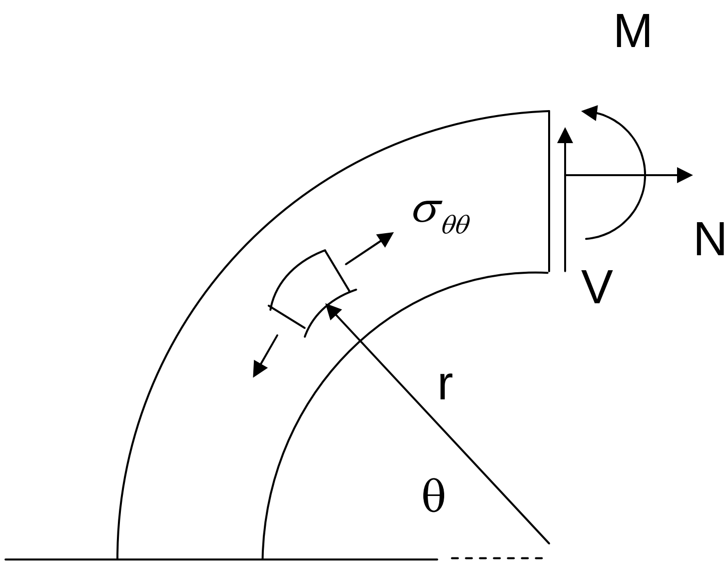
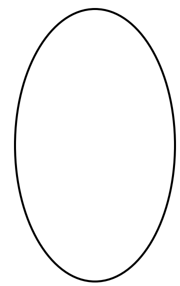
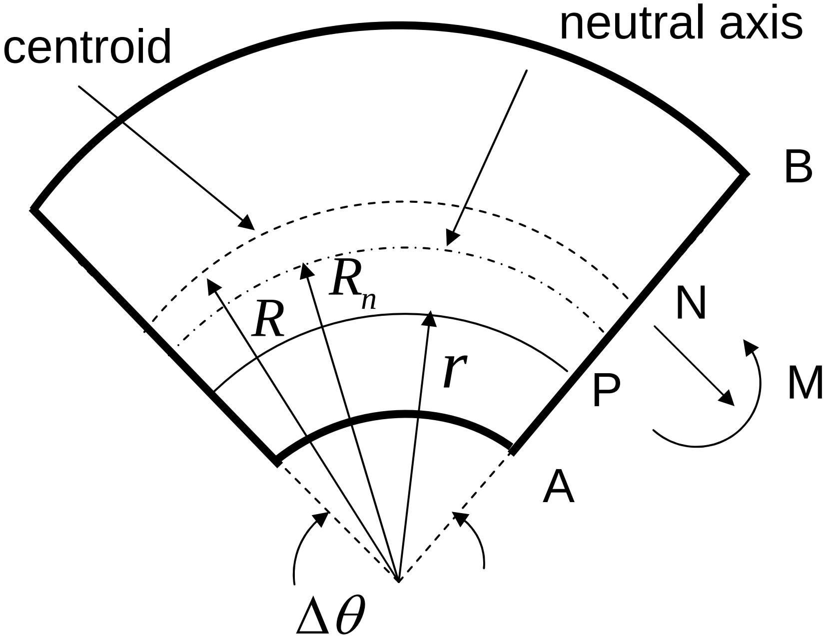
Bending of Curved Beams – Strength of Materials Approach



