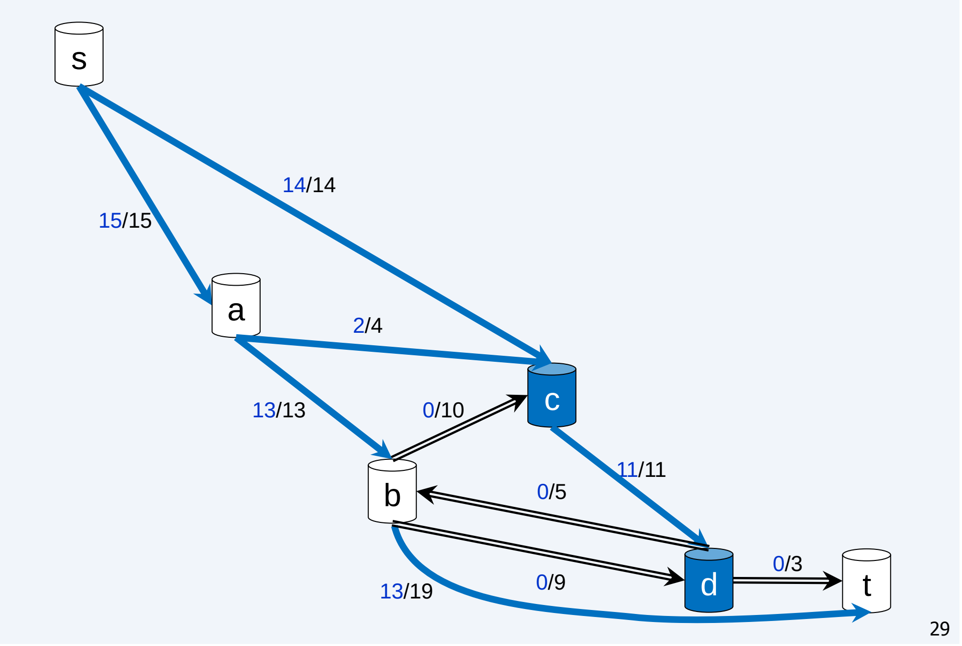
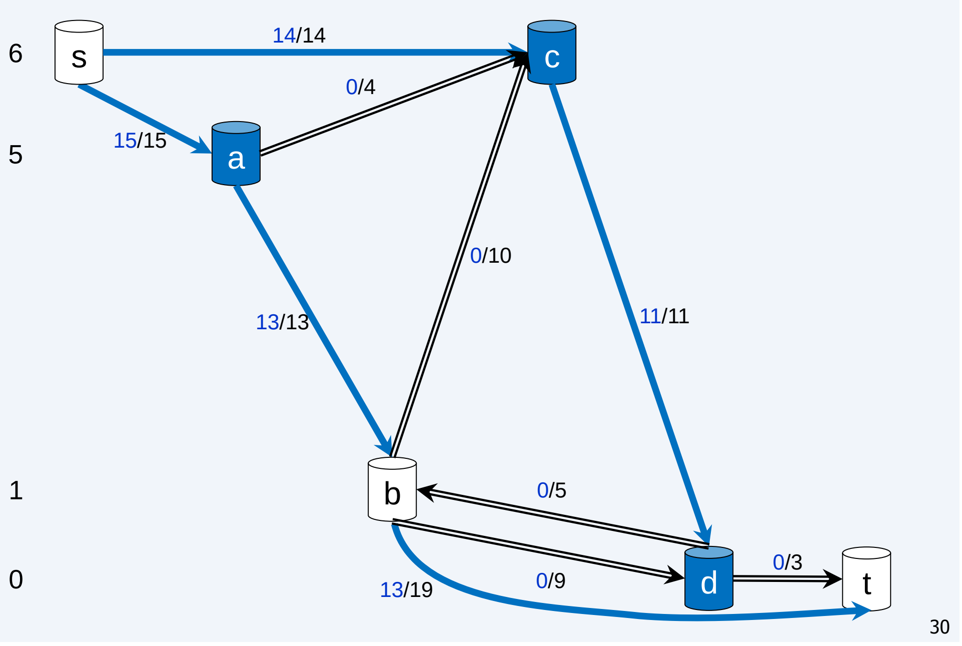
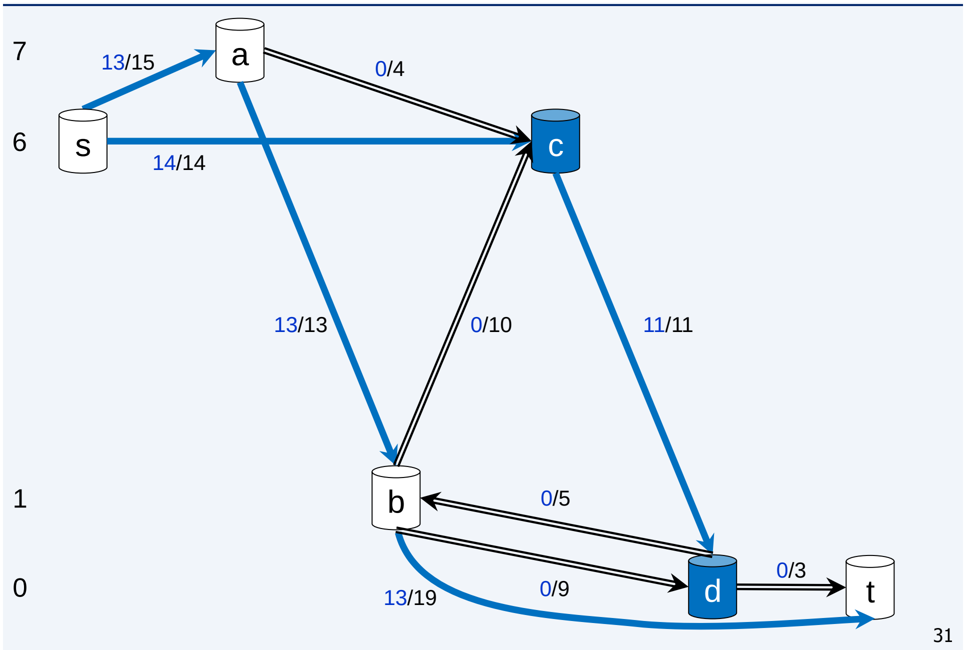
Relabel a & Push Preflow from a to c



