Torsion of Solid Shafts and the Thermal Analogy

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Introduction

The torsion of a circular solid shaft is a common special case, of prismatic bar torsion, that is covered in a basic course in mechanics of materials. Since a circular cross-section has an infinite number of lines of symmetry it was assumed (and verified) that plane cross-sections remain plane during torsion of the circular shafts. It was found that the shear stress is given by

$$\tau = Tc/J,$$

where $c$ is the radius to the point, $T$ is the applied torque, and $J = \pi d^4/32$ is the polar moment of inertia of the cross-section. The corresponding angle of twist, $\theta$, for a shaft of length $L$ having a shear modulus of $G$ is

$$\theta = TL/GJ,$$

and the constant twist per unit length is $\beta = \theta/L$. Note that the above shear stress varies directly with the radial distance from the center of twist. That is not true for the torsion of non-circular shafts. In general, the point of maximum shear stress in a bar under torsion occurs at the point(s) where the largest inscribed circle touches the perimeter of the bar.

Torsion of non-circular shafts

For a non-circular bar the maximum angle of rotation is modified to be expressed as $\theta = TL/GK$. The torsional stiffness for a shaft is defined as the product $GK$, $K \leq J$. The torsional constant of a shaft, $K$, is usually $\ll J$, and equals $J$ only for the circular shaft. For every cross-section the applied torque is proportional to the torsional stiffness: $T/\beta = GK$. Saint-Venant developed an approximate solution for the torsional constant that is accurate for solid shapes except for elongated sections:

$$K \approx \frac{A^4}{40J},$$

which is a useful rule of thumb to remember.

For a non-circular shaft a plane transverse section will warp normal to the plane when loaded by the torque. However, the in-plane motion takes place as a rigid body rotation of the original cross-section. The shear stress is zero at any boundary point that corresponds to an exterior corner. If the cross-section has a sharp (no fillet) re-entrant corner then the warping of the originally plane section causes a local singularity which theoretically gives an infinite shear stress at the corner point.

It can be shown that the only non-zero stresses in the pure twisting of a shaft are $\tau_{xy}$ and $\tau_{xz}$. The maximum shear stress occurs at a point on a traction free boundary of the cross-section and is tangent to that boundary. The two shear stresses are usually represented in terms of the Saint-Venant’s stress function, $\Phi$:

$$\tau_{xy} = \partial \Phi / \partial z \text{ and } \tau_{xz} = -\partial \Phi / \partial y.$$
subject to the boundary conditions that $\Phi$ must be a constant on any free boundary. On the exterior boundary the usually condition is $\Phi = 0$. On any interior free boundaries (holes) special care must be taken to specify the correct non-zero constant on such boundaries. Most shafts are solid and made of a single material so many references multiply through by $G$, but that is not correct in general for shafts of more than one material. It can be shown that the applied torque is related to the stress function by

$$T = 2 \iint \Phi \, dA.$$  

Since $\beta$ is unknown it may not be clear how to solve the last two equations. For a linear elastic material, as we have assumed, any two solutions must be related by a constant scaling factor (linear superposition). Thus, you solve the Poisson equation using a unity value for $\beta$ ($\beta_{\text{unit}} = 1$) and calculate $T_{\text{unit}}$ from the integral. Knowing the actual applied torque, $T_{\text{actual}}$, you simply scale the twist per unit length to be

$$\beta_{\text{actual}} = \beta_{\text{unit}} \frac{T_{\text{actual}}}{T_{\text{unit}}}, \quad \theta_{\text{actual}} = L \beta_{\text{actual}}.$$  

Usually it is difficult to get an analytic solution for a Poisson equation. However, since it is correct to set $\Phi = 0$ on the exterior boundary that makes it easier to generate accurate approximate solutions. This will be illustrated later.

The gradient of the stress function is

$$\vec{\nabla} \Phi = \frac{\partial \Phi}{\partial y} \vec{j} + \frac{\partial \Phi}{\partial z} \vec{k} = -\tau_{xz} \vec{j} + \tau_{xy} \vec{k}.$$  

This shows that the resultant of the two shear stress components at a point gives the stress vector

$$\vec{\tau} = \tau \vec{s},$$  

where $\vec{s}$ is a unit vector perpendicular to the stress function gradient vector, and the magnitude of the shear stress is

$$\tau^2 = \tau_{xy}^2 + \tau_{xz}^2.$$  

Since a boundary has a constant $\Phi$ value, the gradient of $\Phi$ at the boundary is always normal to the boundary. Thus, the maximum shear stress, $\tau_{\text{max}}$, always occurs tangent to a boundary and at one or more points on the boundary. Unlike the circular shaft, where the maximum stress occurs at the maximum distance from the twist center, the non-circular shaft maximum shear stress often occurs at points on the boundary that are closer to the center of twist. In general, the point of maximum shear stress in a bar under torsion occurs at the point(s) where the largest inscribed circle touches the perimeter of the bar. That can often be observed from a free-body-diagram of a section cut through the region nearer the center of twist. Since the shear stresses there have a smaller lever arm they must have larger magnitudes to balance the applied torque, $T$.