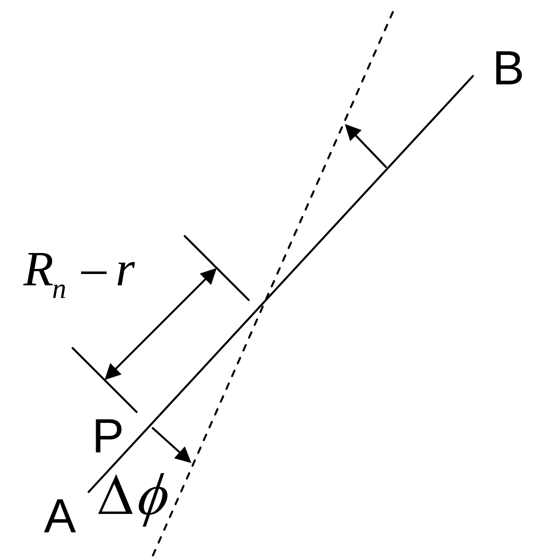


cross-section must be symmetric but does not have to be rectangular

assume plane sections remain plane and just rotate about the neutral axis, as for a straight beam, and that the only significant stress is the hoop stress $\sigma_{\theta\theta}$



Let $\Delta\phi$ = rotation of the cross-section



R = radius to centroid
R_n = radius to neutral axis
r = radius to general fiber in the beam

$$e_{\theta\theta} = \frac{\Delta l}{l} = \frac{(R_n - r)\Delta\phi}{r\Delta\theta} = \omega\left(\frac{R_n}{r} - 1\right)$$

N, M = normal force and bending moment computed from centroid

$$\omega = \frac{\Delta\phi}{\Delta\theta}$$

Reference: Advanced Mechanics of Materials : Boresi, Schmidt, and Sidebottom

From Hooke's law

$$\sigma_{\theta\theta} = Ee_{\theta\theta} = E\omega\left(\frac{R_n}{r} - 1\right)$$

Then the normal force is given by

$$N = \int_A \sigma_{\theta\theta} dA = E\omega\left[R_n \int_A \frac{dA}{r} - \int_A dA\right]$$
$$= E\omega\left(R_n A_m - A\right)$$

where $A_m = \int_A \frac{dA}{r}$ has the dimensions of a length

Similarly, for the moment

$$M = \int_A \sigma_{\theta\theta}\left(R - r\right) dA$$
$$= E\omega \int_A \left(\frac{R_n}{r} - 1\right)\left(R - r\right) dA$$
$$= E\omega\left[R_n R \int_A \frac{dA}{r} - \cancel{R\int_A dA} - R_n \int_A dA + \cancel{\int_A r\, dA}\right]$$
$$= E\omega R_n\left(RA_m - A\right)$$

$$M = E\omega R_n \left(RA_m - A \right) \qquad (1)$$

$$N = E\omega \left(R_n A_m - A \right) \qquad (2)$$

from (1)

$$E\omega R_n = \frac{M}{RA_m - A}$$

from (2)

$$N = \left(E\omega R_n \right) A_m - E\omega A$$

$$= \frac{MA_m}{RA_m - A} - E\omega A$$

so solving for $E\omega$

$$E\omega = \frac{MA_m}{A\left(RA_m - A\right)} - \frac{N}{A}$$

Recall, the stress is given by

$$\sigma_{\theta\theta} = Ee_{\theta\theta} = E\omega\left(\frac{R_n}{r} - 1\right)$$

$$= \frac{E\omega R_n}{r} - E\omega$$

so using expressions for $E\omega R_n, E\omega$

we obtain the hoop stress in the form

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M\left(A - rA_m\right)}{Ar\left(RA_m - A\right)}$$

