Form-finding, soap film and membrane analogy in Engineering

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February 26th, 2007
Outline

- Membrane analogy in torsion
- Form finding in civil engineering
Membrane analogy in torsion

- Prandtl in 1903 identified this analogy for the stress distribution of non-circular shafts under torsion
- A membrane is the 2D equivalent of a string under tension
- Example of membrane: a soap film
- Assume the cross-section of the shaft to be covered by a soap film (membrane) having the same shape of the cross-section, with pressure under it that keeps it in tension
Membrane analogy in torsion (cont’d)

Figure 6.33: The thin membrane attached to the contour $\mathcal{C}$.  

Bauchau and Craig notes, August 2006
Membrane analogy in torsion (cont’d)

\[ \theta_1 > \theta_2 \quad \theta_3 = 0 \]

http://www.ae.msstate.edu/~Emasoud/Teaching/SA2/A6.5_more2.html
## Membrane analogy in torsion (cont’d)

<table>
<thead>
<tr>
<th><strong>Main variables</strong></th>
<th><strong>Membrane Problem</strong></th>
<th><strong>Torsion Problem</strong></th>
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</thead>
<tbody>
<tr>
<td>displacement normal to membrane, ( u_1 )</td>
<td>stress function, ( \Phi )</td>
<td></td>
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<tr>
<td>pressure under membrane, ( p )</td>
<td>shear modulus, ( G )</td>
<td></td>
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<tr>
<td>tension along membrane, assumed constant, ( S )</td>
<td>twist rate, assumed constant, ( \kappa_1 )</td>
<td></td>
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</tbody>
</table>

**Governing PDE**

\[
\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = -\frac{p}{S} \quad \text{on } A.
\]

\[
\frac{\partial u_1}{\partial s} = 0 \quad \text{along } C.
\]

\[
\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = -2G\kappa_1 \quad \text{on } A.
\]

\[
\frac{d\Phi}{ds} = 0 \quad \text{along } C.
\]

**Critical Items**

- Maximum slope, \( \frac{\partial u_1}{\partial n} \)
- Maximum shear stress, \( \tau_s = -\frac{\partial \Phi}{\partial n} \)

**Volume**

- volume under membrane, \( V_0 = \int_{A} u_1 \, dA \)
- resisting moment \( M_1 = 2\int_{A} \Phi \, dA \)
Membrane analogy in torsion (cont’d)

Elastic Membrane Analogy

\[ \theta_1 = \theta_2 \quad \theta_3 = 0 \]

http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more3.html
Membrane analogy in torsion (cont’d)

Elastic Membrane Analogy
\( \theta_1 = \text{Maximum} \)

http://www.ae.msstate.edu/%7Emasoud/Teaching/SA2/A6.5_more4.html