Comp487/587 - Exercise 1

- Questions labeled with **(COMP 587)** are required only for students in COMP 587. Questions labeled with **(Open Problem)** are not required, but may be interesting to consider. All other questions are required both for students in COMP 487 and students in COMP 587.
- Collaboration is allowed and encouraged, but every student must write the solutions in his/her own words. Indicate clearly the name(s) of people you collaborated with.
- Note on the submission the total number of hours spent on this assignment.

Definitions: Let $L \subseteq \Sigma^*$ be a language.

- The *complement* of L is defined to be the set

$$\overline{L} = \{x \in \Sigma^* \mid x \notin \text{L}\},$$

i.e. the set of all words in Σ^* that are not in L.
- The *Kleene star* (or *Kleene closure*) of L is defined to be the set

$$L^* = \{w_1 w_2 \cdots w_k \mid k \geq 0, \forall i \; w_i \in L\},$$

i.e. the set of all words that can be obtained by concatenating finitely many words in L. Note that there is no separator between the w_i terms.

Problem 1: Deterministic Turing Machines

For each of the following parts, write down the full set of states, alphabet and transition function for the deterministic Turing machine you construct. I strongly encourage you to do this with `https://www.turingmachinesimulator.com/`. If so, a link (Click "Share Link") to your solution suffices for the construction.

Part 1: Construct a deterministic Turing machine that erases the tape and outputs 1 on the first tape square. What is the running time of your machine?

Part 2: Construct a deterministic Turing machine whose language is $\{a^n b^n \mid n \in \mathbb{N}\}$. What is the running time of your machine?

Part 3: Construct a deterministic Turing machine that sorts an input string of zeros and ones. That is, given an input $w \in \{0,1\}^*$ containing m zeros and n ones, output $0^m 1^n$. What is the running time of your machine?

(COMP 587) Part 4: Construct a deterministic Turing machine whose language is $\{1^p \mid p \text{ is prime}\}$ i.e. the set of prime numbers encoded in unary. What is the running time of your machine?

Problem 2: Closure of P

Part 1: Prove that P is closed under union, intersection, and concatenation. That is, if $L_1, L_2 \in P$, prove that each of the following are also in P:

1. $L_1 \cup L_2 = \{w \mid w \in L_1 \text{ or } w \in L_2\}$
2. $L_1 \cap L_2 = \{w \mid w \in L_1 \text{ and } w \in L_2\}$
3. $L_1 \circ L_2 = \{w_1 w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2\}$

Part 2: Let $Co\text{-}P = \{L \subseteq \Sigma^* \mid \overline{L} \in P\}$. Prove that $Co\text{-}P = P$.

(COMP 587) Part 3: Prove that P is closed under Kleene star. That is, if $L \in P$, prove that $L^* \in P$.

Problem 3: Closure of NP

Part 1: Prove that NP is closed under union, intersection, and concatenation. That is, if $L_1, L_2 \in NP$, prove that each of the following are also in NP:

1. $L_1 \cup L_2 = \{w \mid w \in L_1 \text{ or } w \in L_2\}$
2. $L_1 \cap L_2 = \{w \mid w \in L_1 \text{ and } w \in L_2\}$
3. $L_1 \circ L_2 = \{w_1 w_2 \mid w_1 \in L_1 \text{ and } w_2 \in L_2\}$

(Open Problem) Part 2: Let $Co\text{-}NP = \{L \subseteq \Sigma^* \mid \overline{L} \in NP\}$. Prove that $Co\text{-}NP = NP$.

Part 3: Prove that NP is closed under Kleene star. That is, if $L \in NP$, prove that $L^* \in NP$.

Problem 4: Properties of Karp Reduction

Prove or disprove each of the following claims about the relation $\leq_P$. Recall that for languages $A, B \subseteq \Sigma^*$, $A \leq_P B$ means A is polynomially reducible (Karp reducible) to B.

1. $\leq_P$ is reflexive (that is, for every language A, $A \leq_P A$).
2. $\leq_P$ is symmetric (that is, for every pair of languages A, B, if $A \leq_P B$ then $B \leq_P A$).
   - Hint: we stated in class that $P \subsetneq EXP$.
3. $\leq_P$ is anti-symmetric (that is, for every pair of languages A, B, if $A \leq_P B$ and $B \leq_P A$ then $A = B$).
4. $\leq_P$ is transitive (that is, for every triple of languages A, B, C, if $A \leq_P B$ and $B \leq_P C$ then $A \leq_P C$).
5. For every pair of languages A, B, if $A \leq_P B$ then $\overline{A} \leq_P \overline{B}$.

Problem 5

Let $L \subseteq \{0,1\}^*$ be a language such that for every n we have $|L \cap \{0,1\}^n| = n$, meaning the number of words of length n that are in L is exactly n. Prove that if $L \in NP$ then $\overline{L} \in NP$.