Selected solutions for "Finite Element Analysis with Error Estimators"

An Introduction to the FEM and Adaptive Error Analysis for Engineering Students

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Selected solutions and examples

Here we will present selected analytic solutions, source codes, and/or data files and corresponding outputs that are associated with the exercises at the end of the various chapters. They are listed in chapter order.

1.1 Problems from chapter 1

Problem 1.9 What is the size of the Boolean array, $\mathcal{B}$, for any element in Fig. 1.4.3? Explain why. Solution: Each node has 2 generalized degrees of freedom. The system has 4 nodes and will therefore have 8 degrees of freedom total. Each element has 2 node and a total of 4 local degrees of freedom. Thus each element Boolean array will have 4 rows, for the local dof, and 8 columns, for the system dof. Of course, each row will contain only one unity term and the rest of the row has null entries.

Multiple choice: 1.12 b, 1.13 e, 1.14 a, 1.15 a, 1.16 c, 1.17 e, 1.18 d.

Problem 1.19 List: S_ mass, S_ time, V_ position, V_ centroid, S_ volume, S_ surface area, V_ displacement, S_ temperature, V_ heat flux, S_ heat source, T_ stress, T_ moment of inertia, V_ force, V_ moment, V_ velocity.

Multiple choice: 1.20 a & d, 1.21 b & c.

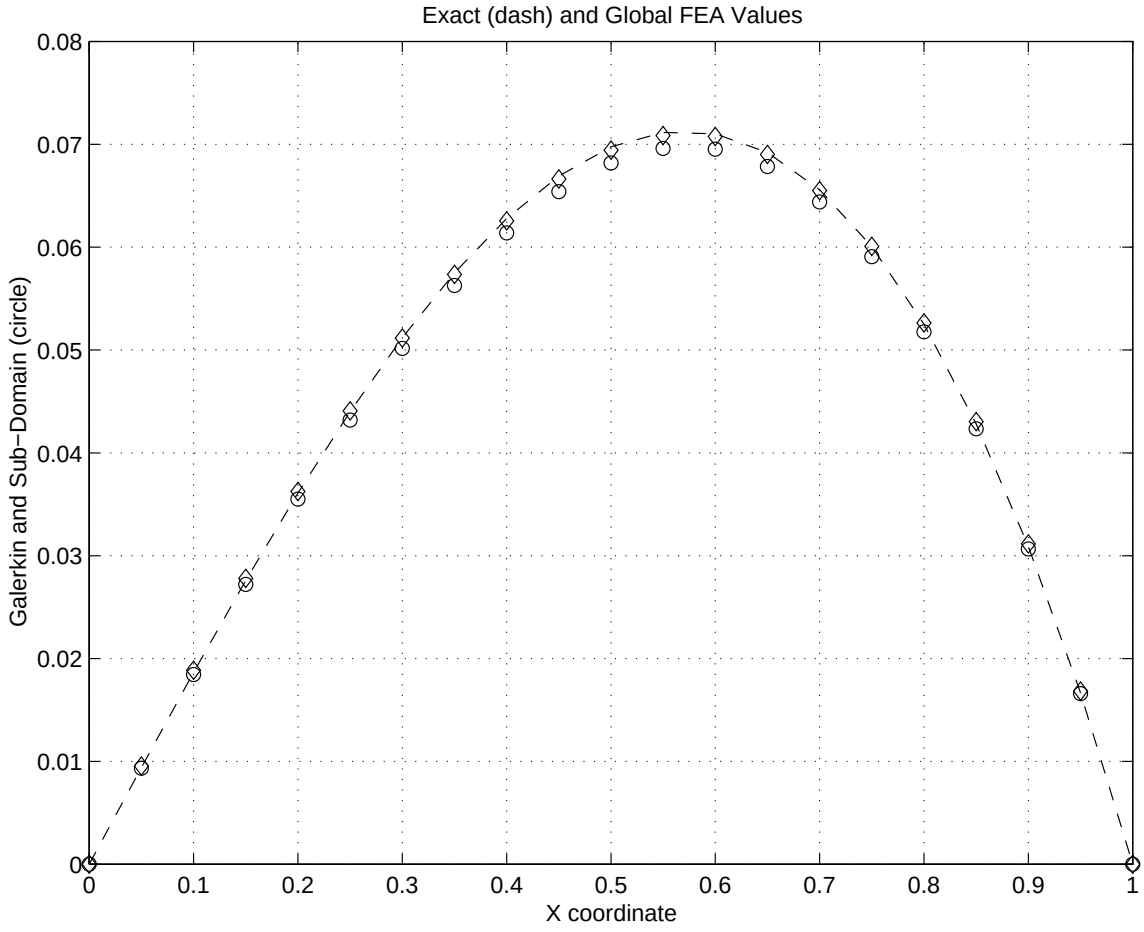
1.2 Problems from chapter 2

Problem 2.1-A Write a program (or spread-sheet) to plot the exact solution and the approximations on the same scale. Solution: This problem is most reasonably solved by using a spread-sheet or a writing a simple code in a plotting environment like Matlab or Maple. Using Matlab a minimal script would be like that of figure P2.1Aa which produces output in P2.1Ab.

However, we will want to do this many times for various meshes and analytic results. Thus we use a much larger code that accepts data files containing the coordinates, connectivity, and result values and produces plots like Fig. 2.10.6. Those files can be created manually, or be given as output from the *MODEL* code. In addition, we can foresee problems with more that one unknown per node, and analytic solutions to be selected from a given library of solutions. Thus the example plot script requires the user to declare the degree of freedom to be plotted and the *exact_case* number that identifies the solution (here and in *MODEL*). It is presented in figure P2.1Ac and d. Note that in lines 16-18 and 26-28 it cites and loads the three ASCII character files that

```
function P2_1A_plot   % for Matlab                              ! 1
% Global (single element) solution comparisons                 ! 2
  clf                              % clear frame               ! 3
  x  = [0:0.05:1.];               % 21 points                  ! 4
  ar = sin(x)/sin(1) - x ;         % analytic result           ! 5
  plot (x, ar, 'k--')              % dash lines                ! 6
  hold ; grid                      % wait for more data        ! 7
  c1 = 0.1880 ; c2 = 0.1695 ;      % sub-domain                ! 8
  y  = x .* (1.0 - x) .* ( c1 + c2 .* x) ; % global fe         ! 9
  plot (x, y, 'ko')                % circles                   !10
  c1 = 0.1924 ; c2 = 0.1707 ;      % Galerkin                  !11
  y  = x .* (1.0 - x) .* ( c1 + c2 .* x) ; % global fe         !12
  plot (x, y, 'kd')                % diamonds                  !13
  title ('Exact (dash) and Global FEA Values')                 !14
  ylabel ('Galerkin and Sub-Domain (circle)')                  !15
  xlabel ('X coordinate')                                      !16
% end function P2_1A_plot                                      !17
```

Problem P2.1Aa *An elementary Matlab plot script*



Problem P2.1Ab *Example Galerkin and sub-domain results*

give the required data to plot. It would be much more efficient on large problems to use binary files but the character files are retained for educational use.

A group of more that 150 Matlab plot scripts, including this one, are provided in the public source library. Many are very similar to each other. They provide for 1-D graphs, mesh plots, boundary condition flags, code, or values, 2-D contour plots, shaded displays, 3-D wireframe, color filled, or hidden surface displays, etc. They are available for data checks, FEA results and fluxes, exact results and fluxes, FEA error estimates, exact errors and error norms, etc. They all access sequential character files such as the three above. The necessary files are produced (or not) by *MODEL*. Below is a list of the most common files and a list of associated keywords.

Name (*.tmp)	Name (*.tmp)
msh_bc_xyz	msh_typ_nodes
node_results	msh_bc_values
el_qp_xyz_fluxes	scp_node_ave_fluxes
el_error_est	pt_ave_error_est
ex_error_est	msh_new_el_size

Partial list of data files for optional plotting

Keywords	File Created (*.tmp)
list_exact	exact_node_solution
list_exact_flux	exact_node_flux
sav_exact	exact_node_solution
sav_exact	pt_ave_ex_error
turn_on_scp	pt_ave_error_est
turn_on_scp	scp_node_ave_fluxes
turn_on_scp	pt_ave_ex_error

List of optional keyword control of data files

Problem 2.6-A Formulate the first order equation $dy/dx + Ay = F$ by: a) least squares, b) Galerkin method. Use analytic integration for the linear line element (L2) to form the two element matrices. Compute a solution for $y(0) = 0$ with $A = 2, F = 10$ for 5 uniform elements over $x \leq 0.5$. Solution: Since this problem involves only the first derivative in the weak form both a Galerkin and a least squares form can utilize C^0 continuity elements. That is, only the solution must be continuous between elements, so we select the simple L2 line element for an analytically integrated set of matrices. For the least square choice we get the matrices shown in P2.6Aa which for the specified data gives the results in P2.6Ab.

Note that one could consider x here to represent time so this becomes an initial value problem, and the boundary condition $y(0) = 0$ is actually the initial condition. That is, the example also corresponds to a finite element least squares integration in time. Likewise, a Galerkin implementation can be thought of as a finite element Galerkin integration in time. The process is easily extended to matrix coefficients where y is a vector, A is a square matrix, and the leading coefficient is the identity matrix. For large matrices we may not have enough memory to take 5 steps in time in one solution but we

```
function  true_result_1d_graph (i_p, Exact)                    !  1
%  ------------------------------------------------------     !  2
%  Graph FEA & Exact_Case i_p-th component values              !  3
%   If i_p = 0, show RMS value                                 !  4
%  ------------------------------------------------------     !  5
                                                              !  6
% nod_per_el = Nodes per element                               !  7
% np         = Number of Points                                !  8
% nr         = Number of results per node                      !  9
% nt         = Number of elements                              ! 10
% t_x        = x coordinates of nod_per_el corners             ! 11
% t_nodes    = nodes on an element                             ! 12
% x          = all x-coordinates     ; ax for analytic         ! 13
% y          = result component i_p  ; ar for analytic         ! 14
                                                              ! 15
% msh_typ_nodes = connectivity list, nt x nod_per_el           ! 16
% msh_bc_xyz    = nodal coordinates, np x 1                    ! 17
% node_results  = nodal result values, np x nr                 ! 18
                                                              ! 19
  if ( nargin < 2 )                                           ! 20
    error ('No solution given for Exact_Case number')         ! 21
  end % if no arguments                                       ! 22
  pre_e  = 0 ;  % items before connectivity list              ! 23
  pre_p  = 1 ;  % items before coordinates (BC flag)          ! 24
                                                              ! 25
%    Read coordinate, connectivity and result files           ! 26
  load msh_bc_xyz.tmp    ;  load  msh_typ_nodes.tmp ;         ! 27
  load node_results.tmp  ;                                    ! 28
                                                              ! 29
%    Set control data:                                        ! 30
  np = size (msh_bc_xyz,1) ;       % number of nodal points  ! 31
  nr = size (node_results, 1) ;    % dof per node            ! 32
  nt = size (msh_typ_nodes,1) ;    % number of elements      ! 33
  nod_per_el = size (msh_typ_nodes,2) - pre_e -1 ; % nodes   ! 34
  max_p = size (node_results, 2) ; % number of results       ! 35
                                                              ! 36
% Optional pre-allocation of arrays, get x coords            ! 37
  x (np) = 0. ; y (np) = 0. ; t_nodes (nod_per_el) = 0 ;      ! 38
  t_x (nod_per_el) = 0 ; t_y (nod_per_el) = 0 ;               ! 39
  x    = msh_bc_xyz (1:np, (pre_p+1)) ; % extract x column   ! 40
  xmax = max (x)  ; xmin = min (x) ;    % x range to plot     ! 41
                                                              ! 42
%  add analytic points (12 per element)                       ! 43
  a_inc = (xmax-xmin)/(10*nt) ; ax = [xmin:a_inc:xmax] ;      ! 44
  [ar] = analytic_1_d_result (i_p, ax, Exact) % result        ! 45
  maxa = max (ar)      ; mina = min (ar) ;    % range         ! 46
                                                              ! 47
  if ( i_p >= 1 )            %  get FEA nodal results        ! 48
    y = node_results(:, i_p) ;                                ! 49
  else % i_p = 0, get root mean square value                  ! 50
    for k = 1:np                                              ! 51
      y (k) = sqrt ( sum (node_results (k, 1:max_p).^2)) ;    ! 52
    end % for k                                               ! 53
  end % if get RMS value                                      ! 54
  maxy = max (y)      ; miny = min (y) ;   % range           ! 55
                                                              ! 56
```