axial stress

bending stress

$$N \neq 0$$

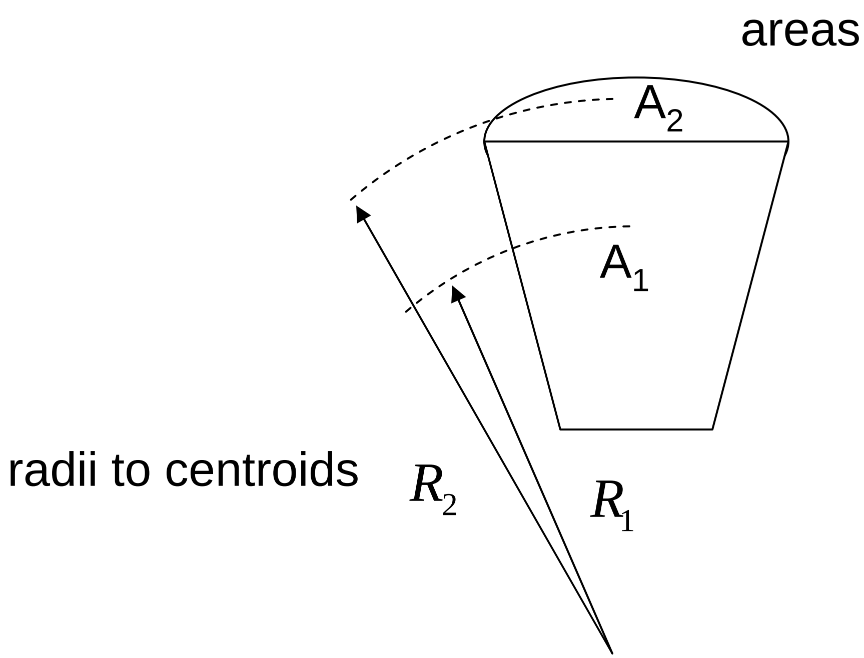
setting the total stress = 0 gives $$r\Big|_{\sigma_{\theta\theta}=0} = \frac{AM}{A_m M + N\left(A - RA_m\right)}$$

$$N = 0$$

setting the bending stress = 0 and $r = R_n$ gives $$R_n = \frac{A}{A_m}$$ location of the neutral axis

