Lazy Strength Reduction*

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Abstract

We present a bit-vector algorithm that uniformly combines code motion and strength reduction, avoids superfluous register pressure due to unnecessary code motion, and is as efficient as standard unidirectional analyses. The point of this algorithm is to combine the concept of lazy code motion of [1] with the concept of unifying code motion and strength reduction of [2, 3, 4]. This results in an algorithm for lazy strength reduction, which consists of a sequence of unidirectional analyses, and is unique in its transformational power.

Keywords: Data flow analysis, program optimization, partial redundancy elimination, code motion, strength reduction, bit-vector data flow analyses.

1 Motivation

Code motion improves the runtime efficiency of a program by avoiding unnecessary recomputations of a value at runtime. Strength reduction improves runtime efficiency by reducing “expensive” recomputations to less expensive ones, e.g., by reducing computations involving multiplication to computations involving only addition. Common to both techniques is replacing the original computations of a program by auxiliary variables (registers) that are initialized at suitable program points. In the case of strength reduction, they are additionally updated at certain points. The similarity between strength reduction and code motion suggests the combination of both techniques, an idea which was first realized by an algorithm of Joshi and Dhamdhere [2, 3] that enhances the code motion algorithm of Morel and Renvoize [5] to capture strength reduction.

In this paper we combine Joshi and Dhamdhere’s approach with the idea of lazy code motion presented in [1]. This results in an algorithm for lazy strength reduction, which uniformly combines code motion and strength reduction, and avoids any unnecessary register pressure. In fact, in contrast to previous algorithms, it does not insert multiplications and additions on the same path, minimizes the lifetimes of moved computations, and limits the insertion of multiple additions on a path to a minimum. Moreover, as our algorithm is composed of a sequence of unidirectional bit-vector analyses, it is as efficient as the standard unidirectional analyses (cf. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]), which drastically improves on the results about the original bidirectional algorithms of [2, 3, 4].

The illustration of the essential new features of our algorithm requires a rather complex program structure. In the example of Figure 1, which is a slight modification of an example of [3], our lazy strength reduction algorithm is unique in yielding the result shown in Figure 2. This transformation is exceptional for the following reasons: It replaces the multiplications of

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i * 10 at node 13 and 14 by moving them to node 7 and 8 and inserting a single addition at node 9. In the “right” part of the loop this reduces the original multiplications to an addition. Furthermore, it does not touch the computation of i * 10 at node 21 that cannot be moved profitably. The example will be discussed in more detail during the development of the paper.

Figure 1: The Motivating Example

Related Work

The point of Morel and Renvoise’s code motion algorithm [5] is to place computations as early as possible in a program, while guaranteeing that every inserted computation is used on every terminating program path leaving the insertion point. In order to capture strength reduction,
Joshi and Dhamdhere’s algorithms [2, 3, 4] allow the values of inserted computations to be updated before their use by the addition of constants, while maintaining the strategy of placing computations as early as possible. This strategy, however, moves computations even if it is unnecessary, i.e., there is no runtime gain, and therefore causes superfluous register pressure, which is a major problem in practice.\footnote{In [16] unnecessary code motion is called redundant.}

Recently the problem of unnecessary code motion was addressed in [1], where an algorithm for lazy code motion was presented. In contrast to all previous code motion algorithms (cf. [17, 18, 19, 20, 5, 21]), this algorithm places computations \textit{as late as possible} in a program, while maintaining computational optimality.\footnote{Here, \textit{computational optimality} means that a program cannot be improved by means of semantics preserving code motion (see [1] for details).} It is unique in that it avoids any unnecessary

Figure 2: The Lazy Strength Reduction Transformation
There are also other, conceptually different, approaches to strength reduction. For example the algorithms of [22, 23, 24] are restricted to loops and require explicit detection of induction variables. Thus, in the example of Figure 1 they would not do anything, since \( i \) is not an induction variable (cf. [25]). In contrast, the semantically based algorithm for code motion and strength reduction of [26] works for arbitrary program and term structures, but does not capture the laziness effect. This also holds for the (significantly different) approach of [27, 28, 29], namely finite differencing, whose major achievement is the generalization of strength reduction to non-numerical applications.

**Structure of the Paper**

After the preliminary definitions in Section 2, and a brief summary of the basic version of the code motion algorithm of [1] together with its extension to strength reduction along the lines of [2] in Section 3, we arrive at the central Section 4, where the lazy strength reduction algorithm is developed. This development is split into three steps which successively enhance the power of the algorithm by adding new predicates that guarantee new properties of the resulting program. The paper closes with Section 5 containing our conclusions.