Problem P2.1Ac *Gather data for a FEA and analytic graph*

```
%    Cite max, min FEA values and locations                    ! 57
  [V_X, L_X] = max (y) ; [V_N, L_N] = min (y) ;                 ! 58
  fprintf ('Max value is %g at node %g \n', V_X, L_X)           ! 59
  fprintf ('Min value is %g at node %g \n', V_N, L_N)           ! 60
  null (1:np) = V_N ; % to locate labels                        ! 61
                                                                ! 62
% finalize axes                                                ! 63
  ymax = max ([maxy, maxa])      ; ymin = min ([miny, mina]) ;! 64
  clf ; hold on ;                   % clear graphics, hold     ! 65
  axis ([xmin, xmax, ymin, ymax])   % set axes                 ! 66
  xlabel ('X, Node number at 45, Element number at 90 deg')     ! 67
                                                                ! 68
  if ( i_p >= 1 )                                               ! 69
    title(['Exact (dash) & FEA Solution Component\_', ...       ! 70
      int2str(i_p),':  ', int2str(nt),' Elements, ', ...        ! 71
      int2str(np),' Nodes'])                                    ! 72
    ylabel (['Component ', int2str(i_p), ' (max = ', ...        ! 73
    num2str(V_X), ', min = ', num2str(V_N), ')'])               ! 74
                                                                ! 75
  else % i_p = 0, get root mean sq                              ! 76
    title(['Exact (dash) & FEA RMS\_value:  ', ...              ! 77
           int2str(nt),' Elements, ', int2str(np),' Nodes'])    ! 78
    ylabel (['Solution RMS Value (max = ', ...                  ! 79
    num2str(V_X), ', min = ', num2str(V_N), ')'])               ! 80
  end % if get RMS value                                        ! 81
                                                                ! 82
  plot (ax, ar, 'r--')    %      add analytic plot first        ! 83
                                                                ! 84
  for it = 1:nt  ;  %    Loop over all elements                 ! 85
                                                                ! 86
%      Extract element connectivity & coordinates               ! 87
   t_nodes = msh_typ_nodes(it,(pre_e+2):(nod_per_el+pre_e+1));! 88
   t_x  = x (t_nodes) ; t_y  = y (t_nodes) ; % coordinates      ! 89
                                                                ! 90
%      Plot the element number, graph element result            ! 91
   x_bar  = sum (t_x' )/nod_per_el ;     % centroid             ! 92
   t_text = sprintf ('   (%g)', it);     % offset # from pt     ! 93
   text (x_bar, V_N, t_text, 'Rotation', 90) % incline          ! 94
   plot (x_bar, V_N, 'k+')               % element number      ! 95
   plot (t_x, t_y)                       % element lines       ! 96
  end % for over all elements                                   ! 97
                                                                ! 98
  for i = 1:np                % plot node points on axis        ! 99
    t_text = sprintf ('   %g', i);      % offset # from pt      !100
    text (x(i), null(i), t_text, 'Rotation', 45) % incline      !101
  end % for all node numbers, Add * at nodes                    !102
  plot (x, null, 'k*') ; grid ; hold off                        !103
% end of true_result_1d_graph                                   !104
```

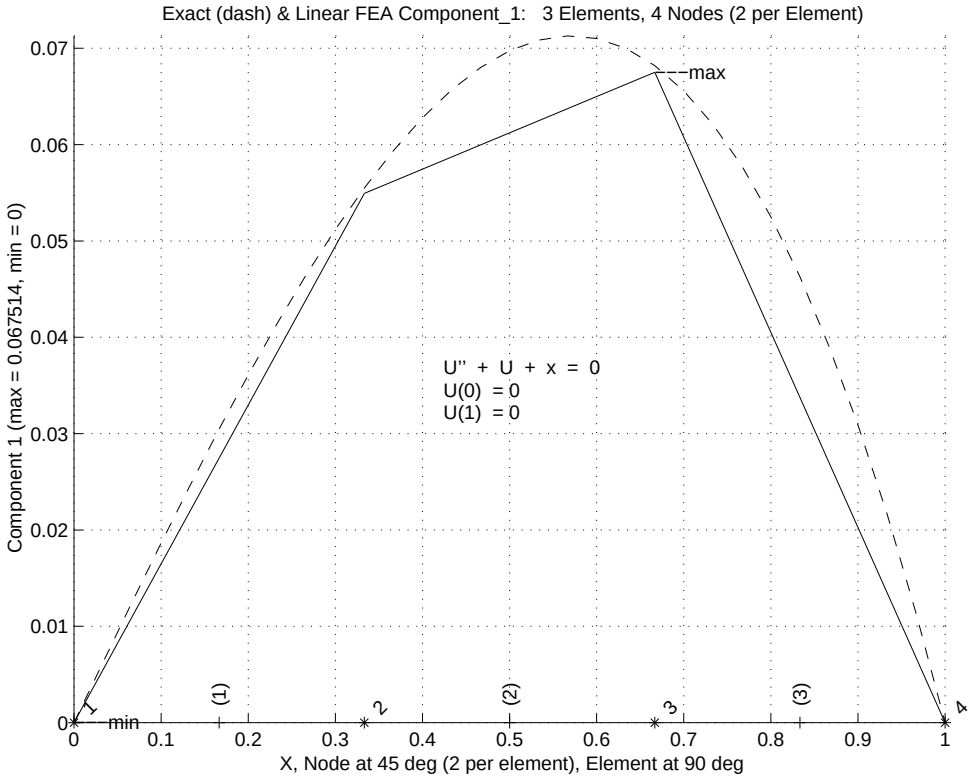
Problem P2.1Ad *Plotting a FEA and analytic graph*

```
! ..............................................................    ! 1
!  ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS FOLLOW ***     ! 2
! ..............................................................    ! 3
!     APPLICATION:  LEAST SQUARES SOLUTION OF Y' + A * Y = F      ! 4
! Exact solution: y(x) = (1 - e (-Ax)) * F / A, with y(0) = 0     ! 5
!          N_SPACE = 1, NOD_PER_EL = 2, N_G_DOF = 1               ! 6
 REAL(DP), SAVE :: A, F, DL  ! GLOBAL DATA, ELEMENT LENGTH        ! 7
                                                                  ! 8
!  RECOVER GLOBAL PROBLEM COEFFICIENTS, A AND F (ON FIRST CALL)   ! 9
  IF ( THIS_EL == 1 ) THEN ! Get coefficients                     !10
    A = GET_REAL_MISC (1) ; F = GET_REAL_MISC (2)                 !12
  END IF ! First call                                             !13
  DL = COORD (2, 1) - COORD (1, 1)  ! ELEMENT LENGTH              !14
                                                                  !15
!    EXACT INTEGRATION SQUARE MATRIX AND SOURCE VECTOR           !16
  S (1, 1) = (3.d0 - 3.d0 * A * DL + A * A * DL * DL) / 3 / DL    !17
  S (2, 2) = (3.d0 + 3.d0 * A * DL + A * A * DL * DL) / 3 / DL    !18
  S (1, 2) = (A * A * DL * DL - 6.d0) / 6.d0 / DL                 !19
  S (2, 1) = S (1, 2)                                             !20
  C (1) = 0.5d0 * F * (A * DL - 2.d0)                             !21
  C (2) = 0.5d0 * F * (A * DL + 2.d0)                             !22
! ***  END ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS ***        !23
```

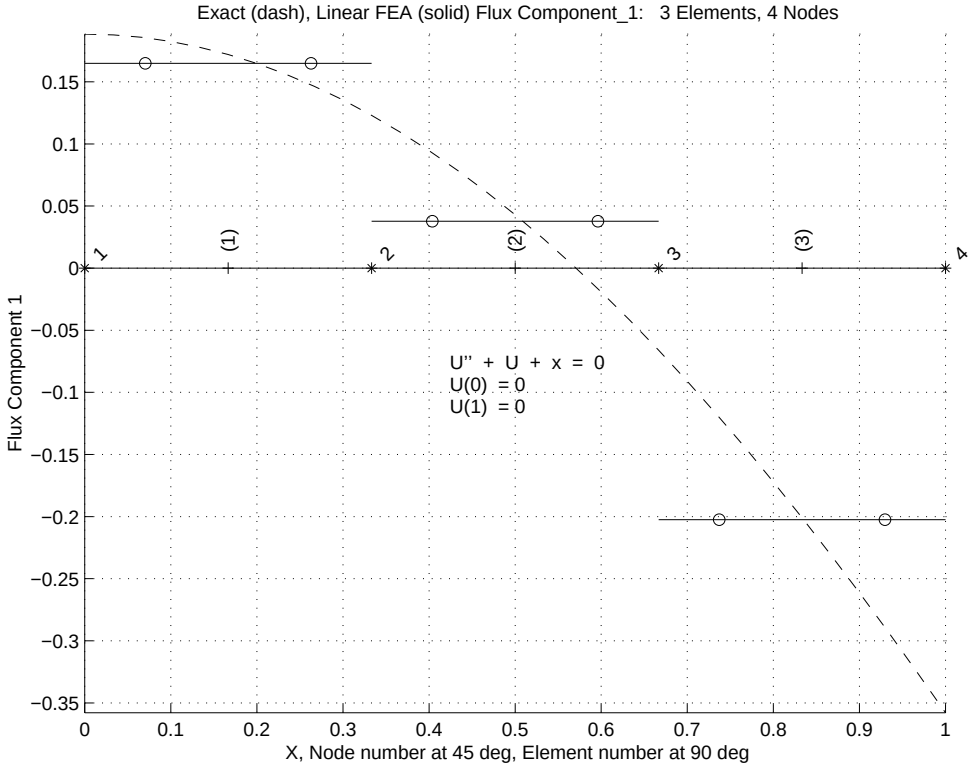
Problem P2.6Aa *Element matrices from exact integration*

```
TITLE "LEAST SQUARES SOLUTION OF Y'+2Y=10, Y(0) = 0"    ! 1
 THE NEXT  3  LINES ARE USER SUPPLIED                   ! 2
 1 LEAST SQUARES SOLUTION OF Y'+2Y=10,  Y(0) = 0        ! 3
 2 Exact solution = 5(1 - e^(-2x))                      ! 4
 3 This is example 103 in the source library            ! 5
*** NODAL POINT DATA ***                                ! 6
   NODE, BC_FLAG, X-Coord,                              ! 7
      1  1    0.0000                                    ! 8
      2  0    0.1000                                    ! 9
      3  0    0.2000                                    !10
      4  0    0.3000                                    !11
      5  0    0.4000                                    !12
      6  0    0.5000                                    !13
*** ELEMENT CONNECTIVITY DATA ***                       !14
ELEMENT,  2 NODAL INCIDENCES.                           !15
      1     1     2                                     !16
      2     2     3                                     !17
      3     3     4                                     !18
      4     4     5                                     !19
      5     5     6                                     !20
*** CONSTRAINT EQUATION DATA ***                        !21
EQ. NO.   NODE_1   PAR_1          A_1                   !22
      1      1      1   0.00000E+00                     !23
***  MISCELLANEOUS SYSTEM PROPERTIES  ***               !24
PROPERTY    REAL_VALUE                                  !25
       1     2.00000E+00                                !26
       2     1.00000E+01                                !27
***  OUTPUT OF RESULTS IN NODAL ORDER  ***              !28
NODE,  X-Coord,    DOF_1,      EXACT1,                  !29
   1  0.0000E+00  0.0000E+00  0.0000E+00                !30
   2  1.0000E-01  9.0749E-01  9.0635E-01                !31
   3  2.0000E-01  1.6502E+00  1.6484E+00                !32
   4  3.0000E-01  2.2580E+00  2.2559E+00                !33
   5  4.0000E-01  2.7554E+00  2.7534E+00                !34
   6  5.0000E-01  3.1624E+00  3.1606E+00                !35
```

Problem P2.6Ab *Results for 5 L2 elements*



Problem P2.13a *Three linear element Galerkin solution results*



Problem P2.13b *Three linear element gradient results*

could solve one step in time repeatedly and use the answer for the first step (here $y(0.1)$) as the initial condition for the next step. Taking more than one step in time in a single solution, when feasible, can reduce the error in the time integration.

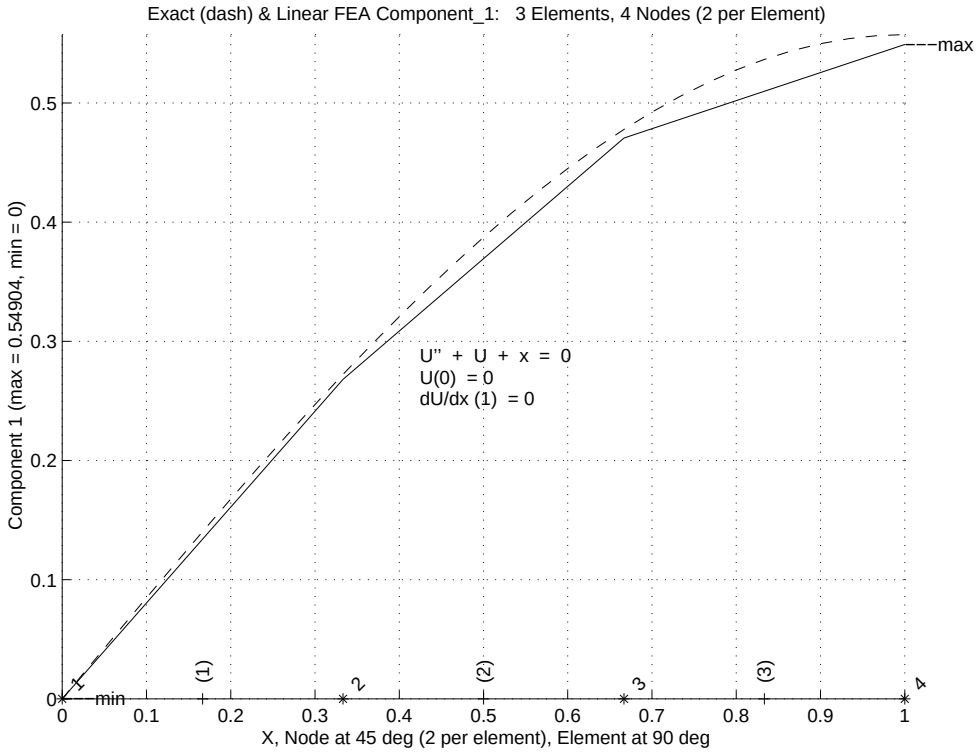
Problem 2.13 Use three equal length elements to solve the problem in Fig. 2.10.6 (L^e = 1/3). Obtain the nodal values, reactions, and post-process for the element gradients. Compare the nodal values to the exact solution. Solution: As expected, the results fall between those seen in Figs. 2.10.6 and 7, and are summarized here in Figs. P2.13 a and 13b. The nodal values are moving closer to the correct values. The solution is most accurate at the nodes, but the slopes (gradients) are discontinuous there. The slopes are most accurate near the center of each element.