which in general is not at the centroid

For composite areas

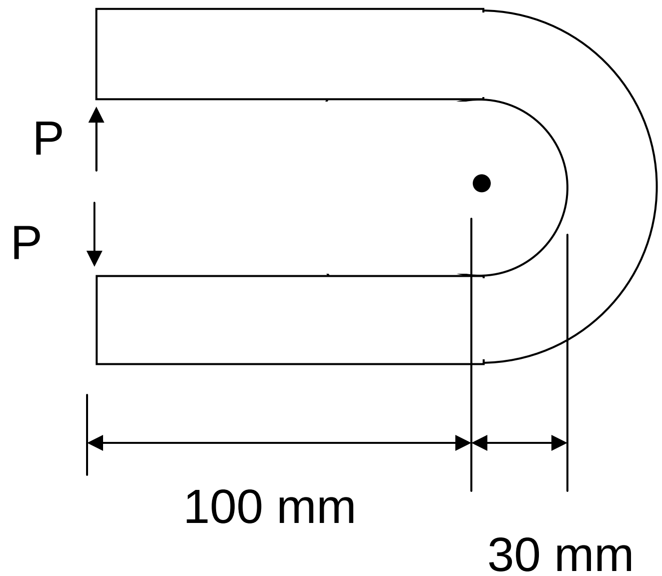


$$A = \sum A_i$$

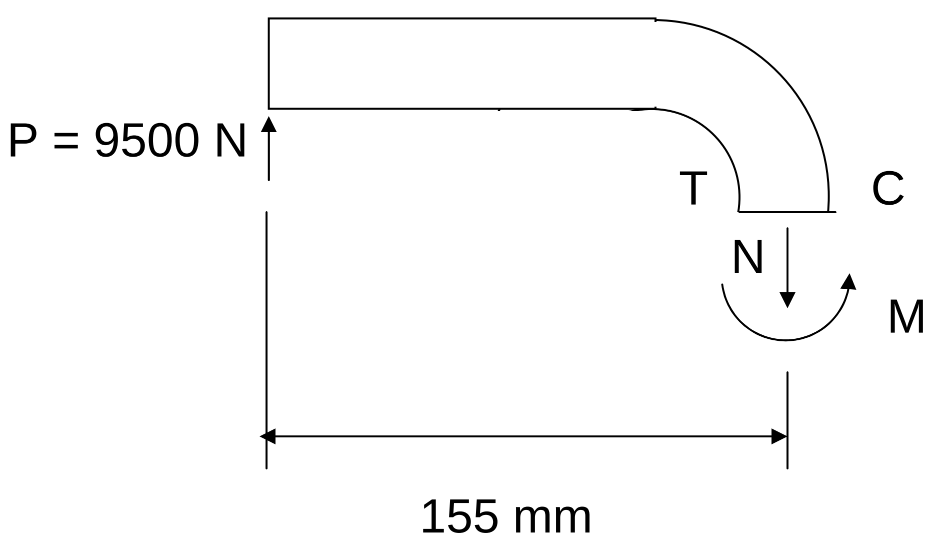
$$A_m = \sum A_{mi}$$

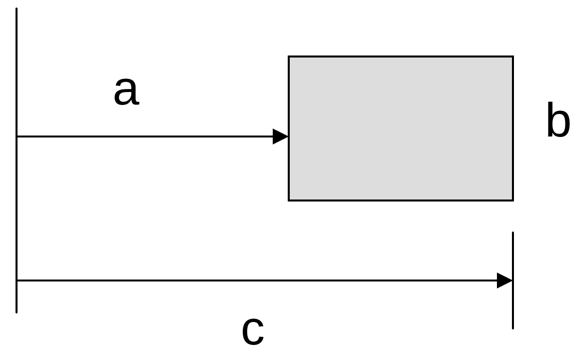
$$R = \frac{\sum R_i A_i}{\sum A_i}$$

Example



For a square 50x50 mm cross-section, find the maximum tensile and compressive stress if P = 9.5 kN and plot the total stress across the cross-section





$$A = b(c - a)$$

$$R = \frac{a + c}{2}$$

$$A_m = b \ln\left(\frac{c}{a}\right)$$

a = 30 mm
b = 50 mm
c = 80 mm

so we have

$$A = (50)(50) = 2500\ mm^2$$

$$A_m = 50 \ln\left(\frac{80}{30}\right) = 49.04\ mm$$

$$R = \frac{80 + 30}{2} = 55\ mm$$

$$R_n = \frac{2500}{49.04} = 51\ mm$$

max tensile stress is at r = 30 mm

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M\left(A - rA_m\right)}{Ar\left(RA_m - A\right)}$$

$$= \frac{9500}{2500} + \frac{(155)(9500)\left[2500 - (30)(49.04)\right]}{(2500)(30)\left[(55)(49.04) - 2500\right]}$$

$$= 106.2\ MPa$$

max compressive stress is at r = 80 mm

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M\left(A - rA_m\right)}{Ar\left(RA_m - A\right)}$$

$$= \frac{9500}{2500} + \frac{(155)(9500)\left[2500 - (80)(49.04)\right]}{(2500)(80)\left[(55)(49.04) - 2500\right]}$$