### 2 Preliminaries

We consider terms \( t \in T \), which are inductively built of variables \( x \in V \), constants \( c \in C \), and operators \( op \in Op \). As usual, we represent an imperative program as a directed flowgraph \( G = (N, E, s, e) \) with node set \( N \) and edge set \( E \). Nodes \( n \in N \) represent assignments of the form \( x := t \). Edges \( (m,n) \in E \) denote the nondeterministic branching structure of \( G \).\(^3\) \( s \) and \( e \) denote the unique start node and end node of \( G \), which are both assumed to represent the empty statement \( \text{skip} \) and not to possess any predecessors and successors, respectively. Every node \( n \in N \) is assumed to lie on a path from \( s \) to \( e \). Finally, \( \text{succ}(n) = \{ m \mid (n,m) \in E \} \) and \( \text{pred}(n) = \{ m \mid (m,n) \in E \} \) denote the sets of all immediate successors and predecessors of a node \( n \), respectively.

**Candidate Expressions for Strength Reduction**

We demonstrate our approach of uniformly combining lazy code motion and strength reduction by means of the classical application of strength reduction, which reduces multiplications to additions. Therefore, we assume two binary operators, + and \( * \), in \( Op \), which we interpret as ordinary addition and multiplication, respectively. Terms of the form \( v \ast c \) with \( v \in V \) and \( c \in C \) are called candidate expressions for strength reduction, because they may give rise to a transformation that eliminates the multiplication. In the following we will develop our algorithm for an arbitrary but fixed candidate expression \( v \ast c \), which allows us to keep our notation simple.

**Use, Transparency, and SR-Transparency**

For every node \( n \equiv x := t \) we define three local predicates indicating, whether \( v \ast c \) is used or modified by the assignment of node \( n \). Here, \( \text{SubTerms}(t) \) denotes the set of all subterms of \( t \), e.g., \( \text{SubTerms}(a + ((v \ast c) - b)) = \{ a, v, c, b, v \ast c, (v \ast c) - b, a + ((v \ast c) - b) \} \).\(^4\)

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\(^3\)We do not assume any structural restrictions on \( G \). In fact, every algorithm computing the fixed-point solution of a unidirectional bit-vector data flow analysis problem may be used to compute the predicates involved in the lazy strength reduction transformation (cf. [25]). However, application of the efficient techniques of [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] requires that \( G \) satisfies the structural restrictions imposed by these algorithms.

\(^4\)Flowgraphs composed of basic blocks can be treated entirely in the same fashion by replacing the predicate Used by the predicate Antiloc (cf. [5]), indicating whether the computation of \( t \) is locally anticipatable at node
- \( \text{Used}(n) = \text{df } v \ast c \in \text{SubTerms}(t) \)
- \( \text{Transp}(n) = \text{df } x \neq v \)
- \( \text{SR-Transp}(n) = \text{df } \text{Transp}(n) \vee t \equiv v + d \text{ with } d \in \mathbb{C} \)

In addition to \( \text{Transp} \), the predicate \( \text{SR-Transp} \) is also valid at those nodes where the effect of an assignment on the value of \( v \ast c \) can be dealt with by means of an update assignment common to strength reduction (cf. Section 3.2). Such nodes will be indicated by the predicate \( \text{Injured} \).\(^5\) Additionally, we define a function \( \text{Eff} : N \to \omega \) by

\[
\forall n \in N. \text{Eff}(n) = \begin{cases} 
c \ast d & \text{if } \text{Injured}(n) \text{ with } n \equiv v := v + d \\
0 & \text{otherwise}
\end{cases}
\]

which provides the amount of updating that is necessary in order to pass node \( n \).\(^6\)

### Inserting Synthetic Nodes

In order to exploit the full power of our algorithm for lazy strength reduction, we assume that in the flowgraph \( G \) to be considered from now on, every edge leading to a node with more than one predecessor has been split by inserting a synthetic node.\(^7\) Inserting synthetic nodes is common for code motion optimizations (cf. [17, 16, 18, 19, 1, 20, 31, 32, 26]) and discussed in more details in Section A.1.

### 3 Simple Strength Reduction

In this section we present a simple algorithm for strength reduction, which evolves straightforwardly as a uniform extension of the basic version of the code motion algorithm of \([1]\). Section 3.1, therefore, recalls the essentials of this code motion algorithm, while Section 3.2 presents the modifications to capture strength reduction, and Section 3.3 presents a discussion of the deficiencies of simple strength reduction.