### The thermal analogy

The membrane analogy gives very good insight into how the shear stresses are distributed in the torsion of a non-circular shaft. The reader should study such descriptions. A particular good presentation is given by Seely and Smith [2]. However, if you wish to compute results for a 2D study then a finite element torsion code would be best [1]. They are less common than 2D heat transfer codes so it is useful to note the analogy between torsion and heat transfer. The torsion relation
\[
\frac{1}{G} \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{G} \frac{\partial^2 \Phi}{\partial z^2} = -2\beta
\]

compares directly to the heat conduction equation
\[
k \frac{\partial^2 u}{\partial y^2} + k \frac{\partial^2 u}{\partial z^2} = -Q
\]

where \(u\) is the temperature, \(k\) is the thermal conductivity of the material, and \(Q\) is the heat generation per unit area. You could simply define a special material where \(k = 1/G\) and set \(Q = \beta\). Some finite element codes do not specifically accept \(Q\), but utilize its corresponding power input, \(P = QV\), where \(V\) is the volume of the component generating the heat.

The contours of \(T\) will correspond to the contours of \(\Phi\) (and the height in the membrane analogy). The temperature gradient vector, \(\nabla T\), at a point will be perpendicular to the shear stress vector, \(\tau = \tau \hat{s}\), at that point. The temperature gradient magnitude corresponds to the resultant shear stress, \(\tau\). The difficulty is that you usually know the torque, but not the twist angle.

The stress function differential equation can be written in another form also. Recall from above that the twist angle per unit length is
\[
\beta = \frac{T}{GK}.
\]

Then
\[
\frac{1}{G} \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{G} \frac{\partial^2 \Phi}{\partial z^2} = - \frac{2T}{GK},
\]

and you see the shear modulus would cancel out for a single material. This means that the spatial distribution of the shear stresses depends only on the shape of the cross-section and not its material. The magnitude of the shear stress is proportional to the torque. Defining a custom material with \(k = 1\), the thermal model becomes:
\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -Q = -\frac{2T}{K}.
\]

If the corresponding power is input, to indirectly define \(Q\), then \(P = -2TV/K\). The torsional constant, \(K\), is known for several shapes but can be approximated as
\[
K \approx \frac{A^4}{40J}
\]

as noted above. Then commonly available 2D heat transfer programs can be employed.

**Equilateral Triangle 2D Thermal Model**

To illustrate the thermal analogy use, a two-dimensional domain was created (see Figure 1) defining an equilateral triangle with sides \(b = 4\) inches in length. The centerline shown was replaced with a “split line” so as to provide a graphing reference in the post-processing. The triangle was given a boundary condition of zero temperature (zero stress function) on the three edges. A unit torque (\(T = 12\) inch-lb) was applied to (a unit thickness of) the bar. That defines the fake heat generation source to be \(Q = -2T/K\) where the exact torsion constant is
\[
K = \sqrt{3} \frac{b^4}{80} = \frac{b^4}{46.2}.
\]
The area, \( A = \sqrt{3}b^2/4 \), and the polar moment of inertia, \( J = \sqrt{3}b^4/48 \), give the approximate torsion constant of \( K_{\text{approx}} = 36\sqrt{3} b^4/2560 = b^4/41.1 \). They also define the volume of \( V = 6.928 \text{ in}^3 \) and a reference power source of \( P = 2.50 \text{ lb} \) (in the units of the torsion problem).

This 2D example was implemented in the Cosmos finite element system as a thermal analysis problem. Cosmos is one of the few codes that does not accept the direct specification of the rate of heat generation, \( Q \) (contrary to its documentation), but instead requires the corresponding power, \( P \), created in the volume. The input of that boundary condition is on a per geometric entity basis. Since the split line created two equal surface areas each was assigned half the power value given above. The fake thermal boundary conditions are illustrated in Figure 2.

The solid was defined to be a "Custom Material" and the thermal conductivity was set to unity (Figure 3). A mesh was created, including the split line, and is given in Figure 4. (Note that symmetry and anti-symmetry were not employed to select only one-sixth of the analysis region.) Then the standard "thermal" solution was executed to determine the temperature (stress function) values and the temperature gradient vector (i.e., vectors orthogonal to the effective shear stress vectors). The contour of the stress function (in lb/in) and a graph of its value along the split line from the top vertex to the base center are given in Figure 5. From the graph it is easily seen that the slope of the stress function goes to zero in the exterior corners, and that the slope is maximum at the mid-point of the straight side(s). (In the membrane analogy that graph would be the height of the inflated membrane.)
The exact solution to this torsion problem is known. It gives the maximum shear stress value, at the center of each straight side, to be

\[
\tau_{\text{max}} = \frac{20\theta}{b^3} = \frac{20(1)}{64} = 0.3125 \text{ psi}.
\]