Problem 2.15 Solve Eq 2.43 for a non-zero Neumann condition of $du/dx(1) = Cotan(1) - 1$ and $u(0) = 0$, and compare the results to the same exact solution. Solution: Changing the Dirichlet boundary condition at the right end to a Neumann natural condition ($q_L = 0$) not only changes the analytic solution but also shows an approximation behavior that is quite different from a boundary value problem. That is because the slope boundary condition at the right is satisfied only in the weak sense. In other words, neither the solution value or slope is exact at the right end. In particular, there is a noticeable difference in P2.15a and P2.15b between the slope of the last element and the specified rightmost slope. This is generally true and the user can partially compensate for that by making the element lengths smaller as they approach (normal to) a Neumann boundary condition. Of course, going to higher order elements also reduces the slope differences at such a boundary. If you used the same number of degrees of freedom in a single cubic Lagrangian element both the solution and slope would be much more accurate. A cubic Hermite element would convert the Neumann condition to a Dirichlet condition on its nodal slope value and thus give an exact slope on the boundary. However, that might over constrain the interior solution if the given data were not smooth over space.

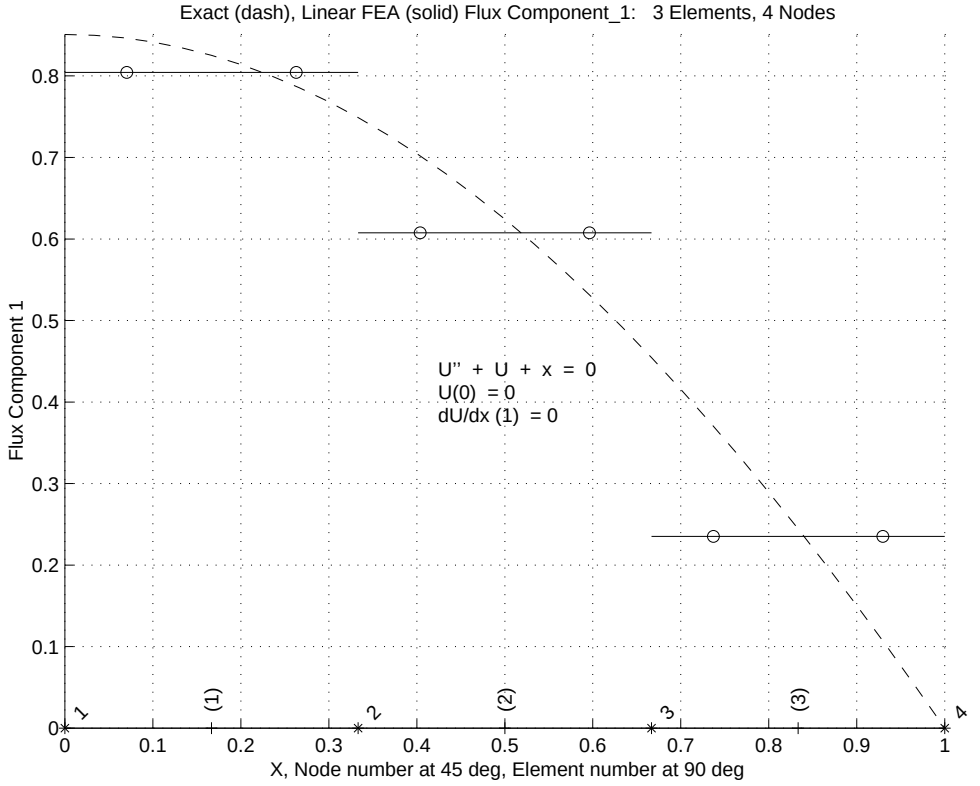
Problem 2.16 The exact solution of $d^2u/dx^2 + Q(x) = 0$ with $u(0) = 0 = u(1)$ with $Q = 1$ for $x \le 1/2$ and 0 for $x > 1/2$ is $u(x) = x(3 - 4x)/8$ for $x \le 1/2$ othewise $U(x) = (1 - x)/8$. The assembled source vector for length ratios of 1:1:2 is $\mathbf{C}^T_Q = [1\ \ 2\ \ 1\ \ 0]/8$ while for three equal length elements it is $\mathbf{C}^T_Q = [4\ \ 7\ \ 1\ \ 0]/24$.

Problem 2.17 Obtain a Galerkin solution of $y'' - 2y'x/g + 2y/g = g$, for $g = (x^2 + 1)$, on $0 \le x \le 1$ with the boundary conditions $y(0) = 2$, $y(1) = 5/3$. Solution: This requires only a C^0 approximation since the weak form involves only the first derivative of y when the first term is integrated by parts. Since the variable coefficients are rational polynomials it would be complicated to try to evaluate the matrices by exact integration so we use numerical integration. For the same reason, Gaussian quadratures will not be exact so we select a moderate number of quadrature points as a balance between accuracy and computational cost. This implementation supports all one-dimension elements in the library and is displayed in P2.17a.

Note, in lines 25-26, that the variable coefficients have been hard coded rather than using an include file or user function. The variable *SOURCE* is a global item that can sometimes be assigned constant values through keyword data controls. It was not



Problem P2.15a *Three element solution for a natural BC*



Problem P2.15b *Three element gradients for a natural BC*

```
! ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS FOLLOW ***    ! 1
! ..........................................................     ! 2
!     APPLICATION DEPENDENT Galerkin MWR FOR EXAMPLE 106         ! 3
!        Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1), or           ! 4
!      E Y'' + f_1 Y'       + f_2 Y      = f_3                   ! 5
! Y(0)=2, Y(1)=5/3, Y=X^4/6 - 3X^2/2 + X + 2 (is EXACT_CASE 15)  ! 6
                                                                 ! 7
  REAL(DP)  :: DL, DX_DR, CONST           ! Length, Jacobian     ! 8
  REAL(DP)  :: f_1, f_2, f_3              ! Coefficients         ! 9
  INTEGER   :: IQ                         ! Loops               !10
                                                                !11
  DL    = COORD (LT_N, 1) - COORD (1, 1)  ! LENGTH              !12
  DX_DR = DL / 2.                         ! CONSTANT JACOBIAN   !13
  E (1, 1) = 1                            ! identity matrix     !14
  CALL STORE_FLUX_POINT_COUNT             ! Save LT_QP          !15
                                                                !16
  DO IQ = 1, LT_QP           ! LOOP OVER QUADRATURES            !17
    CONST = DX_DR * WT (IQ)  ! NET WEIGHT                       !18
                                                                !19
!        GET INTERPOLATION FUNCTIONS, AND X-COORD               !20
    H    = GET_H_AT_QP (IQ)    ! INTERPOLATIONS                 !21
    XYZ  = MATMUL (H, COORD)   ! ISOPARAMETRIC POINT            !22
                                                                !23
!        DEFINE VARIABLE COEFFICIENTS                           !24
    f_3 = 1.d0 + XYZ (1) **2  ; SOURCE = -f_3  ! global source  !25
    f_2 = 2.d0 / f_3          ; f_1 = -XYZ (1) * f_2            !26
                                                                !27
!        LOCAL AND GLOBAL DERIVATIVES, dH/dr, dH/dx             !28
    DLH   = GET_DLH_AT_QP (IQ) ; DGH   = DLH / DX_DR            !29
                                                                !30
!         SQUARE MATRIX                                         !31
    S = S + MATMUL (TRANSPOSE(DGH), DGH) * CONST       &        !32
          - f_1 * OUTER_PRODUCT (H, DGH(1, :)) * CONST &        !33
          - f_2 * OUTER_PRODUCT (H, H) * CONST                  !34
                                                                !35
    C = C + SOURCE * H * CONST ! RESULTANT SOURCE VECTOR        !36
                                                                !37
    CALL STORE_FLUX_POINT_DATA (XYZ, E, DGH) ! for SCP          !38
  END DO  ! QUADRATURE                                          !39
!    ***  END ELEM_SQ_EX_106 PROBLEM DEPENDENT STATEMENTS ***   !40
```

Problem P2.17a *Galerkin variable coefficient test*

actually needed here since f_3 is defined. To test the solution we select a crude mesh of quadratic L3 line elements. The reader should try a solution with linear or cubic elements. The typical input test data are given in P2.17b along with corresponding selected outputs in P2.17c. The true solution is compared to the Galerkin FEA result in P2.17d. There straight lines are plotted between nodal values (rather than actual parabolic curves) so the two plots appear different.

Problem 2.18 For the differential equation in Problem 2.17 if we have one essential boundary condition of $y(0) = 1$ and one Neumann flux boundary condition of $dy/dx(1) = -4/3$ the exact solution is unchanged. Obtain a Galerkin finite element solution and compare it to the exact result. Solution: This problem only requires a change in the data file that reduces the number of essential boundary conditions to one and adds a known flux term to the rhs source vector. These changes are denoted in

```
title "Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)"          ! 1
nodes          7 ! Number of nodes in the mesh             ! 2
elems          3 ! Number of elements in the system        ! 3
dof            1 ! Number of unknowns per node             ! 4
el_nodes       3 ! Maximum number of nodes per element     ! 5
space          1 ! Solution space dimension                ! 6
b_rows         1 ! Number of rows in B (operator) matrix   ! 7
el_react         ! Compute & list element reactions        ! 8
gauss          4 ! Maximum number of quadrature points     ! 9
exact_case    15 ! Exact analytic solution                 !10
scp_neigh_el     ! Default SCP patch group type            !11
unsymmetric      ! Unsymmetric skyline storage is used     !12
list_exact       ! List exact answers at nodes             !13
list_exact_flux  ! List exact fluxes at nodes              !14
bar_chart        ! print-plot result                       !15
remarks        4 ! Number of user remarks                  !16
quit ! keyword input, remarks follow                       !17
Galerkin: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)        !18
with Y(0)=2, Y(1)=5/3, Y(X) = X^4/6 - 3X^2/2 + X + 2       !19
Fausett p. 482, EXACT_CASE = 15                            !20
Here we use three quadratic (L3) line elements.            !21
 1 1 0.           ! node bc_flag x                         !22
 2 0 0.166666667  ! node bc_flag x                         !23
 3 0 0.333333333  ! node bc_flag x                         !24
 4 0 0.5          ! node bc_flag x                         !25
 5 0 0.666666667  ! node bc_flag x                         !26
 6 0 0.833333334  ! node bc_flag x                         !27
 7 1 1.00         ! node bc_flag x                         !28
 1    1    2    3  ! elem j, k, l                          !29
 2    3    4    5  ! elem j, k, l                          !30
 3    5    6    7  ! elem j, k, l                          !31
 1 1 2.             ! Essential BC: node dof_value         !32
 7 1 1.66666667     ! Essential BC: node dof_value         !33
```

Problem P2.17b *Galerkin L2 model test data*

P2.18a. External source vector components are initialized to zero. The keyword *loads* requires the non-zero entries to be input (after the EBC and MPC) by giving the node number, degree of freedom number, and corresponding known flux. The data ends with the last dof flux value (which is usually zero). The output is unchanged except for echoing the above lines and listing the initial external source vector terms, in P2.18b.

1.3 Problems from chapter 3

Problem 3.1 For a one-dimensional quadratic element use the unit coordinate interpolation functions in Fig. 3.4.1 to evaluate the matrices:

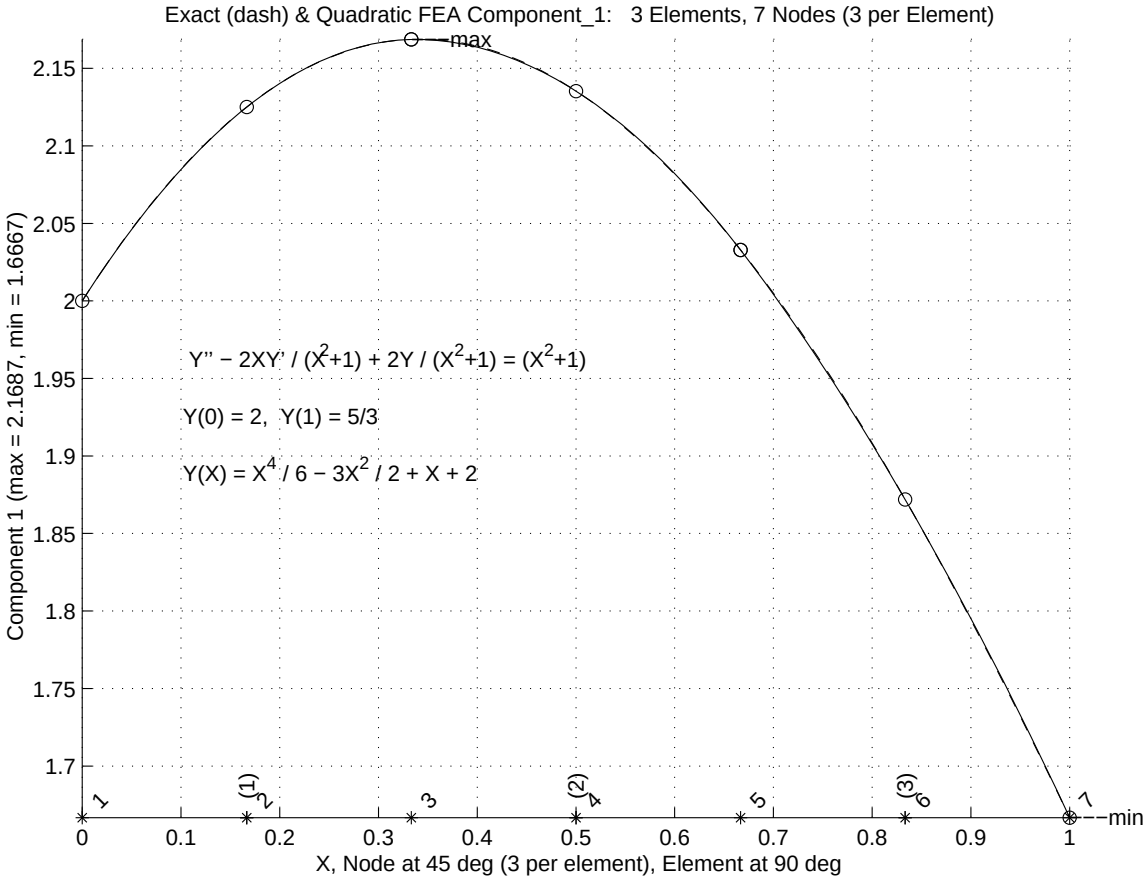
$$a)\quad \mathbf{C}^e = \int_{L^e} \mathbf{H}^T\, dx, \quad b)\quad \mathbf{M}^e = \int_{L^e} \mathbf{H}^T\, \mathbf{H}\, dx\,,$$

$$c)\quad \mathbf{S}^e = \int_{L^e} \frac{d\mathbf{H}^T}{dx}\,\frac{d\mathbf{H}}{dx}\, dx\,, \quad d)\quad \mathbf{U}^e = \int_{L^e} \mathbf{H}^T\,\frac{d\mathbf{H}}{dx}\, dx\,, \text{ and}$$