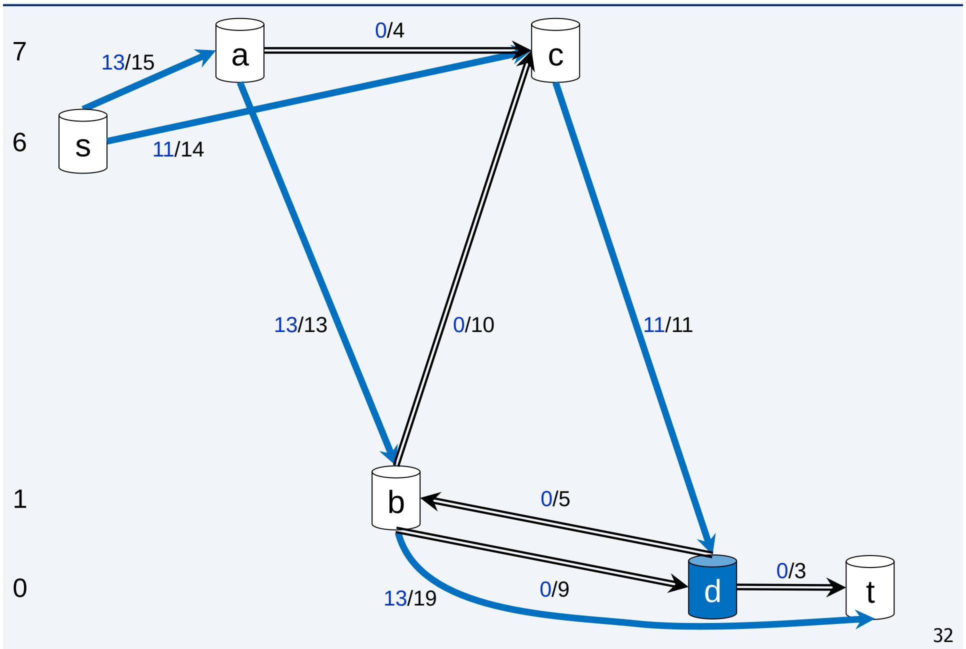
Relabel & Push c, Relabel & Push a, Relabel & Push c