#### 3.1 Code Motion: Down-Safety and Earliestness

The point of the basic version of the code motion algorithm of \([1]\) is to place computations as early as possible within a program while maintaining its semantics. This is achieved by moving computations to program points where they are down-safe and earliest. Intuitively, down-safe means that the inserted value is used on every terminating program path starting with the insertion point. This guarantees that the program semantics is preserved,\(^8\) and earliest means that a placement at an “earlier” position would either not be down-safe or would not always deliver the required value. This is sufficient in order to guarantee that the number of calculations at runtime cannot be reduced any further by means of a safe placement. Clearly, the code motion transformation, which we apply to the candidate expression \( v \ast c \) here, works for arbitrary program terms.

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\(^n\) In the terminology of \([14]\), assignments of the form \( v := v + d \) do not kill the value of \( v \ast c \), but injure it, i.e., establishing the new value of \( v \ast c \) requires only a “small”, cheap to perform modification of the current value. In \([30]\) the term \emph{wounded} is used instead of injured.

\(^6\) Note that for any node \( n \) satisfying \( \text{Injured} \) the value \( \text{Eff}(n) \) can be computed at compile time, since \( c \) and \( d \) are both constants in \( \mathbb{C} \).

\(^7\) In order to keep the presentation of the motivating example simple, we omit synthetic nodes that are not relevant for the lazy strength reduction transformation.

\(^8\) In particular, a down-safe placement does not change the potential for runtime errors, e.g., “division by 0” or “overflow”.

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3.1.1 Down-Safety

$v * c$ can safely be placed at the entry of a node $n \in N$, if it is used on every terminating program path starting with $n$ before $v$ is modified. These down-safe computation points for $v * c$ are characterized by the greatest solution of Equation System 3.1, which specifies a backward analysis of $G$.

**Equation System 3.1 (D-SAFE)**

$$D\text{-SAFE}(n) = \begin{cases} \text{false} & \text{if } n = e \\ \text{Used}(n) \lor (\text{Transp}(n) \land \prod_{m \in \text{succ}(n)} D\text{-SAFE}(m)) & \text{otherwise} \end{cases}$$

3.1.2 Earliestness

Placing computations as "early" as possible is sufficient to obtain computationally optimal programs, i.e., programs that cannot be further improved by means of a safe placement (cf. [1]). The program points enjoying the earliestness property are characterized by the least solution of Equation System 3.2, which is obtained by means of a forward analysis of $G$. Here, $D\text{-Safe}_{CM}$ denotes the greatest solution of Equation System 3.1.

**Equation System 3.2 (EARLIEST)**

$$\text{EARLIEST}(n) = \begin{cases} \text{true} & \text{if } n = s \\ \sum_{m \in \text{pred}(n)} (\neg \text{Transp}(m) \lor (-D\text{-Safe}_{CM}(m) \land \text{EARLIEST}(m))) & \text{otherwise} \end{cases}$$

Let $\text{Earliest}_{CM}$ denote the least solution of Equation System 3.2, which is based upon the following intuition (cf. Section A.2 for an illustration). A placement of $v * c$ at the entry of a node $n$ is "earliest" if there is a path from $s$ to $n$ on which any prior computation of $v * c$

- would not provide the same value as in $n$ due to a subsequent modification

or

- would not be down-safe.

3.1.3 The Code Motion Transformation

Denoting the program points satisfying both $D\text{-Safe}_{CM}$ and $\text{Earliest}_{CM}$ by $\text{Insert}_{CM}$, the following three-step procedure results in a computationally optimal program, i.e., in a program that cannot be improved by means of a safe code motion transformation for $v * c$ (cf. [1]).

1. Introduce a new auxiliary variable $h$ for $v * c$
2. Insert at the entry of every node satisfying $\text{Insert}_{CM}$ the assignment $h := v * c$
3. Replace every (original) occurrence of $v * c$ in $G$ by $h$

**Table 1: The Safe-Earliest Transformation**
3.2 Strength Reduction as Refined Code Motion

Code motion moves the computation $v * c$ backwards as long as $v$ is not modified within a node, i.e., as long as $Transp$ is satisfied. The point of strength reduction is to weaken this transparency requirement, and to move $v * c$ as long as its value can be updated simply by adding a constant to the current value, i.e., as long as $SR-Transp$ is satisfied, or equivalently, as long as the value of $v * c$ is only injured. Subsequently, the injured values can be cured simply by adding the constant $E(n)$ to it.

This change to the notion of transparency also affects the notion of safety: For strength reduction we allow $v * c$ to be placed at the entry of a node $n \in N$ if it is used on every terminating program path starting with $n$ before the value is killed. In order to determine these program points, it is sufficient to replace the predicate $Transp$ in Equation System 3.1 by $SR-Transp$. This directly leads to the definition of $SR$-down-safe and $SR$-earliest computation points for strength reduction.

