Comp487/587 - Exercise 2

- This exercise is due at the **beginning** of class.
- Questions labeled with **(COMP 587)** are required only for students in COMP 587. Questions labeled with **(Open Problem)** are not required, but may be interesting to consider. All other questions are required both for students in COMP 487 and students in COMP 587.
- Collaboration is allowed and encouraged, but every student must write the solutions in his/her own words. Indicate clearly the name(s) of people you collaborated with.
- Note on the submission the total number of hours spent on this assignment.

Problem 1: Co-NP

Consider the following two definitions of Co-NP:

Definition 1. **Co-NP** $= \{L \subseteq \Sigma^* \mid \overline{L} \in NP\}$.

Definition 2. A language $L \subseteq \Sigma^*$ is in **Co-NP** if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time deterministic TM M such that for every $x \in \Sigma^*$:

$$x \in L \iff \forall u \in \Sigma^* \text{ with } |u| \leq p(|x|), M(x,u) = 1$$

Part 1 [5 points]: Prove that these definitions are equivalent.

Part 2 [5 points]: A language L is called Co-NP complete if $L \in$ Co-NP and for every $L' \in$ Co-NP we have $L' \leq_P L$. Prove that L is NP-complete iff $\overline{L}$ is Co-NP complete.

Part 3 [5 points]: Prove that if L is an NP-complete language and $L \in$ Co-NP then $NP =$ Co-NP.

Part 4 [5 points]: Prove that if $NP \neq$ Co-NP then $P \neq NP$.

Problem 2: Feedback Set

Definition 3. A set of vertices V' in an undirected simple graph $G = (V, E)$ is called a **Feedback Set** if every cycle in G contains a vertex in V'.

Part 1 [10 points]: Prove that the following language is NP-complete by reduction from VC:

$$FS = \{(G, k) \mid G \text{ has a Feedback Set of size} \leq k\}$$

Part 2 [5 points]: Prove (or disprove) that the following language is NP-complete:

$$\textit{Quarter-Set} = \{G = (V, E) \mid G \text{ has a Feedback Set of size} \leq |V|/4\}$$

Part 3 [5 points]: Prove (or disprove) that the following language is NP-complete:

$$3\text{-}FS = \{G \mid G \text{ has a Feedback Set of size } 3\}$$

Problem 3 NAE-3SAT:

Definition 4. Let φ be a 3-CNF formula. A truth assignment τ is called a **sat-unsat assignment** for φ if every clause has at least one satisfied literal and at least one unsatisfied literal under τ.

That means that in every clause τ cannot satisfy (or unsatisfy) all three literals.

Part 1 [5 points]: Prove that every 3-CNF formula has an even number of sat-unsat assignments.

Part 2 [15 points]: Prove that the following language is NP-complete:

$$NAE3SAT = \{\varphi \mid \varphi \text{ is a 3-CNF formula that has a sat-unsat assignment}\}$$

Hint: Use extra variables, as in the reduction from SAT to 3SAT.

(COMP 587) Part 3 [10 points]: Describe a direct polynomial reduction from $NAE3SAT$ to 3-Coloring. (By direct we mean, without using Cook-Levin, or any intermediate reduction).

Problem 4: Max-2SAT

Part 1 [20 points]: Prove that the following language is NP-complete:

$$Max\text{-}2SAT = \{(\varphi, k) \mid \varphi \text{ is a 2-CNF formula and } \varphi \text{ has an assignment that satisfies at least } k \text{ clauses of } \varphi\}$$

Hint: Show a reduction from VC by building a formula with a variable for every vertex, clauses for the vertices, and clauses for the edges.

(COMP 587) Part 2 [10 points]: Prove that 2SAT (the language of all satisfiable 2-CNF formulas) is in P.
Hint: Consider a graph construction. Recall that the formula $(a \vee b)$ is logically equivalent to the formula $((\neg a \to b) \wedge (\neg b \to a))$.

Problem 5: Graph Isomorphism

We say that two graphs are isomorphic if they are the same graph with just different names for the vertices. Formally, $G = (V, E)$ and $H = (V', E')$ are isomorphic if there exists a bijection $f : V \to V'$ such that

$$(v, w) \in E \Leftrightarrow (f(v), f(w)) \in E'.$$

Consider the following two languages:

$$GI = \{(G, H) \mid G, H \text{ are isomorphic}\}$$

$$SGI = \{(G, H) \mid G \text{ has a subgraph that is isomorphic to } H\}$$

Part 1 [5 points]: Prove that $GI \leq_P SGI$.

Part 2 [15 points]: Prove that SGI is NP-complete.
Hint: Show a reduction from $CLIQUE$.

(Open Problem) Part 3: Prove or disprove that GI is NP-complete.