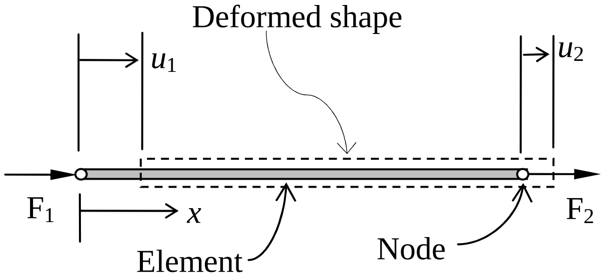
Introduction to Finite Element Dynamics

1. Element Stiffness Matrices

1.1 Bar Element

Consider a bar element shown below. At the two ends of the bar, axially aligned forces [F_1, F_2] are applied producing deflections [u_1,u_2]. What is the relationship between applied force and deflection?



There are typically five key stages in the analysis[1].

(i) ***Conjecture a displacement function***
The displacement function is usually an approximation, that is continuous and differentiable, most usually a polynomial. $u(x)$ is the axial deflections at any point x along the bar element.

$$u(x) = a_1 + a_2 x = [1 \quad x]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = [\mathrm{N}]\underline{a} \qquad (1)$$

(ii) ***Express $u(x)$ in terms of nodal displacements by using boundary conditions.***
i.e. $u(0) = u_1$, $u(L) = u_2$ where L is the bar length. It is clear that with only two nodes only two unknown coefficients for the polynomial can be defined i.e. a linear relationship. Thus if a higher order polynomial is required for $u(x)$ more boundary conditions would be required and hence intermediate nodes along the length of the bar are necessary.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ so } \underline{u} = [\mathrm{A}]\underline{a} \qquad (2)$$

Substituting (2) into (1)

$$u(x) = [\mathrm{N}][\mathrm{A}]^{-1}\underline{u} = [1 \quad x]\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}^{-1}\underline{u}$$

$$u(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}\underline{u} = [\mathrm{C}]\underline{u} \qquad (3)$$

(iii) ***Derive Strain - Displacement Relationship by using Mechanics Theory***[2]
The axial strain $\varepsilon(x)$ is given by the following

$$\varepsilon(x) = \frac{du}{dx} = \frac{d}{dx}([\mathrm{C}]\underline{u}) = [\mathrm{B}]\underline{u} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\underline{u} \qquad (4)$$

(iv) ***Derive Stress - Displacement Relationship by using Elasticity Theory***
The axial stresses $\sigma(x)$ are given by the following assuming linear elasticity and homogeneity of material throughout the bar element. Where E is the material modulus of elasticity.

$$\sigma(x) = E\varepsilon(x) = E[\mathrm{B}]\underline{u} \qquad (5)$$

(v) ***Use principle of Virtual Work***
There is equilibrium between W_I , the internal work done in deforming the bar, and W_E , external work done by the movement of the applied forces. The bar cross sectional area $A = \iint dydz$ is assumed constant with respect to x.

$$\begin{aligned} W_I &= \iiint \sigma(x).\varepsilon(x)dxdydz = \int \sigma(x).\varepsilon(x)dx \iint dydz \\ &= A\int \sigma(x).\varepsilon(x)dx \end{aligned} \qquad (6)$$

$$W_E = [u_1 \quad u_2]\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underline{u}^T \underline{F} \qquad (7)$$

Hence since $W_E = W_I$ by substituting (4) and (5)
Note $[\mathrm{B}]\underline{u}$ is a scalar thus $[\mathrm{B}]\underline{u} = ([\mathrm{B}]\underline{u})^T = \underline{u}^T[\mathrm{B}]^T$

$$\underline{u}^T \underline{F} = EA\int [\mathrm{B}]\underline{u}.[\mathrm{B}]\underline{u}dx = EA\underline{u}^T \int [\mathrm{B}]^T[\mathrm{B}]dx\underline{u}$$

$$\underline{F} = \left\{EA\int [\mathrm{B}]^T[\mathrm{B}]dx\right\}\underline{u} = [\mathrm{K}]\underline{u} \qquad (8)$$