### 3.2.1 SR-Down-Safety and SR-Earliestness

$v * c$ can be placed $SR$-down-safely at the entry of all nodes satisfying the predicate

$$D-Safe_{SR}$$

which denotes the greatest solution of Equation System 3.1, where $Transp$ is replaced by $SR-Transp$. Analogously, the least solution of Equation System 3.2, where $SR-Transp$ and $D-Safe_{SR}$ are used instead of $Transp$ and $D-Safe_{CM}$, respectively, is denoted by

$$Earliest_{SR}$$

$Earliest_{SR}$ specifies the earliest computation points with respect to $D-Safe_{SR}$. Program points satisfying both $D-Safe_{SR}$ and $Earliest_{SR}$ are the computation points of the simple strength reduction transformation and are denoted by the predicate $Insert_{SSR}$.

### 3.2.2 Updating

Similar to the code motion transformation, the simple strength reduction transformation also stores the value of $v * c$ in an auxiliary variable $h$, and replaces all (original) occurrences of $v * c$ by $h$. However, strength reduction additionally requires inserting update assignments for $h$ in some of the nodes satisfying the predicate $Injured$ defined in Section 2. These nodes are characterized by the least solution of Equation System 3.3, which is obtained by means of a backward analysis of $G$.

**Equation System 3.3 (UPDATE)**

$$UPDATE(n) = Used(n) \lor \sum_{m \in succ(n)} (\neg Insert_{SSR}(m) \land UPDATE(m))$$

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11 In contrast to the strong safety requirement of code motion this modification may lead to the introduction of new values on a path, and therefore may enlarge the potential of runtime errors (cf. [26]).

12 Here and in the following, the indices “CM” and “SR” are used in order to distinguish the code motion and strength reduction version of down-safe and earliest.
3.2.3 The Simple Strength Reduction Transformation

Denoting the least solution of Equation System 3.3 by Update\textsubscript{SSR}, the predicate InsUpd\textsubscript{SSR} defined as the conjunction of the predicates Injured and Update\textsubscript{SSR} characterizes those program points where the auxiliary variable must be updated. Together, the predicates Insert\textsubscript{SSR} and InsUpd\textsubscript{SSR} induce the simple strength reduction transformation.

1. Introduce a new auxiliary variable \( h \) for \( v \cdot c \)
2. Insert at the entry of every node satisfying
   (a) Insert\textsubscript{SSR} the assignment \( h := v \cdot c \)
   (b) InsUpd\textsubscript{SSR} the assignment \( h := h + E\textsubscript{ff}(n) \)
3. Replace every (original) occurrence of \( v \cdot c \) in \( G \) by \( h \)

Table 2: The Simple Strength Reduction Transformation

Figure 3 shows the result of this transformation for the flowgraph of Figure 1, which would also be delivered by the algorithm of [2]. In fact, the simple strength reduction transformation and the transformation of [2] coincide. However, the algorithm proposed here is a composition of three unidirectional analyses computing one predicate each, whereas the algorithm of [2] is bidirectional and requires more than twice as many predicates.

3.3 Deficiencies of the Simple Strength Reduction Transformation

The strength reduction algorithm developed so far is particularly simple: It straightforwardly evolves from an extension of a code motion algorithm, and it requires no more than three predicates. However, like the algorithm of [2], it suffers from the following deficiencies:

1. Multiplication-Addition Deficiency:
   There may be paths, on which both multiplications and additions are inserted. In the example of Figure 3 this happens on the path \((13, 16, 10, 4, 7, 11)\), where a multiplication is inserted in node 10 and an addition in node 4. In this example, this even impairs the runtime efficiency of this path, since only one multiplication is saved, but a multiplication and an addition are inserted.

2. Lifetime Deficiency:
   The lifetimes of moved computations may be unnecessarily long due to unnecessary code motion. For example in the flowgraph of Figure 3 the initialization of \( h \) at node 1 is unnecessarily early and should be delayed to node 8 in order to avoid unnecessary register pressure. Moreover, rather than being moved to node 20, the computation of \( i \cdot 10 \) at node 21 should not be touched at all, because a profitable movement is impossible.

3. Multiple-Addition Deficiency:
   There may be paths, on which unnecessarily many additions are inserted. Consider for example the path \((14, 17, 15, 12, 6, 9, 11)\). There, the costs for the three inserted additions may easily exceed the costs for the saved multiplication, and therefore even impair the runtime efficiency.

\[ \text{If both Insert\textsubscript{SSR} and InsUpd\textsubscript{SSR} hold, the initialization statement } h := v \cdot c \text{ must precede the update assignment } h := h + E\textsubscript{ff}(n). \]
4 Lazy Strength Reduction

In this section we stepwise refine the simple strength reduction transformation in order to overcome the three deficiencies mentioned in Section 3.3.