Problem 3.2 Solve the above problem by using the natural coordinate version, $-1 \le n \le 1$, from Fig, 3.4.1. Solution: Either local coordinate system must lead to the

```
Y"-2XY'/(X^2+1)+2Y/(X^2+1)=(X^2+1), Y(0)=2, Y(1)=5/3
** OUTPUT OF RESULTS & EXACT VALUES IN NODAL ORDER **
NODE,  X-Coord,    DOF_1,      EXACT1,
   1  0.0000E+00  2.0000E+00  2.0000E+00
   2  1.6667E-01  2.1251E+00  2.1251E+00
   3  3.3333E-01  2.1687E+00  2.1687E+00
   4  5.0000E-01  2.1353E+00  2.1354E+00
   5  6.6667E-01  2.0329E+00  2.0329E+00
   6  8.3333E-01  1.8719E+00  1.8720E+00
   7  1.0000E+00  1.6667E+00  1.6667E+00

** FE & EXACT FLUX COMPONENTS AT INTEGRATION POINTS **
ELEMENT, PT,  X-Coord,   FX_1,     EX_1,
       1  1  2.314E-02  9.270E-01  9.306E-01
       1  2  1.100E-01  6.723E-01  6.709E-01
       1  3  2.233E-01  3.399E-01  3.374E-01
       1  4  3.102E-01  8.517E-02  8.933E-02
ELEMENT, PT,  X-Coord,   FX_1,     EX_1,
       2  1  3.565E-01 -5.063E-02 -3.923E-02
       2  2  4.433E-01 -2.665E-01 -2.719E-01
       2  3  5.567E-01 -5.482E-01 -5.550E-01
       2  4  6.435E-01 -7.641E-01 -7.529E-01
ELEMENT, PT,  X-Coord,   FX_1,     EX_1,
       3  1  6.898E-01 -8.700E-01 -8.506E-01
       3  2  7.767E-01 -1.008E+00 -1.018E+00
       3  3  8.900E-01 -1.189E+00 -1.200E+00
       3  4  9.769E-01 -1.327E+00 -1.309E+00

** SUPER_CONVERGENT AVERAGED NODAL & EXACT FLUXES **
NODE,   X-Coord,    FLUX_1,     EXACT1,
     1  0.000E+00  1.016E+00  1.000E+00
     2  1.667E-01  4.939E-01  5.031E-01
     3  3.333E-01  1.817E-02  2.469E-02
     4  5.000E-01 -4.187E-01 -4.167E-01
     5  6.667E-01 -8.007E-01 -8.025E-01
     6  8.333E-01 -1.110E+00 -1.114E+00
     7  1.000E+00 -1.360E+00 -1.333E+00

ELEMENT ERROR ESTIMATE GRAPH
EL  VALUE +---+---+---+---+---+---+---+---+---+---+
1 5.89E-3 XXXXXXXXXXXXXXXXX                       +
2 5.14E-3 X                                       +
3 6.95E-3 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
EL  VALUE +---+---+---+---+---+---+---+---+---+---+

NODAL SOLUTION GRAPHS
PT  VALUE +---+---+---+---+---+---+---+---+---+---+
1 2.00E+0 XXXXXXXXXXXXXXXXXXXXXXXXXXX             +
2 2.13E+0 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX   +
3 2.17E+0 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
4 2.14E+0 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX  +
5 2.03E+0 XXXXXXXXXXXXXXXXXXXXXXXXXXXXX          +
6 1.87E+0 XXXXXXXXXXXXXXXXX                       +
7 1.67E+0 X                                       +
PT  VALUE +---+---+---+---+---+---+---+---+---+---+
```

Problem P2.17c *Galerkin variable coefficient results and exact values*



Problem P2.17d *True solution and Galerkin nodal values*

```
loads           ! An initial source vector is input       !11
Galerkin: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)       !18
with Y(0)=2, Y'(1)=-4/3, Y(X) = X^4/6 - 3X^2/2 + X + 2    !19
 7 0 1.00        ! removed EBC flag                        !28
 7 1 -1.33333 ! Enter only (last) non-zero source term     !33
```

Problem P2.18a *Galerkin L2 data changes for Neumann condition*

```
    . . .                                                  !1
*** INITIAL FORCING VECTOR DATA ***                        !2
     NODE   PARAMETER     VALUE    EQUATION               !3
        7           1  -1.33333E+00       7               !4
*** RESULTANTS ***                                         !5
COMPONENT         SUM         POSITIVE      NEGATIVE      !6
 IN_1,       -1.3333E+00    0.0000E+00   -1.3333E+00       !7
```

Problem P2.18b *Galerkin L2 output changes for Neumann condition*

same final result, if one treats the Jacobian correctly. The exact integrals are listed in subroutine form in Problem P3.1. Note that the last square matrix is unsymmetrical. In our usual notation these give:

$$\mathbf{C}^e = \frac{L^e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix},\ \mathbf{M}^e = \frac{L^e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$\mathbf{S}^e = \frac{1}{3\,L^e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix},\ \mathbf{U}^e = \frac{1}{6}\begin{bmatrix} -3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix}.$$

Can you explain why L^e is in the numerator, denominator, or missing in each matrix?

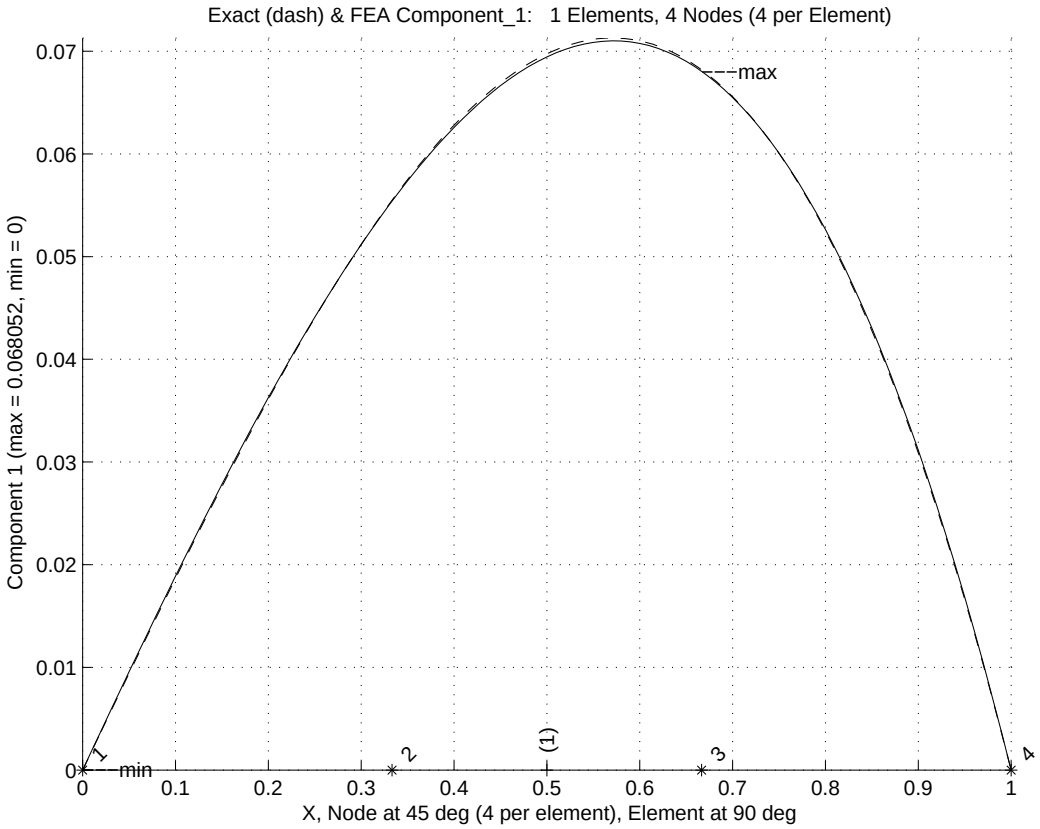
Problem 3.6 A single cubic element solution of Problem 2.13 gives the results $u_2 = 0.055405, u_3 = 0.068052$. Use these computed values with the cubic interpolation functions in Fig. 3.4.1 to plot the single element solution in comparison to the exact solution. Solution: Employing cubic elements quickly smoothes out the approximation, even for this single element case. Figure P3.6 shows little difference compared with the exact solution, but is quite different from the 3 linear element solution given above in Fig. P2.13 even though they involve the same number of unknowns. The main extra cost was in carrying out the numerical integration of the element matrices. The source code in Fig. 2.10.8 was employed with $n_q = 4$ quadrature points specified in the data file so as to exactly integrate the mass matrix contribution on line 54.

Problem P3.9 Solve Problem P2.17 using the least squares finite element method instead. Solution: The approach here is quite different from the Galerkin solution. The good news is that the equation system is always symmetric so we can use a more efficient storage and solver mode. However, we can not use integration by parts and thus the second derivative term will remain in the weak form and that in turn requires a C^1 interpolation function. That is, the solution and its slope (first derivative) must be between elements. Thus we are forced to use a cubic Hermite polynomial with two degrees of freedom per node (value and slope). We have not considered such generalized (or vector) nodal degrees of freedom before. The *MODEL* program is designed to simplify the use of the most common C^0 elements by gathering the quadrature data, evaluating the interpolation functions and their local derivatives there, etc. For C^1 or C^2 inter-element continuity type elements more specific programming details must be supplied to form the element matrices.

Referring to P3.9a we will point out some of the extra details that are not usually needed. In lines 8 and 10 we see that while storage is always allocated for the generalized interpolation functions and their gradient one must specifically assign storage for their second (or higher) derivatives, if needed. The least squares approach is usually most simply programmed by introducing a work space vector that holds the result of the differential operator acting on the generalized interpolations. That is, for every appearance of Y in the differential equation there is a corresponding action on V (the generalized interpolation array) in the work vector. In line 11 we name the workspace

```
SUBROUTINE INTEGRAL_H_ON_L3 (LENGTH, C_E)                          ! 1
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          ! 2
! INTEGRATE SHAPE FUNCTIONS OF A 3 NODE LINE ELEMENT               ! 3
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          ! 4
Use Precision_Module  ! for DP                                     ! 5
 IMPLICIT NONE                                                     ! 6
 REAL(DP), INTENT(IN)  :: LENGTH   ! PHYSICAL LENGTH               ! 7
 REAL(DP), INTENT(OUT) :: C_E (3)  ! INTEGRAL OF H                 ! 8
! LOCAL NODE COORD. ARE -1,0,+1    1-----2-----3                   ! 9
                                                                   !10
  C_E = LENGTH * (/ 1.d0, 4.d0, 1.d0 /) / 6.d0                     !11
END SUBROUTINE INTEGRAL_H_ON_L3                                    !12
                                                                   !13
SUBROUTINE INTEGRAL_HT_H_ON_L3 (LENGTH, MASS_E)                    !14
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !15
! INTEGRATE H-TRANSPOSE H ON A 3 NODE LINE ELEMENT                 !16
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !17
Use Precision_Module  ! for DP                                     !18
 IMPLICIT NONE                                                     !19
 REAL(DP), INTENT(IN)  :: LENGTH          ! PHYSICAL LENGTH        !20
 REAL(DP), INTENT(OUT) :: MASS_E (3, 3)  ! INTEGRAL OF H' H        !21
 REAL(DP), PARAMETER   :: MASS (3, 3) = RESHAPE (       &           !22
   (/ 4, 2, -1, 2, 16, 2, -1, 2, 4 /), (/3, 3/) ) / 30.d0          !23
! LOCAL NODE COORD. ARE -1,0,+1    1-----2-----3                   !24
                                                                   !25
  MASS_E = LENGTH * MASS                                           !26
END SUBROUTINE INTEGRAL_HT_H_ON_L3                                 !27
                                                                   !28
SUBROUTINE INTEGRAL_DHDXT_DHDX_ON_L3 (LENGTH, STIFF_E)             !29
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !30
! INTEGRATE DHDX-TRANSPOSE DHDX ON A 3 NODE LINE ELEMENT           !31
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !32
Use Precision_Module  ! for DP                                     !33
 IMPLICIT NONE                                                     !34
 REAL(DP), INTENT(IN) :: LENGTH         !PHYSICAL LENGTH          !35
 REAL(DP), INTENT(OUT):: STIFF_E (3, 3) !INTEGRAL DHDX' DHDX      !36
 REAL(DP), PARAMETER  :: STIFF (3, 3) = RESHAPE (       &           !37
   (/ 7, -8, 1, -8, 16, -8, 1, -8, 7/), (/3, 3/) ) / 3.d0          !38
! LOCAL NODE COORD. ARE -1,0,+1    1-----2-----3                   !39
                                                                   !40
  STIFF_E = STIFF / LENGTH                                         !41
END SUBROUTINE INTEGRAL_DHDXT_DHDX_ON_L3                           !42
                                                                   !43
SUBROUTINE INTEGRAL_HT_DHDX_ON_L3 (U_E)                            !44
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !45
! INTEGRATE H-TRANSPOSE DHDX ON A 3 NODE LINE ELEMENT              !46
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*          !47
Use Precision_Module  ! for DP                                     !48
 IMPLICIT NONE                                                     !49
 REAL(DP), INTENT(OUT) :: U_E (3, 3) ! INTEGRAL H' DHDX            !50
 REAL(DP), PARAMETER   :: U (3, 3) = RESHAPE (       &             !51
   (/ -3, -4, 1, 4, 0, -4, -1, 4, 3/), (/3, 3/) ) / 6.d0           !52
! LOCAL NODE COORD. ARE -1,0,+1    1-----2-----3                   !53
                                                                   !54
  U_E = U                                                          !55
END SUBROUTINE INTEGRAL_HT_DHDX_ON_L3                              !56
```