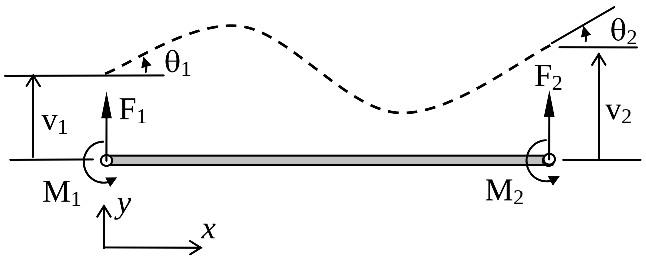
Equation (8) represents the force-displacement relationship for the bar element. The Stiffness matrix [K] is derived by integration thus

$$\begin{aligned} [\mathrm{K}] &= EA\int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}dx \\ &= EA\int_0^L \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \qquad (9)$$

1.2 Beam Element

A similar approach can be adopted to deriving the force-displacement relationship for a beam element, i.e. one that is subject to flexural rather than axial deformation.

Consider a beam element shown below. At the two ends of the bar, normally aligned forces and moments $\underline{F} = [F_1 \quad M_1 \quad F_2 \quad M_2]^T$ are applied to produce deflections and rotations given by $\underline{u} = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$ What is the relationship between applied Force/Moment and Deflection/Rotations?



(i) *Conjecture a displacement function*

The normal displacement *u(x)* is a cubic polynomial. The slope $u'(x)$ of the beam is given by its first derivative.

$$u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$
$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [\mathrm{N}]\underline{a} \qquad (10)$$

$$u'(x) = a_2 + 2a_3 x + 3a_4 x^2$$
$$= \begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix}\underline{a} = [\mathrm{N}']\underline{a} \qquad (11)$$

(ii) *Express $u(x)$ in terms of nodal displacements and rotations by using boundary conditions.*

i.e. $u(0) = v_1$, $u'(0) = \theta_1$, $u(L) = v_2$, $u'(L) = \theta_2$,

$$\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \text{ so } \underline{u} = [\mathrm{A}]\underline{a} \qquad (12)$$

hence substituting (12) into (10) and (11)

$$u(x) = [\mathrm{N}][\mathrm{A}]^{-1}\underline{u} = [\mathrm{C}]\underline{u} \qquad (13)$$
$$u'(x) = [\mathrm{N}'][\mathrm{A}]^{-1}\underline{u}$$

(iii) *Derive "Strain"-Displacement Relationship*

Beam curvature which turns out to be the proxy for "strain" in the virtual work equation. The beam curvature *c(x)* is approximated by the second derivative of the deflection. This assumes that the flexural deflections are small, i.e the beam remains almost flat.

$$c(x) = \frac{d^2u}{dx^2} = \frac{d^2}{dx^2}([\mathrm{C}]\underline{u}) = [\mathrm{N}''][\mathrm{A}]^{-1}\underline{u} = [\mathrm{B}]\underline{u} \qquad (14)$$

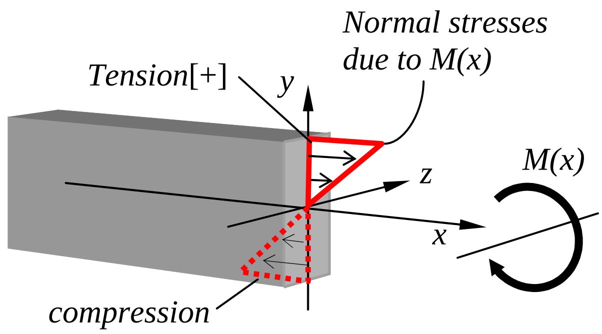
(iv) *Derive "Stress"-Displacement Relationship*

Here, the relationship between moment *m(x)* and curvature *c(x)* is the proxy for "stress"-"strain". For a beam element the Bernoulli-Euler simple beam theory result is used. This assumes no shear deformations are present.

$$M(x) = EI.c(x) = EI[\mathrm{B}]\underline{u} \qquad (15)$$

(v) *Use principle of Virtual Work*

The internal virtual work is given by the same fundamental expression as (6). However it is modified by application of Bernoulli-Euler simple beam theory. The flexurally induced normal stresses $\sigma(x,y) = M(x)y/I_z$ where the origin of the y axis is the centroidal axis and $I_z = \iint y^2 dydz$ is the second moment of area about the z axis; assuming homogeneity. Shear stresses are neglected.