That value should match the magnitude of the maximum gradient found in the thermal analogy. The contour values for the shear stress, in psi, (the thermal gradient magnitude) are shown in Figure 6, along with the graph of the shear strain component parallel to the bottom base \( (\tau_{xy}) \). The maximum shear stress values (0.313 psi) match the exact value given above very closely. As expected, the shear stresses vanish in the corners. For symmetric sections, such as this one, it also vanishes around the center of twist. The graph of the shear stress (psi) from the corner to the center of the baseline is given in Figure 7.
The actual shear stress vectors are orthogonal to the thermal gradient vectors shown in Figure 8. That is, the maximum shear stresses lie parallel to the exterior boundary. The results of this analogy match the exact solution because the exact $K$ value was used to define $Q$ (or $P$). If the approximate $K$ value was used, the stress values would have been about 12% too low.
Appendix 1:

*Torsion of a solid equilateral triangular shaft (INCORRECT FIGURES)*

Consider the equilateral triangular shaft shown in Figure 9. It has a side length of 4 in, a length of 20 in and is made of brass. One end is immovable, while the other is subjected to shear stresses that produce a torque of 1 in-lb. Away from the ends it should be accurately modeled by the above 2-D study.

In this example a different coordinate system was used in CosmosWorks with the z-axis being parallel to the shaft center of twist, and the base of the triangle is parallel to the x-axis. The useful stress results available for plotting are the shear stress component $\tau_{xy}$ and the shear stress intensity ($2\tau_{max}$). The component $\tau_{xy}$ is shown in Figure 10 at the mid-length cross-section. Since the x-axis is parallel to the base that component is parallel to that edge, and thus will show the maximum shear stress location on that edge. From its contour levels you can see that the maximum shear stress is located at the side mid-point, a point tangent to the largest inscribed cylinder. The maximum value is about $\tau_{max} = XXX$. The shear stresses on the other two sides, in the direction parallel to the side should have the same relative distributions.
To see the combined effect of both shear stress components across the full cross-section you need the plot of the principal stress intensity, which is given in Figure 11 at the shaft mid-section. It is shown again, in true shape, in Figure 12. There you can confirm that the maximum shear stress is at the mid-point of each edge and has a value of about $\tau_{max} = XXX$.

Tabulated solutions of the torsion of noncircular bars list the maximum shear stress as $\tau_{max} = 20T/b^3$ and the twist per unit length as $\beta = 46.2 T/G b^4$, where $b = 4$ in, is the length of the triangle sides. That predicts $\tau = 3.125$psi and a twist value of $\beta = XXX$ radians/in.

Since all but the two shear stress components are zero the von Mises effective stress reduces to $\sigma_e = \sqrt{6/4} \tau_{max} = 1.22\tau_{max}$. The von Mises plot is shown in Figure 13. It also shows the maximum stress develops at the mid-face of each face. XXX
Appendix 2: Derivation of the torsion equations

The axial (warping) displacement, $u = f(y,z)$, must be computed while the in-plane displacements are:

$$v = -\beta xz, \quad w = \beta xy,$$

where $x$ is the distance from one end of the bar, and $y,z$ are the coordinates of a point in the section from its center of twist. It can be shown that the only non-zero stresses in the pure twisting of a shaft are $\tau_{xy}$ and $\tau_{xz}$. They both lie in the $y$-$z$ cross-section normal to the $x$-direction shaft axis. The differential equation of equilibrium, in terms of stresses reduces to

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0.$$

The corresponding shear strains $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ and $\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$ and since the in-plane displacements are known $\gamma_{xy} = \frac{\partial u}{\partial y} - \beta z$ and $\gamma_{xz} = \frac{\partial u}{\partial z} + \beta y$. The above equation of equilibrium can be satisfied by introducing a linear stress-strain law $\tau_{xy} = G\gamma_{xy}$, and $\tau_{xz} = G\gamma_{xz}$ and writing the two strains in terms of the stress function gives:

$$\frac{1}{G} \frac{\partial \phi}{\partial z} = \frac{\partial u}{\partial y} - \beta z, \quad \text{and} \quad -\frac{1}{G} \frac{\partial \phi}{\partial y} = \frac{\partial u}{\partial z} + \beta y.$$

To eliminate the unknown $u(y,z)$, take partial derivatives of these with respect to $y$ and $z$, respectively, and subtract the second result from the first:

$$\frac{1}{G} \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{G} \frac{\partial^2 \phi}{\partial z^2} = -2\beta.$$

This is a Poisson equation. Most references assume that the cross-section is made of a single material and multiply through by $G$, but that is not correct in general.