$$= -49.3\ MPa$$

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>> r= linspace(30, 80, 100);
>> stress = 3.8 + 589*(2500 - 49.04.*r)./(197.2*r);
>> plot(r, stress)
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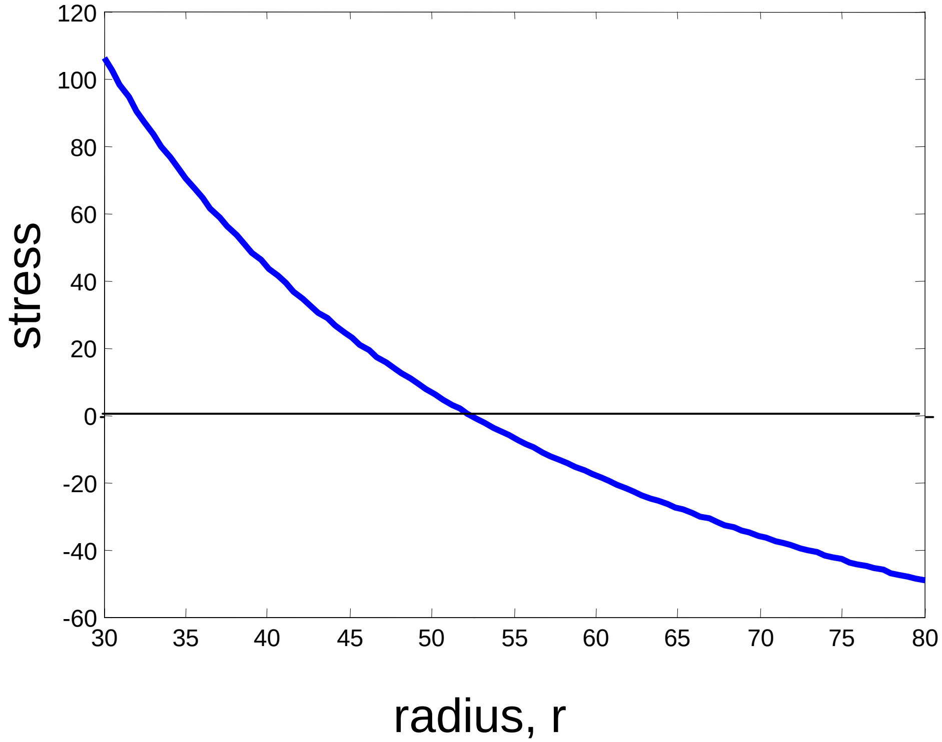
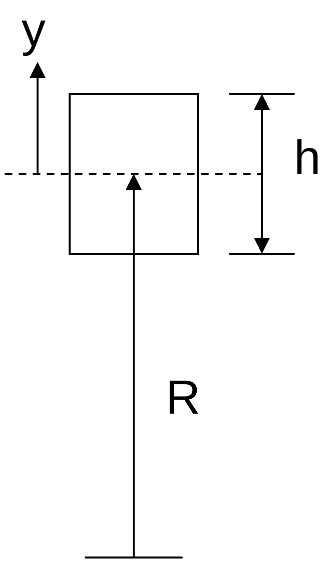
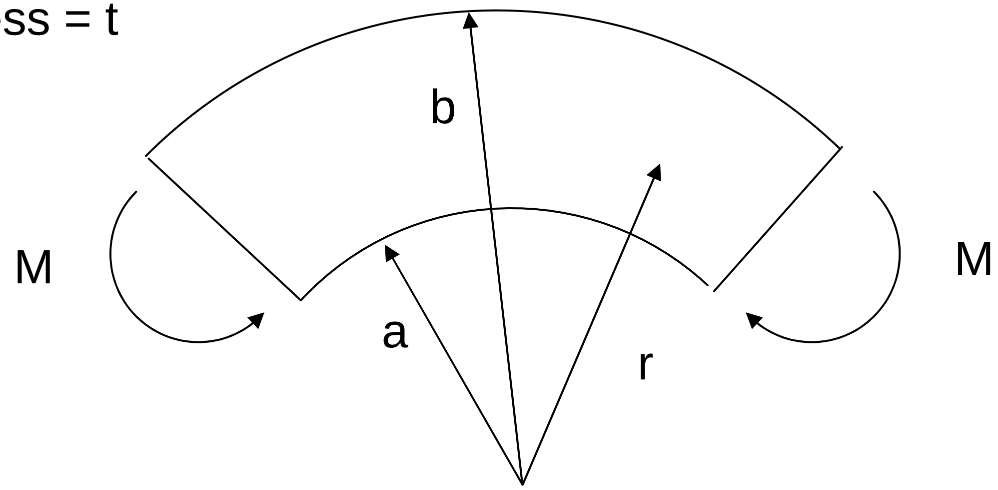


Comparison with Airy Stress Function Results

Consider a rectangular beam under pure moment

thickness = t





Note:

$$R/h = \frac{(b/a+1)}{2(b/a-1)}$$

$$b/a = \frac{(2R/h+1)}{(2R/h-1)}$$

From Airy Stress Function

$$\sigma_{\theta\theta} = \frac{4M}{Nta^2}\left[-\left(\frac{a}{r}\frac{b}{a}\right)^2 \ln\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 \ln\left(\frac{r}{a}\frac{a}{b}\right) - \ln\left(\frac{r}{a}\right) + \left(\frac{b}{a}\right)^2 - 1\right]$$

$$N = \left[\left(\frac{b}{a}\right)^2 - 1\right]^2 - 4\left(\frac{b}{a}\right)^2 \left[\ln\left(\frac{b}{a}\right)\right]^2$$