$$W_I = \iiint \sigma.\varepsilon \, dxdydz = \iiint \frac{M(x)y}{I_z}\frac{M(x)y}{EI_z} dxdydz$$
$$= \int_0^L \frac{M(x)^2}{EI_z^2} dx. \iint y^2 dydz = \int_0^L \frac{M(x)^2}{EI_z} dx$$
$$= \int_0^L M(x)c(x)dx$$

Since $W_E = W_I$ and by substituting (14) and (15)

$$\underline{u}^T \underline{F} = \int M(x)c(x)dx$$

$$\underline{F} = \left\{ EI\int [\mathrm{B}]^T [\mathrm{B}]dx \right\} \underline{u} = [\mathrm{K}]\underline{u} \qquad (16)$$

By comparing (16) and (8) it can be observed that the general form of the stiffness matrix is given by a similar matrix integral equation.

Hence stiffness matrix [K] is derived thus

$$[\mathrm{K}] = EI\left([\mathrm{A}]^{-1}\right)^T \int [\mathrm{N}'']^T [\mathrm{N}'']dx [\mathrm{A}]^{-1}$$

$$\int [\mathrm{N}'']^T [\mathrm{N}'']dx = \int_0^L \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6x \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 6x \end{bmatrix} dx$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}$$

Using matrix algebra

$$[\mathrm{A}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix}$$

Hence

$$[\mathrm{K}] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (17)$$

1.3 Combined Beam/Bar Element

This element contains both axial and flexural displacements. The stiffness matrix that relates action and deformation can be derived by inspection from results (17) and (9). However, it is instructive to derive it using the five stages used previously.

(i) ***Conjecture displacement functions***

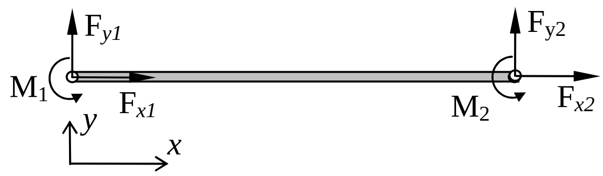
In this problem, because there are two distinct and separate deformations i.e. axial and flexural, there is a need for two separate displacement functions

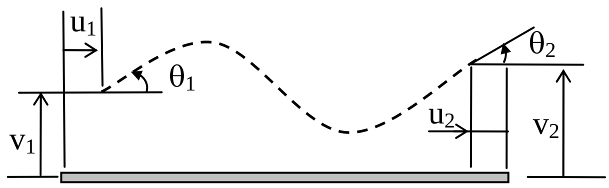
$$u_y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$
$$u_x(x) = a_5 + a_6 x$$

$$\begin{bmatrix} u_y(x) \\ u_x(x) \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} \qquad (18)$$

$$\underline{U(x)} = [\mathrm{N}]\underline{a}$$

Thus a vector of displacement functions is conjectured.





(ii) ***Express*** ***U(x)*** ***in terms of nodal displacements and rotations by using boundary conditions.***

The boundary conditions are the same as those used in the previous sections

$$\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L \\ 1 & L & L^2 & L^3 & 0 & 0 \\ 0 & 1 & 2L & 3L^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \quad \text{so } \underline{u} = [\mathrm{A}]\underline{a}$$

hence $\underline{U(x)} = [\mathrm{N}][\mathrm{A}]^{-1}\underline{u}$ (19)

(iii) ***Derive "Strain"-Displacement Relationship***

$$\begin{bmatrix} c(x) \\ \varepsilon(x) \end{bmatrix} = \begin{bmatrix} \frac{d^2 u_y}{dx^2} \\ \frac{du_x}{dx} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 6x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{a} \qquad (20)$$

$$\underline{\varepsilon} = [\mathrm{N}''][\mathrm{A}]^{-1}\underline{u} = [\mathrm{B}]\underline{u}$$

(iv) ***Derive "Stress"-Displacement Relationship***

$$\begin{bmatrix} M(x) \\ A\sigma(x) \end{bmatrix} = \begin{bmatrix} EI & 0 \\ 0 & EA \end{bmatrix} \begin{bmatrix} c(x) \\ \varepsilon(x) \end{bmatrix} \quad \text{so } \underline{\sigma} = [\mathrm{D}]\underline{\varepsilon} \qquad (21)$$