4.1 First Refinement: Avoiding the Multiplication-Addition Deficiency

4.1.1 Critical Program Points

In order to avoid insertions of multiplications and additions on the same path, we determine the set of critical program points. Intuitively, a program point is critical, if there is a \( v \ast c \)-free
program path from this point to a modification of \( v \). The idea of our modification is to move critical insertion points in the direction of the control flow to "earliest" noncritical positions.

Technically, the set of critical program points is characterized by the least solution of Equation System 4.1, whose computation requires a backward analysis of \( G \).

**Equation System 4.1 (CRITICAL)**

\[
\text{CRITICAL}(n) = \neg \text{Used}(n) \land (\neg \text{Transp}(n) \lor \sum_{m \in \text{succ}(n)} \text{CRITICAL}(m))
\]

### 4.1.2 Substituting Critical Insertion Points

Let \( \text{Critical} \) denote the least solution of Equation System 4.1. The existence of critical insertion points for \( v \cdot c \) (i.e., of nodes \( n \) satisfying the predicate \( \text{CritIns}_{\text{SSR}}(n) = \text{df} \ \text{Insert}_{\text{SSR}}(n) \land \text{Critical}(n) \)) characterizes the situations in which the simple strength reduction transformation would insert multiplications as well as additions on some program paths. In these situations, the critical computations of \( v \cdot c \) must be replaced by noncritical ones. This is realized by moving them in the direction of control flow until all paths to a first use of \( v \cdot c \) are transparent for \( v \) instead of only SR-transparent. Technically, this is accomplished by determining the least solution of Equation System 4.2, which specifies a forward analysis of \( G \).

**Equation System 4.2 (SUBST-CRIT)**

\[
\text{SUBST-CRIT}(n) = \text{CritIns}_{\text{SSR}}(n) \lor \sum_{m \in \text{pred}(n)} (\neg \text{Used}(m) \land \text{SUBST-CRIT}(m))
\]

We denote the least solution of Equation System 4.2 by \( \text{Subst-Crit} \). Code motion insertion points\(^{14}\) satisfying \( \text{Subst-Crit} \) are, in fact, the "earliest" substitutes of critical insertion points of the simple strength reduction transformation guaranteeing that multiplications and additions are not simultaneously inserted on program paths.

### 4.1.3 The First Refinement

Let \( \text{Insert}_{\text{FstRef}} \) be defined by

\[
\forall n \in N. \text{Insert}_{\text{FstRef}}(n) = \text{df} (\text{Insert}_{\text{SSR}}(n) \land \neg \text{Critical}(n)) \lor (\text{Insert}_{\text{CM}}(n) \land \text{Subst-Crit}(n))
\]

and let \( \text{Update}_{\text{FstRef}} \) be defined as the least solution of Equation System 3.3 using \( \text{Insert}_{\text{FstRef}} \) in place of \( \text{Insert}_{\text{SSR}} \). With \( \text{InsUpd}_{\text{FstRef}} \) defined analogously as in the simple strength reduction transformation, i.e., \( \forall n \in N. \text{InsUpd}_{\text{FstRef}}(n) = \text{df} \text{Injured}(n) \land \text{Update}_{\text{FstRef}}(n) \), the following three-step procedure specifies the first refinement, which overcomes the multiplication-addition deficiency (cf. Section 3.3).

1. Introduce a new auxiliary variable \( h \) for \( v \cdot c \)
2. Insert at the entry of every node satisfying
   - \( \text{Insert}_{\text{FstRef}} \) the assignment \( h := v \cdot c \)
   - \( \text{InsUpd}_{\text{FstRef}} \) the assignment \( h := h + \text{Eff}(n) \)
3. Replace every (original) occurrence of \( v \cdot c \) in \( G \) by \( h \)

Table 3: The First Refinement

\(^{14}\)I.e., nodes satisfying \( \text{Insert}_{\text{CM}} \).
Figure 4 shows the result of this transformation for the flowgraph of Figure 1, which coincides with the result of the bidirectional algorithm of [3] that enhances the algorithm of [2]. Here, the initialization of $h$ at node 10, and the update assignments at nodes 4 and 18 of Figure 3 are replaced by a single initialization of $h$ at node 7. However, as in Figure 3, the lifetimes of moved computations are still unnecessarily long, and on path $(14, 17, 15, 12, 6, 9, 11)$ unnecessarily many update assignments for $h$ are inserted.
4.2 Second Refinement: Avoiding the Lifetime Deficiency

In this section we refine the strength reduction transformation further in order to overcome the lifetime deficiency mentioned in Section 3.3.