Strength Approach

$$A_m = t \ln\left(\frac{b}{a}\right)$$

$$A = t(b-a)$$

$$R = (a+b)/2$$

$$\sigma_{\theta\theta} = \frac{-M\left(A - rA_m\right)}{Ar\left(RA_m - A\right)}$$

minus on M since it is opposite to what we had before

$$= \frac{-M\left[\left(b-a\right) - r\ln\left(\frac{b}{a}\right)\right]}{t\,r\left(b-a\right)\left[\frac{\left(a+b\right)}{2}\ln\left(\frac{b}{a}\right) - \left(b-a\right)\right]}$$

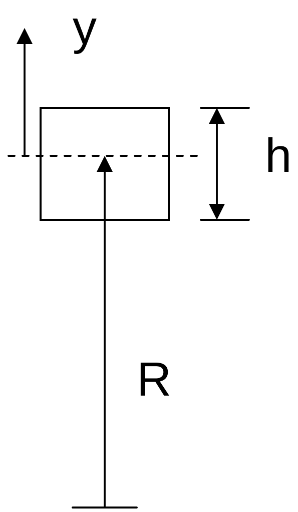
$$= \frac{2M\left[\left(\frac{b}{a}-1\right) - \frac{r}{a}\ln\left(\frac{b}{a}\right)\right]}{t\,a^2\left(\frac{r}{a}\right)\left(\frac{b}{a}-1\right)\left[2\left(\frac{b}{a}-1\right) - \left(\frac{b}{a}+1\right)\ln\left(\frac{b}{a}\right)\right]}$$

If we had used the ordinary straight beam formula instead

$$\sigma_{\theta\theta} = \frac{My}{I} = \frac{M\left[r - \frac{\left(a+b\right)}{2}\right]}{\frac{1}{12}t\left(b-a\right)^3}$$

$$= \frac{M}{ta^2}\frac{6\left[2\frac{r}{a} - \left(\frac{b}{a}+1\right)\right]}{\left(\frac{b}{a}-1\right)^3}$$

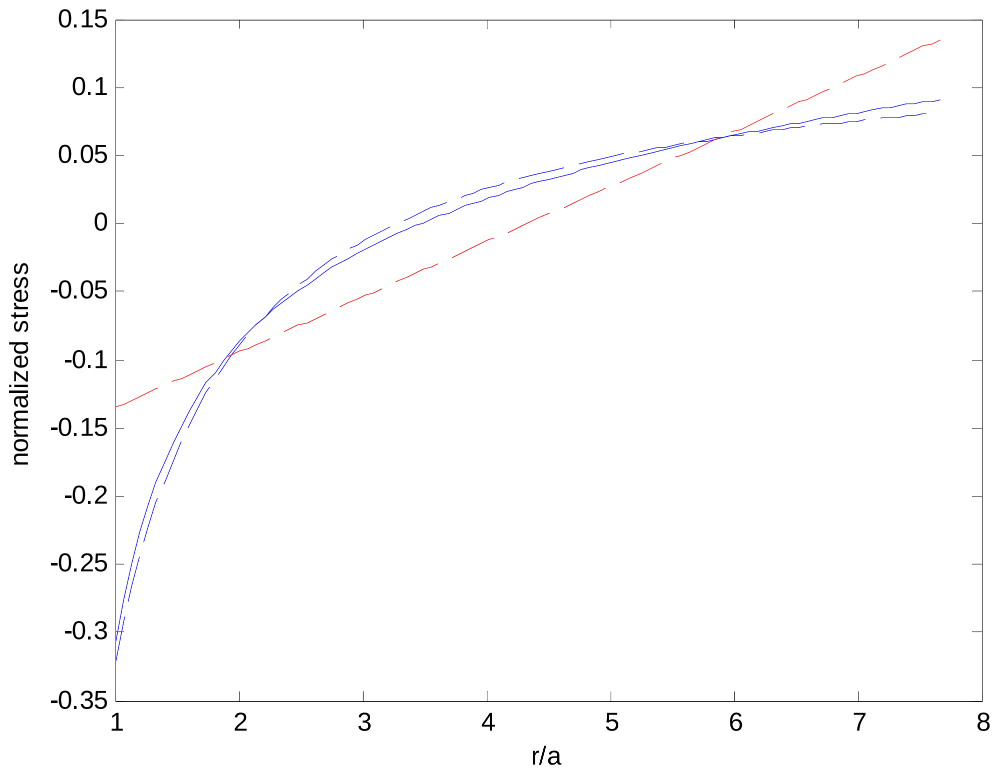
Comparison of the ratio of the max bending stresses