(v) ***Use principle of Virtual Work***

$$\underline{u}^T \underline{F} = \int \underline{\varepsilon}^T \underline{\sigma} dx = \underline{u}^T \int [B]^T [D][B] dx\, \underline{u}$$

$$\underline{F} = \left\{ \int [B]^T [D][B] dx \right\} \underline{u} = [K] \underline{u} \qquad (22)$$

Where $\underline{F} = \begin{bmatrix} F_{x1} & F_{y1} & M_1 & F_{x2} & F_{y2} & M_2 \end{bmatrix}^T$

$$[K] = \left([A]^{-1}\right)^T \int [N'']^T [D][N''] dx [A]^{-1} \qquad (23)$$

Equation (23) can be evaluated by using Maple[3]. The following code will evaluate the stiffness matrix.

```
> with(linalg):
> A:=matrix(6,6,[[0,0,0,0,1,0],
[1,0,0,0,0,0], [0,1,0,0,0,0],
[0,0,0,0,1,L], [1,L,L^2,L^3,0,0],
[0,1,2*L,3*L^2,0,0]]);
```

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L \\ 1 & L & L^2 & L^3 & 0 & 0 \\ 0 & 1 & 2L & 3L^2 & 0 & 0 \end{bmatrix}$$

```
> N2:=matrix(2,6,[[0,0,2,6*x,0,0],
[0,0,0,0,0,1]]);
```

$$N2 := \begin{bmatrix} 0 & 0 & 2 & 6x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> DD:=matrix(2,2,[[EI[z],0],[0,EA]]);
```

$$DD := \begin{bmatrix} EI_z & 0 \\ 0 & EA \end{bmatrix}$$

```
> NDN:=map(int,evalm( transpose(N2) &*
DD &* N2 ), x=0..L);
```

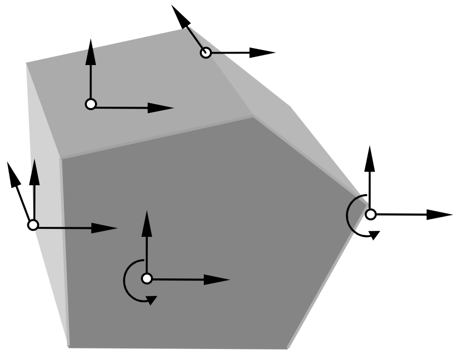
$$NDN := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\,EI_z\,L & 6\,EI_z\,L^2 & 0 & 0 \\ 0 & 0 & 6\,EI_z\,L^2 & 12\,EI_z\,L^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & EA\,L \end{bmatrix}$$