4.2.1 Latestness

In order to minimize the lifetimes of moved computations, they must be placed as late as possible, while maintaining the benefits of the algorithm developed so far. Intuitively, this requires to move computations from their earliest down-safe noncritical computation points in the direction of control flow to “later” computation points. Technically, this is realized by determining the greatest solution of Equation System 4.3, which requires a forward analysis of \( G \).

**Equation System 4.3 (DELAY)**

\[
\text{DELAY}(n) = \text{Insert}_{\text{FstRef}}(n) \lor \begin{cases} 
\text{false} & \text{if } n = s \\
\prod_{m \in \text{pred}(n)} (\neg \text{Used}(m) \land \text{DELAY}(m)) & \text{otherwise}
\end{cases}
\]

The intuition behind the definition of \( \text{DELAY} \) is to move computations from their earliest down-safe noncritical computation points as far as possible in the direction of control flow. Thus, \( \text{Insert}_{\text{FstRef}} \) implies \( \text{DELAY} \). This movement must stop in nodes \( n \) that have a predecessor \( m \) containing a computation of \( v \ast c \), or for which the process of moving is not successful, i.e., where \( \text{DELAY}(m) \) does not hold. In the first case, we would miss replacing an original occurrence of \( v \ast c \), and in the second case a partial redundancy would be introduced into the program.

**Latest Computation Points**

Let \( \text{Delay} \) denote the greatest solution of Equation System 4.3. Then we define

\[
\forall n \in N. \text{Latest}(n) := \text{Delay}(n) \land (\text{Used}(n) \lor \prod_{m \in \text{succ}(n)} \text{Delay}(m))
\]

The second refinement does not affect the insertion of update assignments. Thus, \( \text{InsUpd}_{\text{SndRef}} \) coincides with the version from the first refinement, i.e., \( \text{InsUpd}_{\text{SndRef}} = d_f \text{InsUpd}_{\text{FstRef}} \). Together \( \text{Latest} \) and \( \text{InsUpd}_{\text{SndRef}} \) specify a program transformation, where the resulting program is of the same computational complexity as that of the first refinement. However, it is more economic with respect to the lifetimes of auxiliary variables, as shown in Figure 5. Note, however, that the flowgraph of Figure 5 still contains an unnecessary initialization of \( h \) in node 21, which is only used in the insertion node itself.

4.2.2 Isolation

In order to avoid unnecessary initializations as in node 21 of Figure 5, we determine all program points where an inserted computation would be isolated, i.e., where an inserted computation would only be used in the insertion node itself (cf. [1]). This is achieved by determining the greatest solution of Equation System 4.4, which specifies a backward analysis of \( G \). Note that this analysis does not depend on the number of occurrences of the candidate expression in an insertion node itself, since expression evaluation without multiple calculations of common subexpressions is well understood in code generation (cf. [33]).
Equation System 4.4 (ISOLATED)

\[ \text{ISOLATED}(n) = \prod_{m \in \text{succ}(n)} (\text{Latest}(m) \lor (\neg \text{Used}(m) \land \text{ISOLATED}(m))) \]

4.2.3 The Second Refinement

Nodes satisfying Latest and ¬Isolated, where Isolated denotes the greatest solution of Equation System 4.4, specify the optimal computation points for \( v \cdot c \) in \( G \), and are denoted by the predicate Insert\textsubscript{SndRef}. In contrast to the preceding strength reduction transformations,
the occurrences of $v \times c$ satisfying Latest and Isolated are no longer replaced by $h$, because their corresponding initializations of $h$ are suppressed for efficiency reasons. All other original occurrences of $v \times c$, however, are redundant with respect to the computation points given by InsertSndRef, and can be eliminated. This is indicated by the predicate DeleteSndRef.

Now the second refinement is obtained by the following three-step procedure, which transforms the flowgraph of Figure 1 into the one shown in Figure 6. In fact, our algorithm is unique in performing this transformation.