Problem P3.1 *Typical quadratic C^0 element integrals*



Problem P3.6 *A single cubic element model and exact problem solution*

```
! ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS FOLLOW ***   ! 1
! ..........................................................   ! 2
! LEAST SQ. SOL. OF: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)  ! 3
!         USING 3-RD ORDER HERMITE IN UNIT COORDINATES          ! 4
!             CONVERTED FROM NATURAL COORDINATES                ! 5
!     >>> NOTE NOT SCALAR INTERPOLATION, C_1 ELEMENTS <<<       ! 6
!         ---------------------------------------               ! 7
! Allocated V (LT_FREE) and DGV (N_SPACE, LT_FREE) by MODEL     ! 8
                                                                ! 9
 REAL(DP) :: D2GV (LT_FREE)       ! v'', 2nd derivative         !10
 REAL(DP) :: OP_V (LT_FREE)       ! ODE on H array              !11
 REAL(DP) :: PT_UNIT, WT_UNIT     ! unit coord pt, weight       !12
 REAL(DP) :: DL, X_SQ, X_G        ! physical length, pt         !13
 INTEGER  :: IP                   ! loops                       !14
                                                                !15
! V    = GENERALIZED (VECTOR) INTERPOLATION FUNCTIONS           !16
! DGV  = DERIVATIVE OF GENERALIZED (VECTOR) INTERPOLATION       !17
! D2GV = 2nd DERIVATIVE OF GENERALIZED (VECTOR) INTERPOLATION   !18
                                                                !19
  IF ( DEBUG_EL_SQ ) WRITE (N_BUG, *) 'Enter my_sq_matrix_inc'  !20
                                                                !21
  E  = GET_REAL_IDENTITY (N_R_B)     ! DUMMY CONSTITUTIVE       !22
  DL = COORD (2, 1) - COORD (1, 1)   ! GET THE LENGTH           !23
  CALL STORE_FLUX_POINT_COUNT        ! Save LT_QP               !24
                                                                !25
!      NUMERICAL INTEGRATION LOOP                              !26
  DO IP = 1, LT_QP                                              !27
                                                                !28
!-->     FIND UNIT COORDINATES AND WEIGHT FOR INTEGRATION       !29
    PT_UNIT = (1.d0 + PT(1,IP)) / 2.d0    ! Convert pt          !30
    WT_UNIT = WT (IP) / 2.0d0             ! Convert wt          !31
    XYZ (1) = COORD (1, 1) + PT_UNIT * DL ! Physical pt         !32
    X_SQ    = XYZ (1) * XYZ (1)           ! x ^ 2               !33
    X_G     = X_SQ + 1.d0                 ! scalar g(x)         !34
                                                                !35
!-->     EVALUATE HERMITE SHAPE FUNCTIONS AND DERIVATIVES       !36
    CALL SHAPE_C1_L  (PT_UNIT, DL, V)     ! V (NOT H)           !37
    CALL DERIV_C1_L  (PT_UNIT, DL, DGV)   ! DV / DX             !38
    CALL DERIV2_C1_L (PT_UNIT, DL, D2GV)  ! D^2 V / DX^2        !39
                                                                !40
!     WORK VECTOR, OP_V =  V'' - 2X V'/(X^2+1) + 2 V/(X^2+1)    !41
   OP_V = D2GV - 2.d0 * XYZ (1) * DGV(1,:) / X_G  &             !42
        + 2.d0 * V / X_G                                        !43
                                                                !44
!     COMPLETE THE SQUARE MATRIX AND SOURCE VECTOR              !45
!        S_IJ = S_IJ + WT_UNIT * OP_V_I * OP_V_J * DL           !46
    S = S + WT_UNIT * OUTER_PRODUCT (OP_V, OP_V) * DL           !47
    C = C + WT_UNIT * X_G * OP_V * DL                           !48
                                                                !49
!      STORE DATA FOR SCP OR POST PROCESSING                    !50
    B (1, :) = DGV (1, :) ; B (2, :) = D2GV (:)                 !51
    CALL STORE_FLUX_POINT_DATA (XYZ, E, B)     ! for SCP        !52
  END DO                                                        !53
! ***  END ELEM_SQ_EX_107 PROBLEM DEPENDENT STATEMENTS ***      !54
```

Problem P3.9a *A C^1 unit coordinate least squares version*

```
title "Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)"      ! 1
nodes       4 ! Number of nodes in the mesh             ! 2
elems       3 ! Number of elements in the system        ! 3
dof         2 ! Number of unknowns per node             ! 4
el_nodes    2 ! Maximum number of nodes per element     ! 5
space       1 ! Solution space dimension                ! 6
b_rows      2 ! Number of rows in B (operator) matrix   ! 7
shape       1 ! Shape, 1=line, 2=tri, 3=quad, 4=hex     ! 8
gauss       5 ! Maximum number of quadrature points     ! 9
example   107 ! Source library example number           !10
data_set    1 ! Data set for example (this file)        !11
exact_case 15 ! Exact analytic solution                 !12
scp_neigh_el  ! Default SCP patch group type            !13
list_exact    ! List exact answers at nodes             !14
list_exact_flux ! List exact fluxes at nodes            !15
remarks       4 ! Number of user remarks                !16
quit ! keyword input                                    !17
Least Sq: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)     !18
with Y(0)=2, Y(1)=5/3, Y(X) = X^4/6 - 3X^2/2 + X + 2    !19
Fausett p. 482, EXACT_CASE = 15                         !20
Here we use 3 cubic Hermite (C1_L) line elements.       !21
 1 10 0.          ! node, 2_binary_ bc_flags, x         !22
 2 00 0.333333333 ! node, 2_binary_ bc_flags, x         !23
 3 00 0.666666667 ! node, 2_binary_ bc_flags, x         !24
 4 10 1.00        ! node, 2_binary_ bc_flags, x         !25
 1    1    2    ! elem, j, k                           !26
 2    2    3    ! elem, j, k                           !27
 3    3    4    ! elem, j, k                           !28
 1 1 2.         ! Essential BC: node, dof_value        !29
 4 1 1.66666667 ! Essential BC: node, dof_value        !30
```

Problem P3.9b *Three element least square C^1 data set*

OP_V (for OPerator acting on V) and assign it the same storage allotment as V. Note that at any point, the work vector times the element degrees of freedom plus any source term is the residual error in the differential equation at that point. In other words $OP\_V = \partial R / \partial \mathbf{\Phi}$ which was defined in Eq. 2.25 as the weighting term in a least squares formulation. Using this work vector the element square matrix and source vector are always defined as:

$$\mathbf{S}^e = \int_{\Omega^e} OP\_V^T \, OP\_V \, d\Omega, \quad \mathbf{C}_Q^e = \int_{\Omega^e} OP\_V^T Q \, d\Omega \, .$$

For this specific application the terms required in the work vector are seen by comparing the comments in lines 3 and 41.

From Fig. 3.5.1 we see that the Hermite family of elements are provided here in unit coordinates, not in the natural coordinates used to store the Gaussian quadrature data (for lines). Therefore it is necessary to either convert the quadrature data or re-program all the Hermite interpolations and derivatives. Lines 12 and 30-31 declare and carry out the quadrature conversion. Line 13 does the linear unit coordinate interpolation for the physical X position needed to evaluate the variable coefficients and source term. Lines 37-39 evaluate the Hermite polynomials and the first and second physical derivatives (by using the element length, DL). One only needs lines 22, 24, 50-52 if SCP or other post-

```
 THE NEXT 4 LINES ARE USER REMARKS                                      ! 1
 1 Least Sq: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)                  ! 2
 2 with Y(0)=2, Y(1)=5/3, Y(X) = X^4/6 - 3X^2/2 + X + 2                 ! 3
 3 Fausett p. 482, EXACT_CASE = 15                                      ! 4
 4 Here we use 3 cubic Hermite (C1_L) line elements.                    ! 5
                                                                        ! 6
*** NODAL POINT DATA ***                                                ! 7
NODE, BC_FLAG, X-Coord,                                                 ! 8
   1  10    0.0000  ! note binary BC_FLAG                               ! 9
   2  00    0.3333  ! first is value, second slope                      !10
   3  00    0.6667                                                      !11
   4  10    1.0000                                                      !12
                                                                        !13
*** ELEMENT CONNECTIVITY DATA ***                                       !14
ELEMENT,  2 NODAL INCIDENCES.                                           !15
      1     1     2                                                     !16
      2     2     3                                                     !17
      3     3     4                                                     !18
                                                                        !19
*** CONSTRAINT EQUATION DATA ***                                        !20
EQ. NO.   NODE_1   PAR_1          A_1                                   !21
     1      1      1        2.00000E+00                                 !22
     2      4      1        1.66667E+00                                 !23
                                                                        !24
**  OUTPUT OF RESULTS AND EXACT VALUES IN NODAL ORDER  **               !25
PT,  X-Coord,    DOF_1,     DOF_2,     EXACT1,    EXACT2               !26
 1  0.0000E+0  2.0000E+0  9.9993E-1  2.0000E+0  1.0000E+0               !27
 2  3.3333E-1  2.1687E+0  2.4669E-2  2.1687E+0  2.4691E-2               !28
 3  6.6667E-1  2.0329E+0 -8.0245E-1  2.0329E+0 -8.0247E-1               !29
 4  1.0000E+0  1.6667E+0 -1.3333E+0  1.6667E+0 -1.3333E+0               !30
                                                                        !31
** FE AND EXACT FLUX COMPONENTS AT INTEGRATION POINTS **                !32
ELEM, PT,  X-Coord,   FX_1,     FX_2,     EX_1,     EX_2               !33
   1  1  1.564E-2  9.525E-1 -3.026E+0  9.531E-1 -3.000E+0               !34
   1  2  7.692E-2  7.683E-1 -2.986E+0  7.695E-1 -2.988E+0               !35
   1  3  1.667E-1  5.030E-1 -2.926E+0  5.031E-1 -2.944E+0               !36
   1  4  2.564E-1  2.432E-1 -2.866E+0  2.420E-1 -2.869E+0               !37
   1  5  3.177E-1  6.876E-2 -2.825E+0  6.829E-2 -2.798E+0               !38
ELEM, PT,  X-Coord,   FX_1,     FX_2,     EX_1,     EX_2               !39
   2  1  3.490E-1 -1.910E-2 -2.783E+0 -1.858E-2 -2.756E+0               !40
   2  2  4.103E-1 -1.859E-1 -2.661E+0 -1.847E-1 -2.663E+0               !41
   2  3  5.000E-1 -4.167E-1 -2.481E+0 -4.167E-1 -2.500E+0               !42
   2  4  5.897E-1 -6.313E-1 -2.302E+0 -6.325E-1 -2.304E+0               !43
   2  5  6.510E-1 -7.686E-1 -2.179E+0 -7.691E-1 -2.152E+0               !44
ELEM, PT,  X-Coord,   FX_1,     FX_2,     EX_1,     EX_2               !45
   3  1  6.823E-1 -8.356E-1 -2.096E+0 -8.352E-1 -2.069E+0               !46
   3  2  7.436E-1 -9.578E-1 -1.892E+0 -9.567E-1 -1.894E+0               !47
   3  3  8.333E-1 -1.114E+0 -1.592E+0 -1.114E+0 -1.611E+0               !48
   3  4  9.231E-1 -1.244E+0 -1.293E+0 -1.245E+0 -1.296E+0               !49
   3  5  9.844E-1 -1.317E+0 -1.089E+0 -1.317E+0 -1.062E+0               !50
LARGEST FLUX RMS_VALUE = 3.1728 AT ELEM = 1, POINT = 1                  !51
                                                                        !52
** SUPER_CONVERGENT AVERAGED NODAL & EXACT FLUXES **                    !53
PT, X-Coord,    FLUX_1,  FLUX_2,    EXACT1,   EXACT2                    !54
 1  0.000E+0  9.126E-1 -3.241E+0  1.000E+0 -3.000E+0                     !55
 2  3.333E-1  1.896E-2 -2.765E+0  2.469E-2 -2.778E+0                     !56
 3  6.667E-1 -7.782E-1 -2.099E+0 -8.025E-1 -2.111E+0                     !57
 4  1.000E+0 -1.480E+0 -1.241E+0 -1.333E+0 -1.000E+0                     !58
```

Problem P3.9c *Three element least square C^1 results and exact values*

processing is desired. For the cubic Hermite used here, that already has accurate continuous nodal slopes, the SCP averaging process would only give useful estimates of the nodal second derivatives (item *FLUX*_2 column in output lines 54-58 of P3.9c). The SCP estimated nodal slopes (*FLUX*_1 lines 54-58 of P3.9c) will usually be less accurate that the computed nodal slopes (item *DOF*_2 in lines 25-30 of P3.9c). The SCP process and associated error estimator could be improved by adding specific options for Hermite elements but such coding is not provided here.