```
> K:=evalm(transpose(inverse(A)) &* NDN
&* inverse(A));
```

$$K := \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} & 0 & -12\frac{EI_z}{L^3} & 6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} & 0 & 12\frac{EI_z}{L^3} & -6\frac{EI_z}{L^2} \\ 0 & 6\frac{EI_z}{L^2} & 2\frac{EI_z}{L} & 0 & -6\frac{EI_z}{L^2} & 4\frac{EI_z}{L} \end{bmatrix} \qquad (24)$$

1.4 General Element

Consider an abstract, general, element with *n* nodes and *m* nodal displacements/rotations, known as *degrees of freedom*. The process of deriving the stiffness matrix for this element is identical to that adopted in the previous sections. Nodes do not have to be at edges or corners, though normally they are so that elements can be easily matched together in some mesh that will describe a structure. Also Nodes need not have the same number of degrees of freedom though normally they do for a similar reason.



(i) ***Conjecture displacement functions***

$$\underline{U(x,y,z)} = [N]\underline{a} \qquad (a)$$

(ii) ***Express U(x,y,z) in terms of nodal displacements and rotations by using boundary conditions***

$$\underline{u} = [A]\underline{a} \qquad (b)$$

hence $\quad \underline{U(x,y,z)} = [N][A]^{-1}\underline{u} = [C]\underline{u} \qquad (c)$

(iii) ***Derive "Strain"-Displacement Relationship***

$$\underline{\varepsilon} = [N'']\underline{a} = [N''][A]^{-1}\underline{u} = [B]\underline{u} \qquad (d)$$

(iv) ***Derive "Stress"-Displacement Relationship***

$$\underline{\sigma} = [D]\underline{\varepsilon} = [D][B]\underline{u} \qquad (e)$$

(v) ***Use principle of Virtual Work***

$$\underline{u}^T \underline{F} = \iiint \underline{\varepsilon}^T \underline{\sigma}\, dxdydz = \underline{u}^T \iiint [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}]\, dxdydz\, \underline{u}$$

$$\underline{F} = \left\{ \iiint [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}]\, dxdydz \right\} \underline{u} = [\mathrm{K}]\underline{u} \qquad \text{(f)}$$

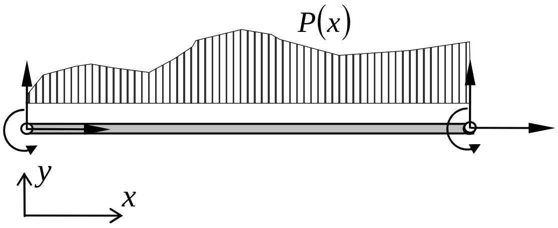
$$[\mathrm{K}] = \iiint [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}]\, dxdydz \qquad \text{(g)}$$

2. Generalised Actions

In the previous sections, $\underline{F}$, the applied actions on the element are always applied at nodes. However in a more general case what will $\underline{F}$ be if actions are applied at any point on the element.

2.1 Distributed Load for Beam Element

The aim is to express the distributed load $P(x)$ in terms of $\underline{F}$ the nodal actions.



Let us reformulate the equation for the external virtual work, thus

$$W_E = \int_0^L u(x)P(x)\,dx = \int_0^L [\mathrm{N}][\mathrm{A}]^{-1}\underline{u}\, P(x)\,dx$$
$$= \underline{u}^T \left([\mathrm{A}]^{-1}\right)^T \int_0^L [\mathrm{N}]^T\, P(x)\,dx = \underline{u}^T \underline{F} \qquad (25)$$

2.1.1 Example 1, linearly increasing distributed load

Consider as an example, $P(x) = w\,x$ i.e. a linear distributed load then (25) becomes $\underline{F} = \left([\mathrm{A}]^{-1}\right)^T \int_0^L [\mathrm{N}]^T\, w\,x\,dx$ which can be evaluated using Maple thus.

```
> with(linalg):
> A:=matrix(4,4,[[1,0,0,0], [0,1,0,0],
[1,L,L^2,L^3], [0,1,2*L,3*L^2]]);
```

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}$$

```
> N:=matrix(1,4,[1,x,x^2,x^3]);
```

$$N := [1 \quad x \quad x^2 \quad x^3]$$

```
> F:=evalm( transpose(inverse(A)) &*
map(int, evalm( w*x * transpose(N)),
x=0..L));
```

$$F := \begin{bmatrix} \frac{3}{20} w L^2 \\ \frac{1}{30} w L^3 \\ \frac{7}{20} w L^2 \\ -\frac{1}{20} w L^3 \end{bmatrix}$$

Using Maple it is not difficult to derive nodal action $\underline{F}$ in terms of any $P(x)$.

2.1.2 Example 2, Point load *P* at position $L/2$

If $P(x)$ is discrete and discontinuous, i.e. the case of an applied point load, the integrals drop out of (25) thus

$$W_E = u(L/2)P = [N(L/2)][A]^{-1}\underline{u}\, P$$
$$= \underline{u}^T \left([A]^{-1}\right)^T [N(L/2)]^T\, P = \underline{u}^T \underline{F} \qquad (26)$$

Continue the Maple file from the previous section by adding the following line, thus