1. Introduce a new auxiliary variable $h$ for $v \times c$
2. Insert at the entry of every node satisfying
   
   (a) InsertSndRef the assignment $h := v \times c$
   
   (b) InsUpdSndRef the assignment $h := h + Eff(n)$
3. Replace every (original) occurrence of $v \times c$ in nodes satisfying DeleteSndRef by $h$

Table 4: The Second Refinement

4.3 Third Refinement: Avoiding the Multiple-Addition Deficiency

The flowgraph of Figure 6 still contains unnecessarily many update assignments for $h$ (see path $(14, 17, 15, 12, 6, 9, 11)$). Inside extended basic blocks, however, the effect of additive modifications of $v$ onto the value of $v \times c$ can be accumulated in update assignments inserted at use sites of $v \times c$ or at the end of the blocks. The effect of such accumulation is illustrated by means of the differences in Figure 7 and 8, where it is assumed that the original righthand side term of the assignments at node 1 and 2 is $i \times 10$. Figure 7 shows the result of inserting an update assignment for every additive modification of $i$, as it is realized by the strength reduction transformation developed so far, and which still suffers from the multiple-addition deficiency. In contrast, Figure 8 shows the result of accumulating the effects of the update assignments.

Our algorithm for lazy strength reduction is unique in overcoming the multiple-addition-deficiency by accumulating the effects of update assignments. In [3] for every modification of the candidate expression, an update assignment is inserted. However, in contrast to [2] the insertion of update assignments is controlled by a machine-dependent parameter indicating, which number of updates is faster than a recomputation of the value. If this number is exceeded on a path, a recomputation of the value is inserted instead of the sequence of updates. In [4] this parameter is set to 1. Thus, in the example of Figure 1 Dhandhere’s algorithm would insert the assignment $h := i \times 10$ instead of $h := h + 80$ in node 9, and therefore would not achieve any strength reduction.\footnote{In fact, in the example of Figure 1 the algorithm of [4] delivers the result of the first refinement except that the updates in the right part of the loop are replaced by the multiplication in node 9.}

4.3.1 Accumulation

The accumulation of update assignments requires the predicate Accumulating that characterizes the set of program points, where an accumulating update assignment must potentially be inserted.

$$\forall n \in N. \; \text{Accumulating}(n) = \text{UpdateSndRef}(n) \land (\text{Used}(n) \lor \text{ExitExtBscBlck}(n))$$

\footnote{A basic block is a maximal sequence of code, where at most the first node has more than one predecessor, and at most the last node more than one successor (cf. [25]). An extended basic block is a maximal sequence of code, in which at most the first node has more than one predecessor, and all sons of nodes with more than one successor have a unique predecessor. Thus, an extended basic block is a maximal tree of nodes, such that control can enter the tree only at its root, and a basic block is a maximal tree with just one leaf (cf. [30]).}
Here, the predicate $ExitExtdBscBlck$ characterizes exit nodes of extended basic blocks. It is defined by

$$\forall n \in N. \ ExitExtdBscBlck(n) = \neg (n = e) \land \bigvee_{m \in \text{succ}(n)} EntryExtdBscBlck(m)$$

where $EntryExtdBscBlck$ characterizes entry nodes of extended basic blocks:

$$\forall n \in N. \ EntryExtdBscBlck(n) = \neg (n = s) \lor \bigvee_{m \in \text{pred}(n)} |\text{pred(succ(m))}| > 1$$

The accumulation process has to be terminated in nodes satisfying the predicate $EntryExtdBscBlck$ or having a predecessor satisfying $Used$. Denoting these nodes by the pred-

Figure 6: The Second Refinement
Figure 7: Illustration of the Multiple-Addition Deficiency

icate $AccumTerm$, the function $AccumEff : N \rightarrow \omega$ determines the accumulated effect of a sequence of update assignments.

$$\forall n \in N. \ AccumEff(n) = df \begin{cases} 0 & \text{if } \neg Update_{SndRef}(n) \\ Eff(n) & \text{if } Update_{SndRef}(n) \land AccumTerm(n) \\ Eff(n) + AccumEff(m) & \text{otherwise } (\text{pred}(n) = \{m\} ) \end{cases}$$

**Remark 4.5** In the example of Figure 8, we can even accumulate the effect of consecutive extended basic blocks in single update assignments, as shown in Figure 11 in Section A.4. The point here is that in all predecessors of the entry nodes of “sibling” extended basic blocks the same update assignment is inserted (in Figure 8: $h := h + 50$). This pattern can be captured in general by means of a refined version of the Accumulating predicate introduced above.

4.3.2 Third Refinement: The Complete Transformation

The third refinement affects only the insertion points of update assignments. Thus, the predicates characterizing the nodes where the auxiliary variable must be initialized, and where an original occurrence of $v \star c$ can be deleted coincide with the corresponding predicates in the second refinement, i.e., $Insert_{LSR} = df Insert_{SndRef}$ and $Delete_{LSR} = df Delete_{SndRef}$. Update assignments must be inserted in nodes satisfying the predicate $InsUpd_{LSR}$. This predicate is true for nodes $n$ satisfying the predicate Accumulating with $AccumEff(n) \neq 0$. 