In closing, it should also be pointed out that the essential boundary condition flags have also generalized to account for multiple unknowns per node. In general it is a packed integer flag with as many digits as unknowns per node. The unknowns are counted from left to right. Thus the boundary condition flags (in lines 22-26 in P3.9b and lines 7-12 of P3.9c) indicate that only the first unknown at nodes 1 and 4 have known essential boundary conditions applied. Thus only the first *dof* (*PAR*_1 in lines 21-23 of P3.9c) are cited in the data. In a different example the slope (second *dof*) might be specified instead.

For example, we have the same exact solution if we change the boundary condition at $x = 1$ to a slope condition, $dY/dx(1) = -4/3$. For the least squares C^1 form that is an essential boundary condition (but it is a Neumann condition in the Galerkin implementation). Thus, simply changing the data to have a different essential boundary condition give essentially the same results as before, as summarized in P3.9d.

1.4 Problems from chapter 4

Problem 4.1 In Tables 4.1 and 4.3 the sum of the weights is exactly 2, but in Table 4.2 the sum is exactly 1. Explain why. Solution: The quadrature rule must be able to exactly integrate the measure of the local parametric domain which is simply the sum of the weight functions. In other words, if the integrand is simply unity then

$$\int_{-1}^{1} dn \;=\; 2 \;=\; \sum_{i=1}^{n_q} W_i \;.$$

Likewise, in unit coordinates the local measure changes from 2 to 1.

```
Least Sq: Y'' - 2XY'/(X^2+1) + 2Y/(X^2+1) = (X^2+1)            !18
Y(0)=2, Y'(1)=-4/3, so  Y(X) = X^4/6 - 3X^2/2 + X + 2          !19
 4 01 1.00      ! flag slope as known                          !25
 4 2 -1.3333333 ! Essential BC: node dof slope_value           !30
                                                               !
                                                               !
**  OUTPUT OF RESULTS AND EXACT VALUES IN NODAL ORDER  **      !25
PT,  X-Coord,    DOF_1,     DOF_2,     EXACT1,    EXACT2       !26
 1  0.0000E+0  2.0000E+0  9.9986E-1  2.0000E+0  1.0000E+0      !27
 2  3.3333E-1  2.1687E+0  2.4600E-2  2.1687E+0  2.4691E-2      !28
 3  6.6667E-1  2.0329E+0 -8.0251E-1  2.0329E+0 -8.0247E-1      !29
 4  1.0000E+0  1.6666E+0 -1.3333E+0  1.6667E+0 -1.3333E+0      !30
```

Problem P3.9d *Least square C^1 slope condition changes*

Problem 4.2 Explain why quadrature points on the ends of a line may be undesirable. Solution: Axisymmetric integrals often require dividing by the global radial position and other applications my require dividing by interpolated items. For a node on an axis of revolution a Lobatto rule would clearly lead to division by zero, but a Gauss rule would avoid that numerical problem. (Such problems often involve 0/0.)

1.5 Problems from chapter 5

Problem 5.3 should yield the same results as presented in problem 3.1.

1.6 Problems from chapter 6

Problem 6.1 Develop a function, say *GET_SCP_PT_AT_XYZ*, to find where an element quadrature point or node is located within an element. Solution: As seen from Fig. 6.2.1 the patch is nothing other than a single large parametric element designed to have a constant diagonal Jacobian. That is, its sides or faces are generally constructed parallel to the global axes. This assures that the inverse mapping from Ω to $\square$ exists and the non-dimensional coordinates, here *POINT*, are obtained as a linear scaling of the global coordinates, *XYZ*. A typical function listing is given in P6.1.

Problem 6.2 Develop a routine, say *EVAL_PT_FLUX_IN_SCP_PATCH*, to evaluate the flux components at any global coordinate point in a patch. Solution: A typical source code is given in P6.2. Given the physical coordinates of a node the previous function is first used to convert the location into a non-dimensional patch parametric coordinate. Then the patch type, shape, parametric dimension, etc. are passed to a general parametric interpolation routine, *GEN_ELEM_SHAPE*, that extracts a matching element from the programs element interpolation library. Then those shape functions are used to form the flux components at the local point. Those components are scattered to the system nodes for later averaging. Each node that has received such a scattered flux has a logical flag set so that some optional patch smoothing features can be selected elsewhere to reduce the cost of the SCP fit.

Problem 6.3 Employ the preceding patch routine to create a related subroutine, say *EVAL_SCP_FIT_AT_PATCH_NODES*, that will interpolate patch fluxes back to all mesh nodes inside the patch. Solution: These calculations are relatively simply. Most of the routine involves setting up the interfaces, restricting the intended operations on the arguments, and commenting on the variable names as seen in P6.3a. The actual loops, in figure P6.3b, utilize the previous subroutine to scatter the continuous patch nodal flux components to the system nodes and to update the contribution count for each node. As seen in Chapter 6, these counts are then used to get a final averaged set of flux components at each system node. They in turn are gathered at the element level to be used in computing the error estimates and/or to compute system norms of interest.

Problem 6.4 Develop a routine, say *FORM_NEW_EL_SIZES*, that can implement the estimated new element sizes for an h-adaptivity based on Eqs. 5.28 and 29. Solution: Most current mesh generators allow one to control the element sizes created by beginning with a group of control points where you specify the desired element size at that point. Here we simply use the desired element size, from the error estimator, averaged at each node to be output to a file so that it can be read as input for the next mesh generation

```
FUNCTION GET_SCP_PT_AT_XYZ (XYZ, XYZ_MIN, XYZ_MAX) RESULT (POINT)   ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 2
!      FIND LOCAL POINT IN PATCH FOR GIVEN XYZ POSITION             ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 4
Use System_Constants ! for N_SPACE, SCP_SHAP = patch shape          ! 5
Use Elem_Type_Data                                                  ! 6
IMPLICIT NONE                                                       ! 7
REAL(DP), INTENT (IN) :: XYZ     (N_SPACE)                          ! 8
REAL(DP), INTENT (IN) :: XYZ_MIN (N_SPACE), XYZ_MAX (N_SPACE)       ! 9
REAL(DP)              :: POINT   (N_SPACE)                          !10
                                                                    !11
! SCP_PARM = NUMBER OF PARAMETRIC SPACES FOR PATCH                  !12
! SCP_SHAP = PATCH SHAPE FLAG NUMBER                                !13
! XYZ_MIN _MAX = LOWER / UPPER BOUNDS FOR SCP GEOMETRY              !14
                                                                    !15
! . S     PATCH MAPPING: ELEMENT AND ITS NEIGHBORS ARE BOUNDED BY   !16
! .       XYZ_MIN < XYZ < XYZ_MAX, WHICH MATCHES THE NATURAL PATCH  !17
! |\      AT -1 < ABC < 1,  OR THE FULLY INTERIOR REGION OF THE     !18
! |  \    UNIT PATCH AT 0 < RST < (1/N_SPACE).  THIS MEANS THAT     !19
! |    \   MAKING THE PATCH AXES PARALLEL TO XYZ AXES WILL CAUSE A  !20
! |      \  CONSTANT DIAGONAL JACOBIAN BETWEEN XYZ AND THE PATCH.   !21
! |        \                                                        !22
! |----------\ XYZ_MAX, (1,1), (1/M, 1/M); M = N_SPACE             !23
! |   .B     | \                                                    !24
! |   .      |   \  SO, COMPUTING THE LOCAL COORDINATE OF A SAMPLE  !25
! |   ...A   |     \  POINT IS A SIMPLE LINEAR TRANSFORMATION.      !26
! |       *P |       \    NOTE: 0 <= SUM(RST) <= 1.                !27
! |----------|---------\ .....R                                     !28
!XYZ_MIN, (-1,-1), (0,0)   [ONLY WEDGE FAMILY CURRENTLY EXCLUDED.]  !29
                                                                    !30
 IF (SCP_SHAP == 1 .OR. SCP_SHAP == 3 .OR. SCP_SHAP == 4 ) THEN     !31
!   A NATURAL COORDINATE PATCH TRANSFORMATION,                      !32
!   FOR A: LINE, QUADRILATERAL, OR HEXAHEDRAL PATCH                 !33
   POINT = (2.D0 * XYZ - (XYZ_MAX + XYZ_MIN)) / (XYZ_MAX - XYZ_MIN) !34
                                                                    !35
 ELSE IF (SCP_SHAP == 2 .OR. SCP_SHAP == 5 ) THEN                   !36
!   A UNIT COORDINATE PATCH TRANSFORMATION,                         !37
!   FOR A: TRIANGULAR, OR TETRAHEDRAL PATCH                         !38
   POINT = (XYZ - XYZ_MIN) / (SCP_PARM * (XYZ_MAX - XYZ_MIN))       !39
                                                                    !40
 ELSE                                                               !41
   PRINT *, 'ERROR, NO CONVERSION FOR SHAPE ', SCP_SHAP             !42
   STOP 'SHAPE ERROR IN GET_SCP_PT_AT_XYZ'                          !43
 END IF ! PARAMETRIC PARENT PATCH SHAPE'                            !44
                                                                    !45
!   VALIDATE POINT LIES IN PARENT SPACE                             !46
 IF (SCP_SHAP == 2 .OR. SCP_SHAP == 5 ) THEN                        !47
!    A UNIT COORDINATE PATCH TRANSFORMATION                         !48
   IF ( ANY (POINT < 0.d0) ) STOP &                                 !49
     'PATCH MAP OUT OF BOUNDS, GET_SCP_PT_AT_XYZ'                   !50
   IF ( SUM (POINT) > 1.d0 ) STOP &                                 !51
     'PATCH MAP OUT OF BOUNDS, GET_SCP_PT_AT_XYZ'                   !52
 ELSE                                                               !53
!   A NATURAL COORDINATE PATCH TRANSFORMATION                       !54
   IF ( ANY (ABS(POINT) > 1.d0) ) STOP &                            !55
     'PATCH MAP OUT OF BOUNDS, GET_SCP_PT_AT_XYZ'                   !56
 END IF                                                             !57
END FUNCTION GET_SCP_PT_AT_XYZ                                      !58
```

Problem P6.1 *Coordinate scaling via a diagonal Jacobian*

```
SUBROUTINE EVAL_PT_FLUX_IN_SCP_PATCH (NP_SYS, X, XYZ_MIN, XYZ_MAX,& ! 1
                                      PATCH_FIT, SCP_AVERAGES,    & ! 2
                                      SCP_COUNTS, NODE_DONE)        ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 4
! CALCULATE THE SUPER_CONVERGENCE_PATCH FLUXES AT A NODE INSIDE     ! 5
!    THE PATCH. (AVERAGE VALUES FROM ALL PATCHES LATER)             ! 6
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 7
Use System_Constants ! for MAX_NP, NEIGH_L, N_ELEMS,                ! 8
                     ! SCP_FIT, U_SCPR, N_SPACE                     ! 9
Use Elem_Type_Data                                                  !10
Use SCP_Type_Data    ! for SCP_H (SCP_N)                            !11
Use Interface_Header ! for function GET_SCP_PT_AT_XYZ               !12
IMPLICIT NONE                                                       !13
INTEGER,  INTENT (IN)    :: NP_SYS  ! CURRENT GLOBAL NODE           !14
REAL(DP), INTENT (IN)    :: X       (MAX_NP, N_SPACE)               !15
REAL(DP), INTENT (IN)    :: XYZ_MIN (N_SPACE), XYZ_MAX (N_SPACE)    !16
REAL(DP), INTENT (IN)    :: PATCH_FIT    (SCP_N , N_R_B  )          !17
REAL(DP), INTENT (INOUT) :: SCP_AVERAGES (MAX_NP, SCP_FIT)          !18
INTEGER,  INTENT (INOUT) :: SCP_COUNTS   (MAX_NP)                   !19
LOGICAL,  INTENT (INOUT) :: NODE_DONE    (MAX_NP)                   !20
                                                                    !21
REAL(DP) :: XYZ     (N_SPACE), POINT   (N_SPACE)  ! SCP POINT       !22
REAL(DP) :: FLUX_PT (N_R_B)                       ! PT FLUX         !23
                                                                    !24
INTEGER            :: IS              ! LOOPS                       !25
INTEGER, PARAMETER :: ONE = 1                                       !26
                                                                    !27
! FLUX_PT    = FLUX AT THE NODE INTERPOLATED FROM A SCP              !28
! MAX_NP     = NUMBER OF SYSTEM NODES                                !29
! NODE_DONE  = TRUE, IF IT HAS ONE CONTRIBUTION FROM PATCH           !30
! NP_SYS     = CURRENT GLOBAL NODE                                   !31
! PATCH_FIT  = LOCAL PATCH VALUES FOR FLUX AT ITS NODES              !32
! POINT      = LOCAL POINT IN PATCH INTERPOLATION SPACE              !33
! SCP_AVERAGES = AVERAGED FLUXES AT ALL NODES IN MESH               !34
! SCP_FIT    = NUMBER OF TERMS BEING AVERAGED, = N_R_B MIN           !35
! SCP_H      = INTERPOLATION FUNCTIONS FOR PATCH, USUALLY IS H       !36
! SCP_N      = NUMBER OF NODES PER PATCH                             !37
! X          = COORDINATES OF SYSTEM NODES                           !38
! XYZ        = SPACE COORDINATES AT A POINT                          !39
! XYZ_MAX    = UPPER BOUNDS FOR SCP GEOMETRY                         !40
! XYZ_MIN    = LOWER BOUNDS FOR SCP GEOMETRY                         !41
                                                                    !42
!         CONVERT IQ XYZ TO LOCAL PATCH POINT                       !43
  XYZ   = X (NP_SYS, :)                                             !44
  POINT = GET_SCP_PT_AT_XYZ (XYZ, XYZ_MIN, XYZ_MAX)                 !45
                                                                    !46
!     EVALUATE PATCH INTERPOLATION AT LOCAL POINT                   !47
  CALL GEN_ELEM_SHAPE (POINT, SCP_H, SCP_N, N_SPACE, ONE)           !48
                                                                    !49
!     INTERPOLATE PATCH NODE FIT TO PHYSICAL NODE                   !50
  FLUX_PT (1:N_R_B) = MATMUL (SCP_H (:), PATCH_FIT (:, 1:N_R_B))   !51
                                                                    !52
!      SCATTER NODE VALUES TO SYSTEM & INCREMENT COUNTS             !53
  SCP_COUNTS   (NP_SYS)           = SCP_COUNTS   (NP_SYS) + 1       !54
  SCP_AVERAGES (NP_SYS, 1:N_R_B) = SCP_AVERAGES (NP_SYS, 1:N_R_B) & !55
                                 + FLUX_PT      (1:N_R_B)          !56
  NODE_DONE    (NP_SYS)           = .TRUE.                          !57
END SUBROUTINE EVAL_PT_FLUX_IN_SCP_PATCH                            !58
```