```
> F:=evalm( P* transpose(inverse(A)) &*
transpose(map2(subs,x=L/2,N)));
```

$$F := \begin{bmatrix} \frac{1}{2} P \\ \frac{1}{8} P L \\ \frac{1}{2} P \\ -\frac{1}{8} P L \end{bmatrix}$$

2.2 General loads for a general element

In a completely general case $\underline{P(x,y,z)}$ will be a vector of loads, in different directions that are distributed across the element.

$$W_E = \iiint \underline{U(x,y,z)}^T\, \underline{P(x,y,z)}\, dxdydz$$
$$= \underline{u}^T \left([\mathrm{A}]^{-1}\right)^T \iiint [\mathrm{N}]^T\, \underline{P(x,y,z)}\, dxdydz = \underline{u}^T \underline{F}$$

$$\underline{F} = \left([\mathrm{A}]^{-1}\right)^T \iiint [\mathrm{N}]^T\, \underline{P(x,y,z)}\, dxdydz \qquad \text{(h)}$$

3. Element Mass Matrices

3.1 Bar Element

In order to compute the element mass matrices it is necessary to consider the work done by the inertial accelerations.

$$W_A = \iiint \rho \ddot{u}(x) u(x) dxdydz = \int \ddot{u}(x) u(x) dx \iint \rho dydz = \rho A \int \ddot{u}(x) u(x) dx \qquad (27)$$

Where $\rho\ddot{u}$ is the inertial force per unit volume and ρ is the density; which is assumed constant, throughout the bar, in this example.

$$\ddot{u}(x) = \frac{d^2u}{dt^2} = [\mathrm{C}]\underline{\ddot{u}}$$
$$W_A = \rho A \int ([\mathrm{C}]\underline{u})([\mathrm{C}]\underline{\ddot{u}}) dx = \underline{u}^T \int [\mathrm{C}]^T [\mathrm{C}] dx \underline{\ddot{u}} \qquad (28)$$

In the case of free vibrations

$$W_A + W_I = 0$$
$$\left\{\int [\mathrm{C}]^T [\mathrm{C}] dx\right\} \underline{\ddot{u}} + \left\{EA \int [\mathrm{B}]^T [\mathrm{B}] dx\right\} \underline{u} = 0 \qquad (29)$$
$$[\mathrm{M}]\underline{\ddot{u}} + [\mathrm{K}]\underline{u} = 0$$

$$[M] = \rho A \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} dx = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (30)$$

The mass matrix (30) is often termed a *consistent mass matrix*. The mass is distributed across the element and is not lumped at any particular places along the element.

3.2 Beam Element

If rotational accelerations of beam are neglected then the formula (29) can be applied directly to the beam element with an alternate $[C]$ matrix.

$$[\mathrm{M}] = \rho A \int [\mathrm{C}]^T [\mathrm{C}] dx = \rho A \left([\mathrm{A}]^{-1}\right)^T \int [\mathrm{N}]^T [\mathrm{N}] dx [\mathrm{A}]^{-1}$$

Using Maple this can be evaluated

```
> with(linalg):
> A:=matrix(4,4,[[1,0,0,0], [0,1,0,0], [1,L,L^2,L^3], [0,1,2*L,3*L^2]]);
```

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}$$

```
> N:=matrix(1,4,[1,x,x^2,x^3]);
```

$$N := [1 \quad x \quad x^2 \quad x^3]$$

```
> NTN:=map(int,evalm( transpose(N) &* N),x=0..L);
```

$$NTN := \begin{bmatrix} L & \frac{1}{2}L^2 & \frac{1}{3}L^3 & \frac{1}{4}L^4 \\ \frac{1}{2}L^2 & \frac{1}{3}L^3 & \frac{1}{4}L^4 & \frac{1}{5}L^5 \\ \frac{1}{3}L^3 & \frac{1}{4}L^4 & \frac{1}{5}L^5 & \frac{1}{6}L^6 \\ \frac{1}{4}L^4 & \frac{1}{5}L^5 & \frac{1}{6}L^6 & \frac{1}{7}L^7 \end{bmatrix}$$