16
The Lazy Strength Reduction Transformation

The lazy strength reduction transformation, which overcomes all three deficiencies mentioned in Section 3.3, proceeds in three steps.

1. Introduce a new auxiliary variable $h$ for $v * c$

2. Insert at the entry of every node satisfying
   
   (a) $\text{Insert}_{\text{LSR}}$ the assignment $h := v * c$
   
   (b) $\text{InsUpd}_{\text{LSR}}$ the assignment $h := h + \text{AccumEff}(n)$

3. Replace every (original) occurrence of $v * c$ in nodes satisfying $\text{Delete}_{\text{LSR}}$ by $h$

<p>| | | |</p>
<table>
<thead>
<tr>
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</thead>
</table>
| 1 | $h := h + 20$ | $a := h$
| 2 | $h := h + 40$ | $h := i * 10$

Table 5: The Lazy Strength Reduction Transformation

Table 6 in Section A.3 summarizes the values of the predicates considered during the development of the lazy strength reduction algorithm for the term $i * 10$ in the flowgraph in Figure 1.

Application of the lazy strength reduction transformation to the flowgraph in Figure 1 results in the promised flowgraph in Figure 2, which in fact is free of all the deficiencies discussed above.

5 Conclusions

We have presented a bit-vector algorithm for lazy strength reduction, which is unique in its transformational power in that it uniformly combines code motion and strength reduction, and
completely avoids superfluous register pressure due to unnecessary code motion. Moreover, like its underlying algorithm for lazy code motion ([1]), it is composed of a sequence of unidirectional analyses. This allows us to apply the efficient algorithms for unidirectional bit-vector analyses to deal with all program terms simultaneously (cf. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) as well as to interleave strength reduction and copy propagation using the slotwise approach of [18]. Additionally, its modular structure supports further extensions. For example, following the lines of [34, 20] it is straightforward to generalize this algorithm to programs with (mutually) recursive procedures, global and local variables, and formal (value) parameters.

References


In order to exploit the full power of our lazy strength reduction algorithm, “critical” edges, i.e., edges leading from nodes with more than one successor to nodes with more than one predecessor, must be removed from the flowgraph, since they may block the process of code motion and strength reduction (cf. [17, 16, 18, 19, 1, 20, 31, 32, 26]), as illustrated in Figure 9.

In Figure 9(a) the computation of “a + b” at node 3 is partially redundant with respect to the computation of “a + b” at node 1. However, this partial redundancy cannot safely be eliminated by moving the computation of “a + b” to its preceding nodes, because this may introduce a new computation on a path leaving node 2 on the right branch. On the other hand, it can safely be eliminated after inserting a synthetic node 4 in the critical edge (2, 3), as illustrated in Figure 9(b). Inserting a synthetic node on every edge leading to a node with more than one predecessor certainly implies that all critical edges are removed, and additionally, it allows us to insert all computations uniformly at the entries of nodes.
A.2 Illustrating Earliestness

Figure 10: Illustrating Down-Safety and Earliestness

Figure 10 shows the predicate values of $\text{Earliest}_{\text{CM}}$ for a small example. This illustrates that $\text{Earliest}_{\text{CM}}$ is valid at the start node and additionally at those nodes that are reachable by a path on which a prior computation of $v * c$ would not be down-safe or delivers a different value due to a subsequent modification of $v$. Of course, $v * c$ cannot be placed earlier than in the start node, which justifies $\text{Earliest}(1)$. Moreover, every computation of $v * c$ in a node on the paths $(1, 2)$ and $(1, 3, 7)$ would not be down-safe. Thus $\text{Earliest}_{\text{CM}}([2, 3, 4, 5, 7, 9, 12])$ holds. Finally, evaluating $v * c$ at node $1, 2$ and $5$ delivers a different value as in node $8$. This yields $\text{Earliest}(8)$. 

\[ s = 1 \]
\[ 2 \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 3 \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 4 \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 5 \]
\[ v := v / w \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 6 \]
\[ x := v * c \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ 7 \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 8 \]
\[ y := v * c \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 9 \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ \text{Earliest}_{\text{CM}} \]
\[ 10 \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ 11 \]
\[ z := v * c \]
\[ \text{D-Safe}_{\text{CM}} \]
\[ e = 12 \]
\[ \text{Earliest}_{\text{CM}} \]
### Relevant Predicate Values for the Motivating Example

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<th>Predicate</th>
<th>Node Number</th>
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<td>Isolated</td>
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</tr>
</tbody>
</table>

Table 6: Relevant Predicate Values for the Motivating Example
A.4 Refined Accumulation

Figure 11: Refined Accumulation of the Effects of Update Assignments