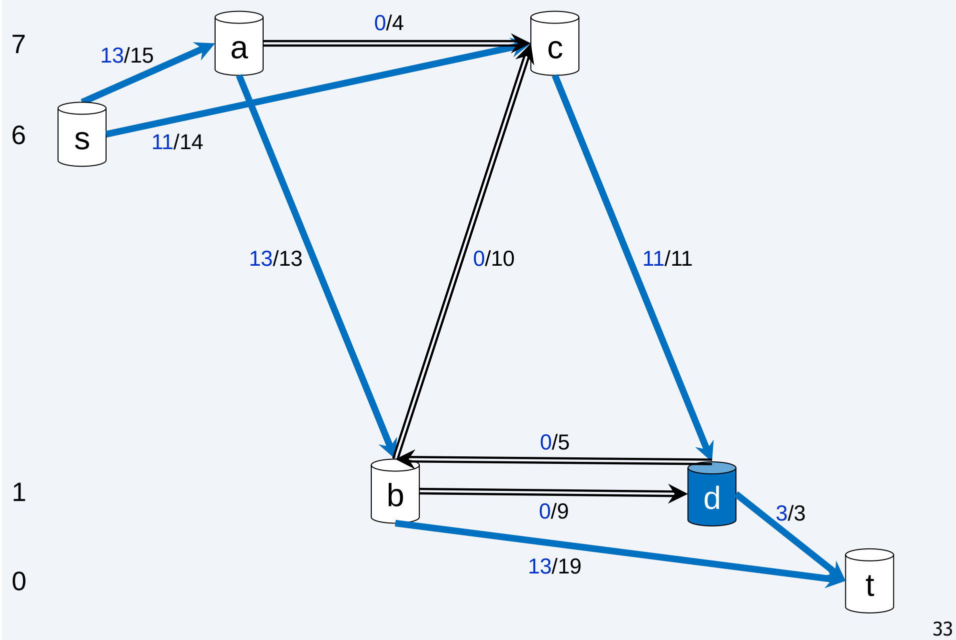
Relabel a & Push Preflow from a to s



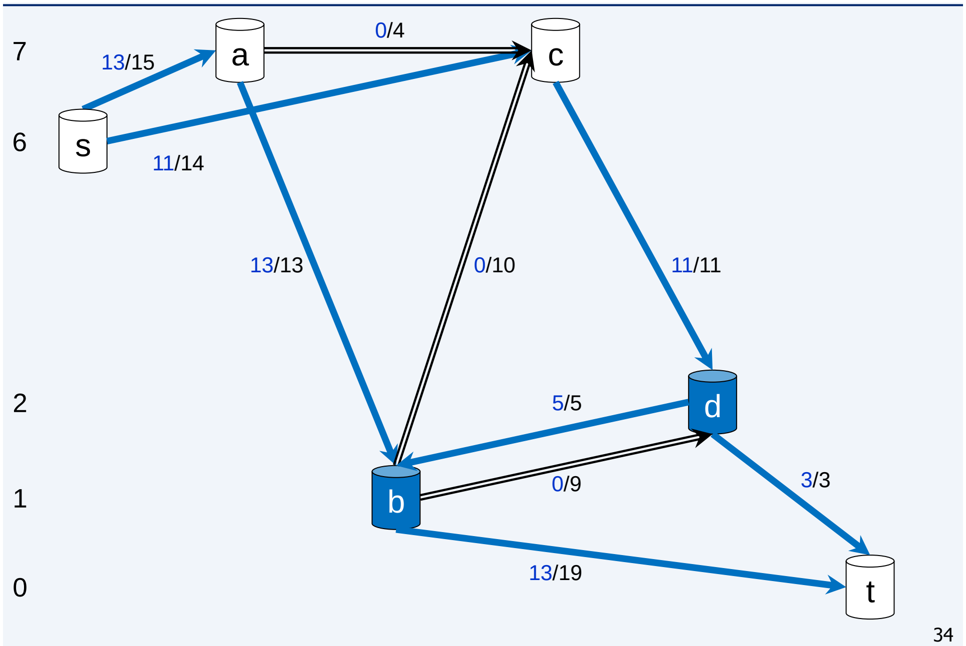
Relabel c & Push Preflow from c to s



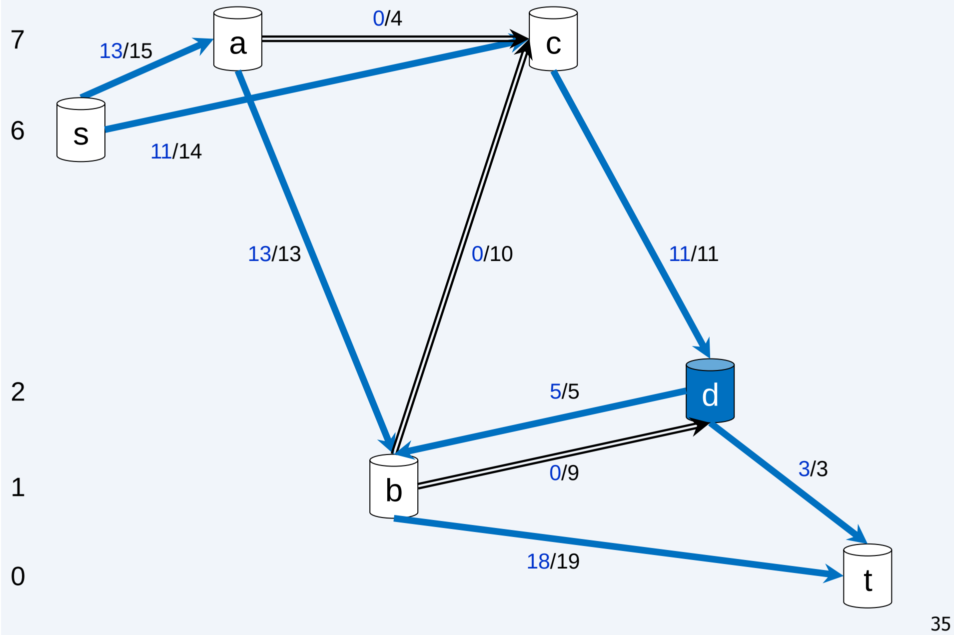
Relabel d & Push Preflow from d to t



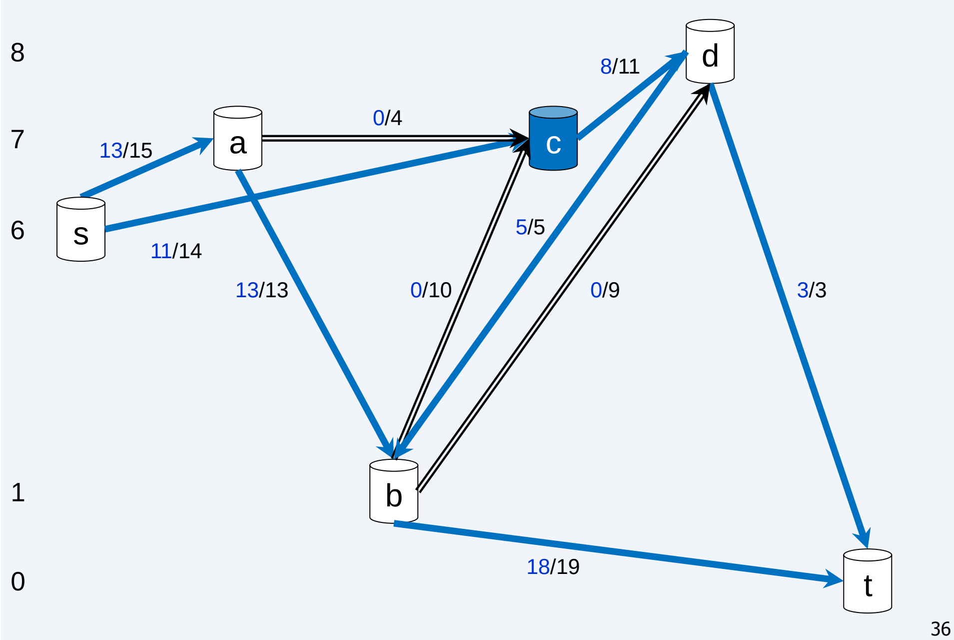
Relabel d & Push Preflow from d to b



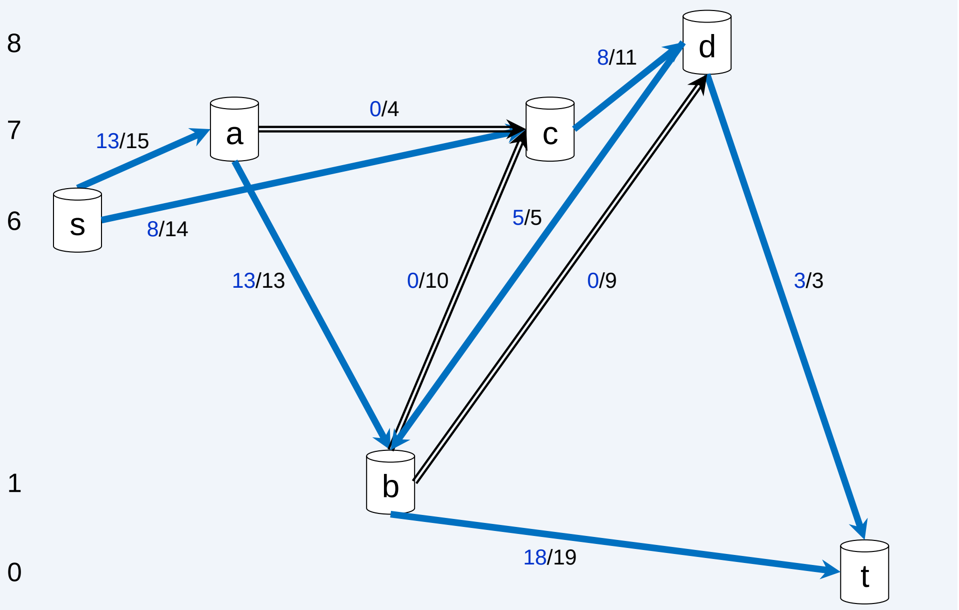
Push Preflow from b to t



Relabel d & Push Preflow from d to c



Push Preflow from c to s



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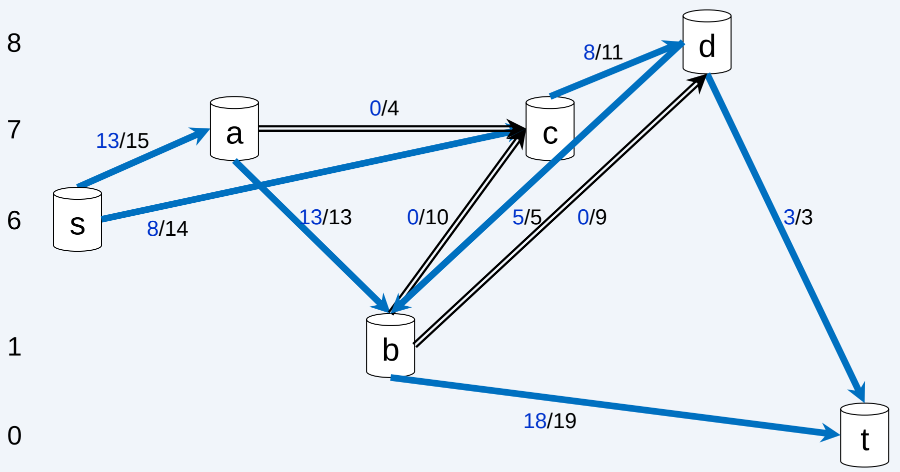