```
> M:=evalm(rho*a* transpose(inverse(A)) &* NTN &* inverse(A) );
```

$$M := \frac{1}{420} \rho a L \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad (31)$$

3.3 General Element

For a general element with n nodes and m degrees of freedom, the work done by inertial accelerations (27) is characterised thus

$$W_A = \iiint \rho \underline{\ddot{u}(x,y,z)}^T \underline{u(x,y,z)} dxdydz$$

Hence the mass matrix is given by

$$[\mathrm{M}] = \iiint \rho [\mathrm{C}]^T [\mathrm{C}] \, dxdydz \qquad \text{(i)}$$

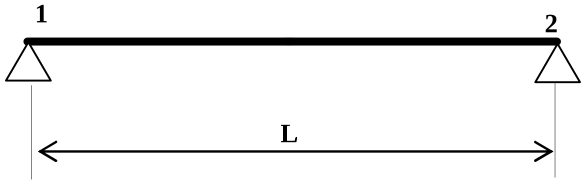
hence the equation of free vibration is given by the following $W_A + W_I = 0$

$$[\mathrm{M}]\underline{\ddot{u}} + [\mathrm{K}]\underline{u} = 0 \qquad \text{(j)}$$

4 Free Vibration Examples

4.1 Simply supported beam

Determine periods of vibration of a simply-supported beam with constant flexural rigidity *EI* and weight *w* per unit length



(17) and (31) are substituted into equation (j) to gives the following. By applying support restraints i.e. $v_1 = v_2 = 0$

$$\frac{wL}{420g}\begin{bmatrix} 156 & 22L & 54 & 13L \\ 54 & 4L^2 & 13L & -3L^2 \\ -12 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}\begin{bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = 0$$

$$\frac{wL^3}{420g}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$

Hence dynamic matrix [D] can be found. The resulting eigenvalue problem on [D] is solved.

$$[\mathrm{D}] = [\mathrm{M}]^{-1}[\mathrm{K}] = \frac{420EIg}{wL^4}\begin{bmatrix} \frac{22}{7} & \frac{20}{7} \\ \frac{20}{7} & \frac{22}{7} \end{bmatrix}$$

$$[\Lambda] = \frac{420EIg}{wL^4}\begin{bmatrix} \frac{2}{7} & 0 \\ 0 & 6 \end{bmatrix} \quad [\Phi] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The natural frequencies for the first two modes of vibration are approximated from the eigenvalues in the standard way $f = \sqrt{\lambda}/2\pi$.

$$f_1 = 1.74\sqrt{\frac{EIg}{wL^4}}\ , \quad f_2 = 4.61\sqrt{\frac{EIg}{wL^4}} \quad [\mathrm{Hz}] \quad (32)$$

The mode shapes can be extracted from the eigenvectors. In this case the first mode is given by $\underline{\phi}_1 = [1 \quad -1]^T$ and if the degrees of freedom that have been removed are now reinstated we get

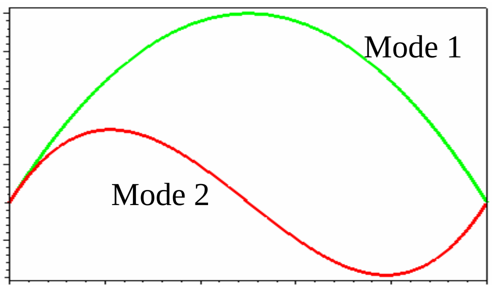
$$\underline{\phi}_1 \equiv [0 \quad 1 \quad 0 \quad -1]^T \text{ and } \underline{\phi}_2 \equiv [0 \quad 1 \quad 0 \quad 1]^T$$

Thus from equation (13) the modal shape is given by the following where the nodal displacement vector $\underline{u}$ is replaced by $\underline{\phi}$. Remember that as the magnitude of the eigenvector is unknown and is normalised. Thus, in general, mode shapes have unknown magnitude. Only with knowledge of some initial conditions, velocity and displacement at time *t*, can the relative magnitude of each mode shape be assessed for an actual free vibration problem.

$$u(x) = [\mathrm{N}][\mathrm{A}]^{-1}\underline{\phi}$$

$$u_1(x) = [\mathrm{N}][\mathrm{A}]^{-1}\underline{\phi}_1 = x\left(1 - \frac{x}{L}\right) \quad (33)$$

$$u_2(x) = [\mathrm{N}][\mathrm{A}]^{-1}\underline{\phi}_2 = x\left(1 - \frac{x}{L}\right)\left(1 - \frac{2x}{L}\right) \quad (34)$$