R/h	Strength/elasticity Curved beam formula	Strength/elasticity Straight beam formula My/I
0.65	1.0455	0.4390
0.75	1.0124	0.5262
1.0	0.9970	0.6545
1.5	0.9961	0.7737
2.0	0.9973	0.8313
3.0	0.9986	0.8881
5.0	0.9994	0.9331

Compare bending stress distributions at the smallest R/h value

R/h = 0.65 (b/a = 7.667)



solid curve – Airy stress function (elasticity)
dashed blue – Curved Strength formula
dashed red – Straight beam formula

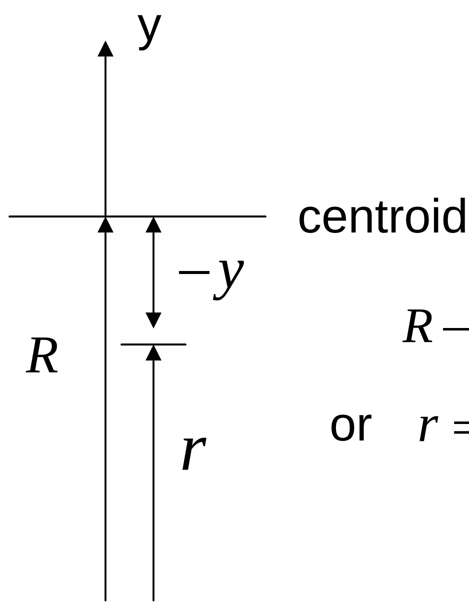
```
% beam_compare.m
m=1;
Rhvals = [0.65 0.75 1.0 1.5 2.0 3.0 5.0];  % R/h ratios to consider
for Rh = Rhvals
ba = (1+2*Rh)/(2*Rh -1); %corresponding b/a values
ra= linspace(1, ba, 100);  % r/a values
N=(ba^2-1)^2 -4*ba^2*(log(ba))^2;
% Airy function flexure stress expression
pa =4*(-(ba./ra).^2.*log(ba)+(ba)^2.*log(ra./ba) - log(ra) +(ba)^2 -1)./N;
% Curved beam strength expression for flexure stress
ps = 2*((ba-1)-ra.*log(ba))./(ra.*(ba-1).*(2.*(ba-1) -(ba+1).*log(ba)));
% Straight beam flexure formula  My/I
pb = 6*(2*ra-(ba+1))./((ba-1)^3);
%obtain ratio of max stresses Curved beam Strength formula/Airy
ratio1(m) = max(abs(ps))/max(abs(pa));
%obtain ratio of max stresses: Straight beam strength formula/Airy
ratio2(m) = max(abs(pb))/max(abs(pa));
m=m+1;
end
% Now plot stress distributions for smallest R/h value
Rh=0.65
ba = (1+2*Rh)/(2*Rh -1); %corresponding b/a values
ra= linspace(1, ba, 100);
N=(ba^2-1)^2 -4*ba^2*(log(ba))^2;
pa =4*(-(ba./ra).^2.*log(ba)+(ba)^2.*log(ra./ba) - log(ra) +(ba)^2 -1)./N;
ps = 2*((ba-1)-ra.*log(ba))./(ra.*(ba-1).*(2.*(ba-1) -(ba+1).*log(ba)));
pb = 6*(2*ra-(ba+1))./((ba-1)^3);
plot(ra, pa)
hold on
plot(ra, ps, '--b')
plot(ra, pb, '--r')
xlabel('r/a')
ylabel( 'normalized stress')
hold off
```