Problem P6.2 *Computing and scattering flux components at a node*

```
SUBROUTINE EVAL_SCP_FIT_AT_PATCH_NODES (IP, NODES, X, L_IN_PATCH, &! 1
                          MEMBERS, XYZ_MIN, XYZ_MAX, PATCH_FIT,  &! 2
                          SCP_AVERAGES, SCP_COUNTS)               ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *    ! 4
!     CALCULATE THE SUPER_CONVERGENCE_PATCH AVERAGE FLUXES         ! 5
!         AT ALL ELEMENT NODES INSIDE THE PATCH                    ! 6
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *    ! 7
Use System_Constants ! for MAX_NP, N_ELEMS, L_S_TOT, N_SPACE,      ! 8
                     ! SCP_FIT, SCP_N , N_R_B                      ! 9
Use Elem_Type_Data   ! for ELEM_NODES, LAST_LT, LT_*                !10
Use Interface_Header ! for GET_ELEM_NODES function                  !11
IMPLICIT NONE                                                       !12
INTEGER,  INTENT (IN)    :: IP           ! CURRENT PATCH NUMBER     !13
INTEGER,  INTENT (IN)    :: NODES        (L_S_TOT, NOD_PER_EL)      !14
REAL(DP), INTENT (IN)    :: X            (MAX_NP, N_SPACE)          !15
INTEGER,  INTENT (IN)    :: L_IN_PATCH                              !16
INTEGER,  INTENT (IN)    :: MEMBERS      (L_IN_PATCH)               !17
REAL(DP), INTENT (IN)    :: XYZ_MIN (N_SPACE), XYZ_MAX (N_SPACE)    !18
REAL(DP), INTENT (IN)    :: PATCH_FIT    (SCP_N , N_R_B  )          !19
REAL(DP), INTENT (INOUT) :: SCP_AVERAGES (MAX_NP, SCP_FIT)          !20
INTEGER,  INTENT (INOUT) :: SCP_COUNTS   (MAX_NP)                   !21
                                                                    !22
INTEGER  :: IN, LM, LP, LT, NP_SYS              ! LOOPS            !23
LOGICAL  :: NODE_DONE (MAX_NP)                                      !24
                                                                    !25
! ELEM_NODES = THE NOD_PER_EL INCIDENCES OF THE ELEMENT             !26
! IP         = CURRENT PATCH NUMBER                                 !27
! L_IN_PATCH = NUMBER OF ELEMENTS IN CURRENT SCP                    !28
! L_S_TOT    = TOTAL NUMBER OF ELEMENTS & THEIR SEGMENTS            !29
! L_TYPE     = ELEMENT TYPE NUMBER FOR ALL ELEMENTS                 !30
! LAST_LT    = LAST ELEMENT TYPE CREATED                            !31
! LT         = CURRENT ELEMENT TYPE NUMBER                          !32
! LT_N       = MAXIMUM NUMBER OF NODES FOR ELEMENT TYPE             !33
! MAX_NP     = NUMBER OF SYSTEM NODES                               !34
! MEMBERS    = ELEMENT NUMBERS MACKING UP A SCP                     !35
! NOD_PER_EL = MAXIMUM NUMBER OF NODES PER ELEMENT                  !36
! NODES      = NODAL INCIDENCES OF ALL ELEMENTS                     !37
! NODE_DONE  = TRUE, IF IT HAS ONE CONTRIBUTION FROM PATCH          !38
! N_SPACE    = DIMENSION OF SPACE                                   !39
! PATCH_FIT  = LOCAL PATCH VALUES FOR FLUX AT ITS NODES             !40
! SCP_AVERAGES = AVERAGED FLUXES AT ALL NODES IN MESH               !41
! SCP_FIT    = NUMBER IF TERMS BEING AVERAGED, = N_R_B MIN          !42
! X          = COORDINATES OF SYSTEM NODES                          !43
! XYZ_MAX    = UPPER BOUNDS FOR SCP GEOMETRY                        !44
! XYZ_MIN    = LOWER BOUNDS FOR SCP GEOMETRY                        !45
                                                                    !46
```

Problem P6.3a *Establishing the interface for continuous flux scatters*

```
 NODE_DONE = .FALSE.                    ! INITIALIZE                 !47
                                                                     !48
!    INTERPOLATE AVERAGES TO ALL NODES IN THE PATCH                  !49
 DO LP = 1, L_IN_PATCH                  ! PATCH MEMBER LOOP          !50
   LM = MEMBERS (LP)                    ! ELEMENT IN PATCH           !51
   IF ( N_L_TYPE > 1) LT = L_TYPE (LM) ! GET ELEMENT TYPE NUMBER    !52
   IF ( LT /= LAST_LT ) THEN            ! NEW ELEMENT TYPE           !53
     CALL GET_ELEM_TYPE_DATA (LT)       ! CONTROLS FOR ELEM TYPE     !54
     IF ( TYPE_APLY_ALLOC ) CALL DEALLOCATE_TYPE_APPLICATION        !55
     CALL ALLOCATE_TYPE_APPLICATION     ! for ELEM_NODES (LT_N)      !56
     LAST_LT = LT                                                    !57
   END IF ! a new element type                                       !58
                                                                     !59
!-->        EXTRACT ELEMENT NODE NUMBERS                           !60
   ELEM_NODES = GET_ELEM_NODES (LM, LT_N, NODES)                     !61
                                                                     !62
!       LOOP OVER ELEMENT'S NODE, INTERPOLATE FLUX                  !63
   DO IN = 1, LT_N                                                   !64
     NP_SYS = ELEM_NODES (IN)    ! SYSTEM NODE NUMBER                !65
     IF ( NP_SYS < 1 ) CYCLE ! TO ACTIVE NODE                        !66
                                                                     !67
!        EVALUATE ITEMS AT THIS NODE                                !68
     CALL  EVAL_PT_FLUX_IN_SCP_PATCH (NP_SYS, X, XYZ_MIN,   &       !69
                          XYZ_MAX, PATCH_FIT, SCP_AVERAGES, &       !70
                          SCP_COUNTS, NODE_DONE)                    !71
   END DO ! LOCAL NODES                                              !72
 END DO ! OVER PATCH MEMBERS                                         !73
END SUBROUTINE EVAL_SCP_FIT_AT_PATCH_NODES                           !74
```

Problem P6.3b *Patch loops for sums and counts to average*

stage. The number of elements created in this way can grow unreasonably fast if the initial mesh is very crude (and very large local errors are estimated). Thus the routine in P6.4 uses some hueristics to try to actually slow the growth of adaptive mesh refinement.

```
SUBROUTINE FORM_NEW_EL_SIZES (NODES, X, MEASURE, ELEM_REFINEMENT)  ! 1
! ------------------------------------------------------------------  ! 2
!   AVERAGE NEW ELEMENT SIZES FOR ADAPTIVE MESH GENERATOR            ! 3
!    H_NEW = H_OLD / (ELEM_REFINEMENT ** (1. / EL_DEG))              ! 4
! ------------------------------------------------------------------  ! 5
Use System_Constants     ! for MAX_NP, NOD_PER_EL, N_ELEMS,         ! 6
                         ! N_SPACE, L_SHAPE, AVE_H_WT, U_NEW_H      ! 7
Use Elem_Type_Data       ! for LT_N, LT_SHA, ELEM_NODES (LT_N)      ! 8
Use Geometric_Properties ! for VOLUME                               ! 9
IMPLICIT NONE                                                      !10
INTEGER,  INTENT (IN) :: NODES            (L_S_TOT, NOD_PER_EL)    !11
REAL(DP), INTENT (IN) :: X                (MAX_NP,  N_SPACE)       !12
REAL(DP), INTENT (IN) :: MEASURE          (N_ELEMS)                !13
REAL(DP), INTENT (IN) :: ELEM_REFINEMENT  (N_ELEMS)                !14
INTEGER  :: NODE_AVE_COUNT   (MAX_NP)                              !15
REAL(DP) :: NODE_AVE_EL_SIZE (MAX_NP), H_SCALE (N_ELEMS)           !16
INTEGER  :: IE, LT, J, K, EL_DEG, GET_SCALAR_DEGREE                !17
REAL(DP) :: H_OLD, H_OLD_AVE, H_OLD_MAX, H_OLD_MID, H_OLD_MIN      !18
REAL(DP) :: H_NEW, H_NEW_AVE, H_NEW_MAX, H_NEW_MID, H_NEW_MIN      !19
REAL(DP) :: H_LIMIT, REFINE, WEIGHT, AVE_VOL, EL_VOL               !20
REAL(DP) :: R_MAX, R_MIN, R_MID, R_AVE, LOW_R = 0.7d0              !21
REAL(DP), PARAMETER :: TRI_WEDGE = 2.309d0, TET = 7.994d0, &       !22
                       AVE_TO_LIMIT = 1.d-2                        !23
                                                                   !24
! AVE_H_WT = (KEYWORD) WEIGHT NEW SIZES. 1.0-REDUCE AVERAGE AND    !25
!            LARGER SIZES, 0.0-REDUCE ALL ELEMENT SIZES            !26
! H_LIMIT  = SMALLEST NEW SIZE, = H_OLD_AVE * AVE_TO_LIMIT         !27
! H_SCALE  = ELEMENT SCALE, NEW SIZE = OLD SIZE * H_SCALE          !28
! LT_SHAP  = ELEMENT TYPE SHAPE FLAG NUMBER                        !29
! L_SHAPE  = ELEM SHAPE, 1=LINE 2=TRI 3=QUAD 4=HEX 5=TET 6=WEDGE   !30
! NODE_AVE_EL_SIZE = NEW ELEMENT SIZES AVERAGED AT ALL NODES       !31
! REFINE   = FINAL SCALED, WEIGHTED REFINEMENT VALUE               !32
! U_NEW_H  = UNIT TO RECEIVE ELEMENT SIZES AFTER ERROR ESTIMATE    !33
                                                                   !34
! INITIALIZE REFINEMENT VALUES. REFINE IF > 1                      !35
 NODE_AVE_EL_SIZE = 0.d0 ; NODE_AVE_COUNT = 0 ; H_SCALE = 1.d0     !36
 R_MAX = MAXVAL(ELEM_REFINEMENT) ; R_MIN = MINVAL(ELEM_REFINEMENT) !37
 R_MID = (R_MIN + R_MAX) * 0.5d0                                   !38
 R_AVE = SUM (ELEM_REFINEMENT) / N_ELEMS                           !39
 IF ( R_AVE > 1.d0 ) THEN ! SCALE DOWN TO SLOW MESH GROWTH         !40
   R_MAX = R_MAX / R_AVE                                           !41
   R_MID = R_MID / R_AVE                                           !42
   R_MIN = R_MIN / R_AVE                                           !43
 END IF ! Scaling                                                  !44
                                                                   !45
 AVE_VOL = VOLUME / N_ELEMS           ! Average volume per element !46
 IF ( L_SHAPE == 2 .OR. L_SHAPE == 6 ) AVE_VOL = AVE_VOL*TRI_WEDGE !47
 IF ( L_SHAPE == 5 )                   AVE_VOL = AVE_VOL*TET       !48
 SELECT CASE (N_SPACE)         ! GET AVERAGE MESH SIZE             !49
   CASE (1) ; H_OLD_AVE = AVE_VOL                                  !50
   CASE (2) ; H_OLD_AVE = SQRT (AVE_VOL)                           !51
   CASE DEFAULT ; H_OLD_AVE = AVE_VOL **(1.d0 / N_SPACE)           !52
 END SELECT ! for ave size                                         !53
 H_OLD_MIN = HUGE (H_OLD_MIN) ; H_OLD_MAX = -HUGE (H_OLD_MAX)      !54
 H_NEW_MIN = HUGE (H_NEW_MIN) ; H_NEW_MAX = -HUGE (H_NEW_MAX)      !55
 H_LIMIT   = H_OLD_AVE * AVE_TO_LIMIT                              !56
 H_OLD_AVE = 0.d0 ; H_NEW_AVE = 0.d0                               !57
                                                                   !58
```