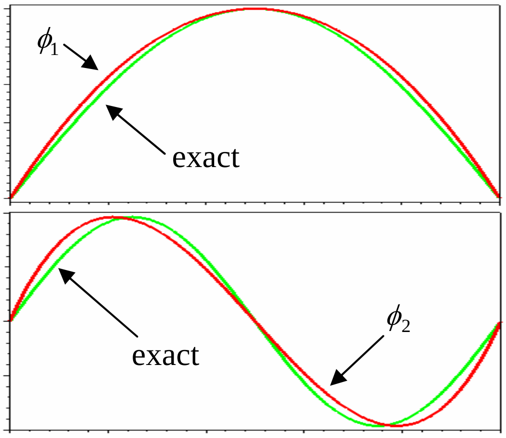
4.1.1 Accuracy of results

The accuracy of these results is dependant chiefly on the how near the cubic displacement function (10) is to the actual dynamic displacement function.

The exact shape for the first mode is $\sin(\pi x/L)$ and for the second mode $\sin(2\pi x/L)$; exact values for first and second frequencies are

$$f_1 = 1.571\sqrt{\frac{EIg}{wL^4}}\ , \quad f_2 = 6.283\sqrt{\frac{EIg}{wL^4}} \quad [\mathrm{Hz}]$$

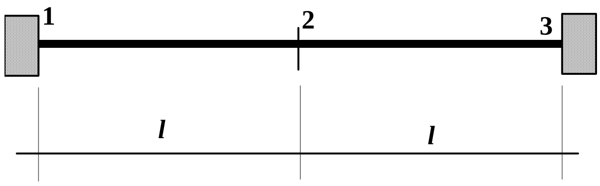
This means (32) overestimates the first mode by 11% and underestimates the second mode by 26.6%. The first mode is more accurate than the second mode.
The discrepancy of (33) and (34) from the exact solutions are displayed by the following graphs.



It is clear that the accuracy of the FE approach is dependant the displacement shape function. This is true if only one element is employed. To improve accuracy there are two alternate strategies you could adopt. (a) increase order of polynomial displacement shape function; or (b) use more that one element. (b) is the most commonly adopted strategy and is the basis of the finite element method[4]

4.2 Encastré beam

Determine periods of vibration of a fixed-end beam with constant flexural rigidity *EI* and weight per unit length *w*. The span *L=2l*



$$\frac{wl}{420g}\begin{bmatrix} 156 & 22l & 54 & -13l & 0 & 0 \\ & 4l^2 & 13 & -3l^2 & 0 & 0 \\ & & 312 & 0 & 54 & -13l \\ & & & 8l^2 & 13l & -3l^2 \\ & & & & 156 & -22l \\ & & & & & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{\theta}_1 \\ \ddot{y}_2 \\ \ddot{\theta}_2 \\ \ddot{y}_3 \\ \ddot{\theta}_3 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2 \\ & & & & 12 & -6l \\ & & & & & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ y_3 \\ \theta_3 \end{bmatrix} = 0$$

(17) and (31) are substituted into equation (j) to gives the above. By applying support restraints i.e. $v_1 = v_3 = \theta_1 = \theta_3 = 0$ these degrees of freedom are removed from the problem.

$$\frac{wl}{420g}\begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{bmatrix} \ddot{y}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = 0$$

$$[D] = [M]^{-1}[K] = \frac{420EIg}{wl^4}\begin{bmatrix} 1/13 & 0 \\ 0 & 1 \end{bmatrix}$$

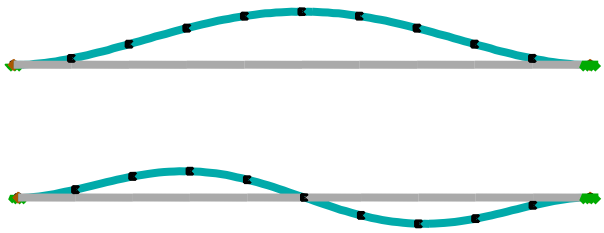
The resulting eigenvalue problem on [D] is solved and the approximate natural frequencies of vibration are determined; note that span *L* is reintroduced.

$$[\Lambda] = \frac{420EIg}{wl^4}\begin{bmatrix} \frac{1}{13} & 0 \\ 0 & 1 \end{bmatrix} \quad [\Phi] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f_1 = 3.618\sqrt{\frac{EIg}{wL^4}}, \quad f_2 = 13.05\sqrt{\frac{EIg}{wL^4}} \quad [\text{Hz}] \quad (35)$$

A more exact answer for these frequencies are given below from a 10 element model.

$$f_1 = 3.56\sqrt{\frac{EIg}{wL^4}} \quad f_2 = 9.81\sqrt{\frac{EIg}{wL^4}} \,[\text{Hz}]$$



This means (35) overestimates the first mode by 16.3% and overestimates the second mode by 33%. The first mode is more accurate than the second mode.

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