Comparison with Bickford’s expression (pure bending)

$$\sigma_{\theta\theta} = -\frac{kM}{A} - \frac{My}{(1+ky)I_2}$$

$$k = 1/R$$

Here, y is distance from the centroid



$$R - r = -y$$

or $r = y + R$

First note that

$$\int_A y\,dA = 0$$

so $$\int_A \frac{y(1+y/R)}{1+y/R}dA = \int_A \frac{y\,dA}{1+y/R} + \frac{1}{R}\int_A \frac{y^2 dA}{1+y/R} = I_1 + \frac{I_2}{R} = 0$$

$$I_1 = \int_A \frac{y\,dA}{1+y/R}$$

$$I_2 = \int_A \frac{y^2 dA}{1+y/R}$$

$\Rightarrow$ $$I_1 = -\frac{I_2}{R}$$

$$RA_m = R\int_A \frac{dA}{r} = R\int_A \frac{dA}{y+R}$$

$$A = \int_A dA = \int_A \frac{(y+R)}{(y+R)} dA = \underbrace{\int_A \frac{y}{y+R} dA}_{I_1/R} + R\int_A \frac{dA}{y+R}$$

$$= \frac{I_1}{R} + RA_m$$

Thus,

$$R\left(RA_m - A\right) = -I_1 = I_2 / R$$

Now, start with Bickford's expression $\sigma_{\theta\theta} = -\frac{kM}{A} - \frac{My}{(1+ky)I_2}$

$$k = 1/R$$

$$\sigma_{\theta\theta} = -M\left[\frac{1}{AR} + \frac{yR}{(y+R)I_2}\right]$$

$$= -M\left[\frac{(y+R)I_2 + yAR^2}{(y+R)ARI_2}\right]$$

same terms added in and subtracted out

$$= -M\left[\frac{(y+R)I_2 + (y+R)AR^2}{(y+R)ARI_2} - \frac{AR^3}{(y+R)ARI_2}\right]$$

$$= -M\left[\frac{I_2 + AR^2}{ARI_2} - \frac{R^2}{(y+R)I_2}\right]$$

$$= M\left[\frac{R^2}{(y+R)I_2} - \frac{I_2 + AR^2}{ARI_2}\right]$$

$$= M\left[\frac{A - (y+R)(I_2 + AR^2)/R^3}{(y+R)AI_2/R^2}\right]$$

$$\sigma_{\theta\theta} = M\left[\frac{A-(y+R)\left(I_2+AR^2\right)/R^3}{(y+R)AI_2/R^2}\right] \qquad y+R=r$$

but $RA_m - A = I_2/R^2$ so $A_m = \left(I_2 + AR^2\right)/R^3$

and we find

$$\sigma_{\theta\theta} = \left[\frac{A-rA_m}{r\left(RA_m - A\right)A}\right]$$

which agrees with our previous expression from Bickford’s expression

$$\sigma_{\theta\theta} = -\frac{kM}{A} - \frac{My}{(1+ky)I_2}$$

$$k = 1/R$$

it is easy to see, as $R \to \infty$, $k \to 0$

and

$$I_2 = \int_A \frac{y^2 dA}{1 + y/R} \to \int_A y^2 dA = I$$

and we recover the straight beam flexure expression

$$\sigma_{\theta\theta} = -\frac{My}{I}$$