Correctness Overview



Is this a flow?

Is it maximum? – What is the residual graph?

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Correctness Depends on Height Invariant

Invariant: Residual edges go down in height by at most one.

But, height(s) – height(t) = |V|, longer than any s-t path.

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Running Time Overview

Max height = $2|V| - 1$.

Overall algorithm is $O(V^2E)$.

- Requires amortized analysis.
- See text & homework.

Choosing operation order wisely, can reduce to $O(V^3)$.

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Two Other Approaches

Blocking flows:

- Each s-t path in blocking flow contains a saturated edge.
- Dinitz/Dinic (1970)
  $O(VE \log V)$
- Goldberg & Rao (1997)
  $O(\min(V^{2/3},E^{1/2})\ E \log(V^2/E) \log(\max_{(u,v)\in E} c(u,v)))$

Combinatorial game:

- Cheriyan, Hagerup (1989)
  $O(VE + V^2 \log^2 V)$ expected
- King, Rao, & Tarjan (1994)
  $O(VE \log_{E/(V \log V)} V)$

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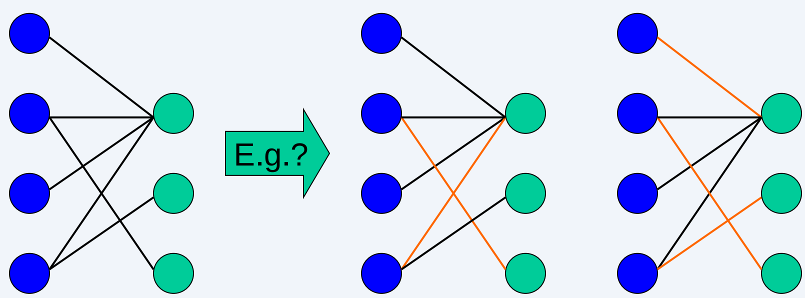
An Application of Max-Flow

Maximum bipartite matching problem

Matching in a graph is a subset *M* of edges such that each vertex has at most one edge of *M* incident on it. It puts vertices in pairs.



Bipartite: V = L+R
$E \subseteq L \times R$

A maximum

E.g., dating agency

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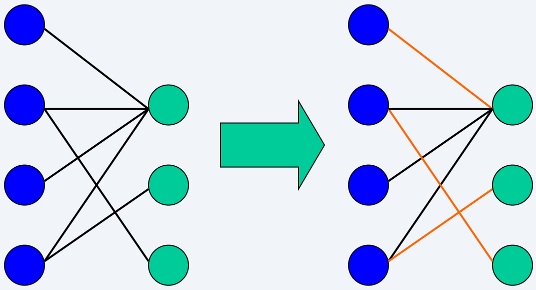
Maximum Bipartite Matching



V = L+R

E ⊆ L×R

- How can we reformulate as a max-flow problem?
- What is the running time, using Edmonds-Karp?