Problem P6.4a *Find average refinement values and current element sizes*

```
!     LOOP OVER ALL STANDARD ELEMENTS                                  ! 59
 LT = 1 ; LAST_LT = 0                                                  ! 60
 DO IE = 1, N_ELEMS  ! LOOP OVER ALL ELEMENTS  ------------------      ! 61
   CALL SET_THIS_ELEMENT_NUMBER (IE)                                   ! 62
                                                                       ! 63
!-->     GET ELEMENT TYPE NUMBER                                       ! 64
   IF ( N_L_TYPE > 1) LT = L_TYPE (IE)  ! SAME AS LAST TYPE ?          ! 65
   IF ( LT /= LAST_LT ) THEN            ! this is a new type           ! 66
     CALL GET_ELEM_TYPE_DATA (LT)       ! CONTROLS FOR THIS TYPE       ! 67
     LAST_LT = LT                                                      ! 68
     IF ( TYPE_APLY_ALLOC ) CALL DEALLOCATE_TYPE_APPLICATION           ! 69
     CALL ALLOCATE_TYPE_APPLICATION ! for ELEM_NODES (LT_N)            ! 70
     EL_DEG = GET_SCALAR_DEGREE ()                                     ! 71
   END IF ! a new element type                                         ! 72
                                                                       ! 73
!     GET AVERAGE ELEMENT SIZE (or recompute its volume)              ! 74
   EL_VOL = MEASURE (IE)                                               ! 75
   IF ( LT_SHAP == 2 .OR. LT_SHAP == 6 ) EL_VOL=EL_VOL*TRI_WEDGE       ! 76
   IF ( LT_SHAP == 5 )                    EL_VOL=EL_VOL*TET            ! 77
   SELECT CASE (N_SPACE)                                               ! 78
     CASE (1) ; H_OLD = EL_VOL                                         ! 79
     CASE (2) ; H_OLD = SQRT (EL_VOL)                                  ! 80
     CASE DEFAULT ; H_OLD = EL_VOL **(1.d0 / N_SPACE)                  ! 81
   END SELECT ! for old size                                           ! 82
   H_OLD_AVE = H_OLD_AVE + H_OLD                                       ! 83
   IF ( H_OLD > H_OLD_MAX ) H_OLD_MAX = H_OLD                          ! 84
   IF ( H_OLD < H_OLD_MIN ) H_OLD_MIN = H_OLD                          ! 85
                                                                       ! 86
!    GET NEW ELEMENT SIZE, WEIGHT TO CONTROL GROWTH                   ! 87
   WEIGHT = R_MIN * (1.d0 - AVE_H_WT) + R_AVE * AVE_H_WT               ! 88
   REFINE = ELEM_REFINEMENT (IE) / WEIGHT                              ! 89
   IF ( REFINE > LOW_R ) THEN                                          ! 90
     SELECT CASE (EL_DEG)                                              ! 91
       CASE (1) ; H_SCALE (IE) = 1.d0 / REFINE                         ! 92
       CASE (2) ; H_SCALE (IE) = 1.d0 / SQRT (REFINE)                  ! 93
       CASE DEFAULT ; H_SCALE (IE) = 1.d0/REFINE **(1.d0/EL_DEG)       ! 94
     END SELECT                                                        ! 95
   END IF ! reduce size (no increases)                                 ! 96
   H_NEW = H_OLD * H_SCALE (IE)                                        ! 97
   IF ( H_NEW < H_LIMIT ) H_NEW = H_LIMIT                              ! 98
   IF ( H_NEW > H_NEW_MAX ) H_NEW_MAX = H_NEW                          ! 99
   IF ( H_NEW < H_NEW_MIN ) H_NEW_MIN = H_NEW                          !100
   H_NEW_AVE = H_NEW_AVE + H_NEW                                       !101
                                                                       !102
!     SEND SIZES FOR AVERAGING AT NODES                               !103
   ELEM_NODES = GET_ELEM_NODES (IE, LT_N, NODES)                       !104
   DO J = 1, LT_N                                                      !105
     K = ELEM_NODES (J)                                                !106
     IF ( K < 1 ) CYCLE ! to a valid node                              !107
       NODE_AVE_EL_SIZE (K) = NODE_AVE_EL_SIZE (K) + H_NEW             !108
       NODE_AVE_COUNT   (K) = NODE_AVE_COUNT   (K) + 1                 !109
   END DO ! over element nodes                                         !110
 END DO ! over elements  ---------------------------------------       !111
```

Problem P6.4b *Get each new element size and scatter to its nodes*

```
 H_OLD_AVE = H_OLD_AVE / N_ELEMS                                          !112
 H_NEW_AVE = H_NEW_AVE / N_ELEMS                                          !113
 H_NEW_MID = (H_NEW_MIN + H_NEW_MAX) * 0.5d0                              !114
 H_OLD_MID = (H_OLD_MIN + H_OLD_MAX) * 0.5d0                              !115
                                                                          !116
!  AVERAGE SIZES AT NODES FOR MESH GENERATOR                             !117
 WHERE ( NODE_AVE_COUNT > 0 )                                             !118
   NODE_AVE_EL_SIZE = NODE_AVE_EL_SIZE / NODE_AVE_COUNT                   !119
 ELSEWHERE                                                                !120
   NODE_AVE_EL_SIZE = H_NEW_AVE                                           !121
 ENDWHERE                                                                 !122
                                                                          !123
!    SAVE RESULTS FOR PLOTS AND/OR MESH GENERATOR                        !124
 WRITE (U_NEW_H, '(5I12)') MAX_NP, (K, K = 1, N_SPACE) ! for plot         !125
 DO J = 1, MAX_NP  ! save averaged sizes at nodes                         !126
   WRITE (U_NEW_H, '(3(1PE12.4))') X (J, :), NODE_AVE_EL_SIZE (J)         !127
 END DO ! over nodes                                                      !128
END SUBROUTINE FORM_NEW_EL_SIZES                                          !129
```

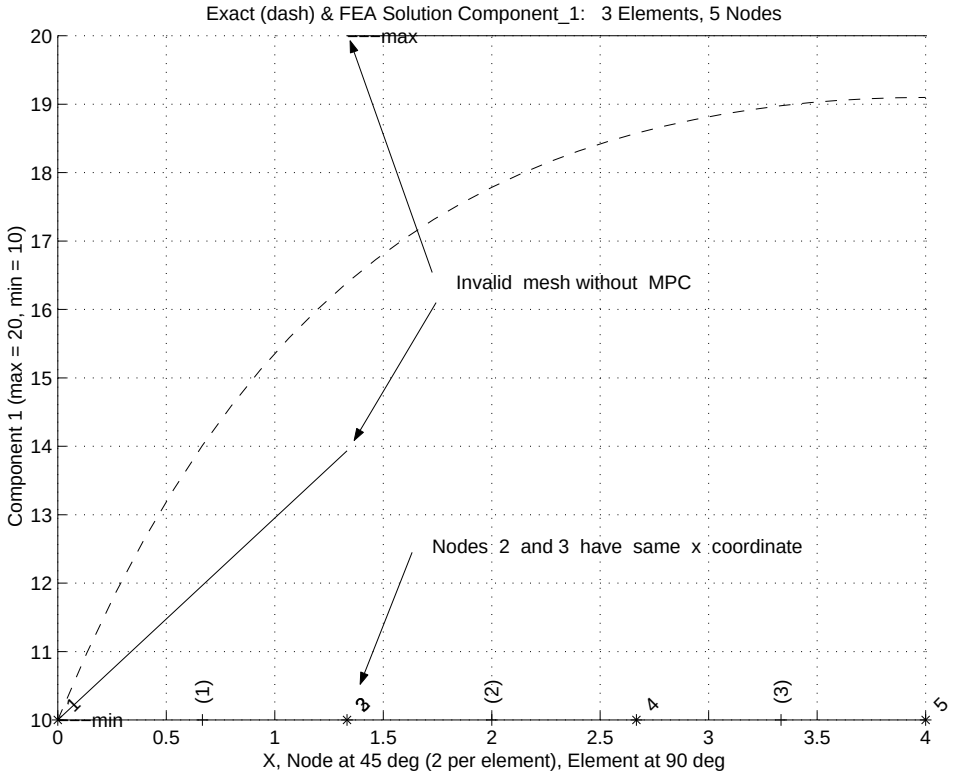
Problem P6.4c *Average all element sizes at mesh nodes and save*

1.7 Problems from chapter 7

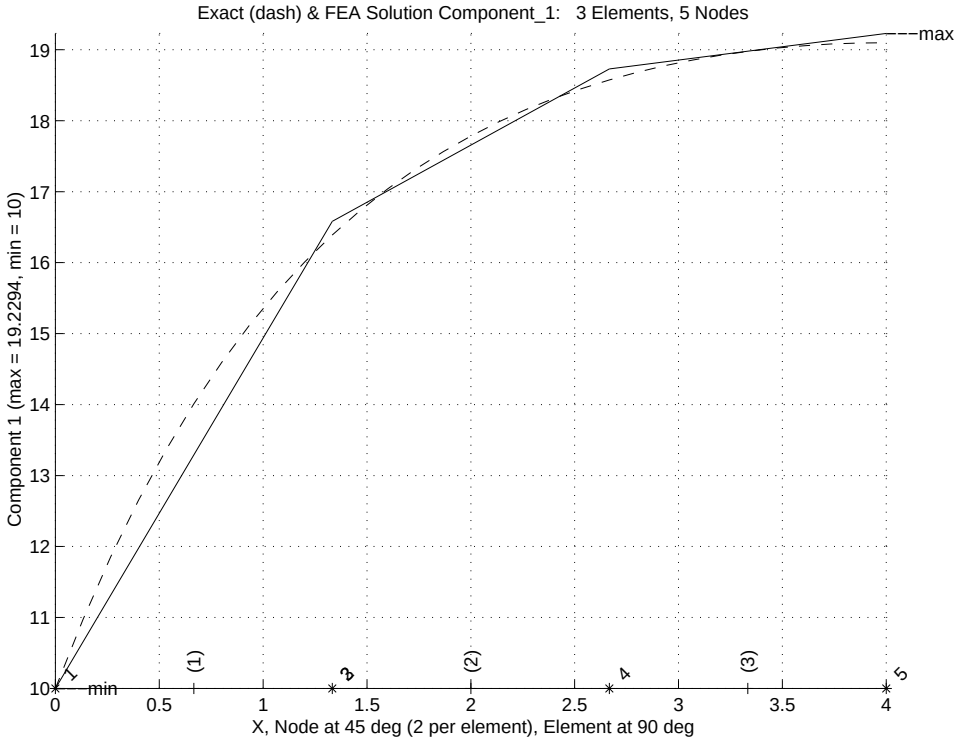
Problem 7.10 Resolve the heat transfer problem in Fig. 7.7.1 with the external reference temperature increased from 0 to 20 F ($U\_ext = 20$). Employ 3 equal length L2 linear elements, and 5 (not 4) nodes with nodes 2 and 3 both at $x = 4/3$. Connect element 1 to nodes 1 and 2 and (incorrectly) connect element 2 to nodes 3 and 4. a) Solve the incorrect model, sketch the FEA and true solutions, and discuss why the left and right ($x \le 4/3$ and $x \ge 4/3$) domain FEA solutions behave as they do. b) Enforce the correct solution by imposing a so called "multiple point constraint" (MPC) that requires $1 \times t_2 - 1 \times t_3 = 0$. Solution: (a) This only requires changing the data in Fig.7.7.3, as described above, and applying the matrix definitions in Fig. 7.7.2, or the simplified L2 version thereof. The invalid (discontinuous) FEA solution is given in P7.10a.

Solution: b) One must either correct the mesh to have the proper connectivity, or apply a MPC to enforce continuity. Here the latter is employed to reinforce the concept of MPC's which are often needed in real applications to correct data errors (as here) or for other analysis reasons. The corrected solution is in P7.10b and is reasonable for a 3 element solution. The data for this solution are given in P7.10c. In the *MODEL* code a MPC between 2 degrees of freedom is referred to as a "Type 2" constraint. In lines 45 and 46 of P7.10c the normally zero boundary constraint flag has been set to 2 for each of the constrained degrees of freedom. For each such pair of flags there must be one "Type 2" constraint equation. In general its form would be $a \times t_j + b \times t_k = c$. Here we want 2 *dof* to be identical so we set $a = 1, b = -1, c = 0$. In addition we must provide the equation numbers, j and k. Here j is for node 2, parameter 1, while k is for node 3, and also parameter 1. Those data are give after the essential boundary conditions (the "Type 1" conditions) and appear at line 53 of the data set. The selected output in P7.10d echos these constraint data again and gives the FEA and exact nodal values.

Problem 7.11 Implement the exact integral matrices for the linear line element version of Eq. 7.34 and the matrix for recovering the convection heat loss, Eq. 7.38, in the post



Problem P7.10a *Invalid (discontinuous) L2 convection solution results*



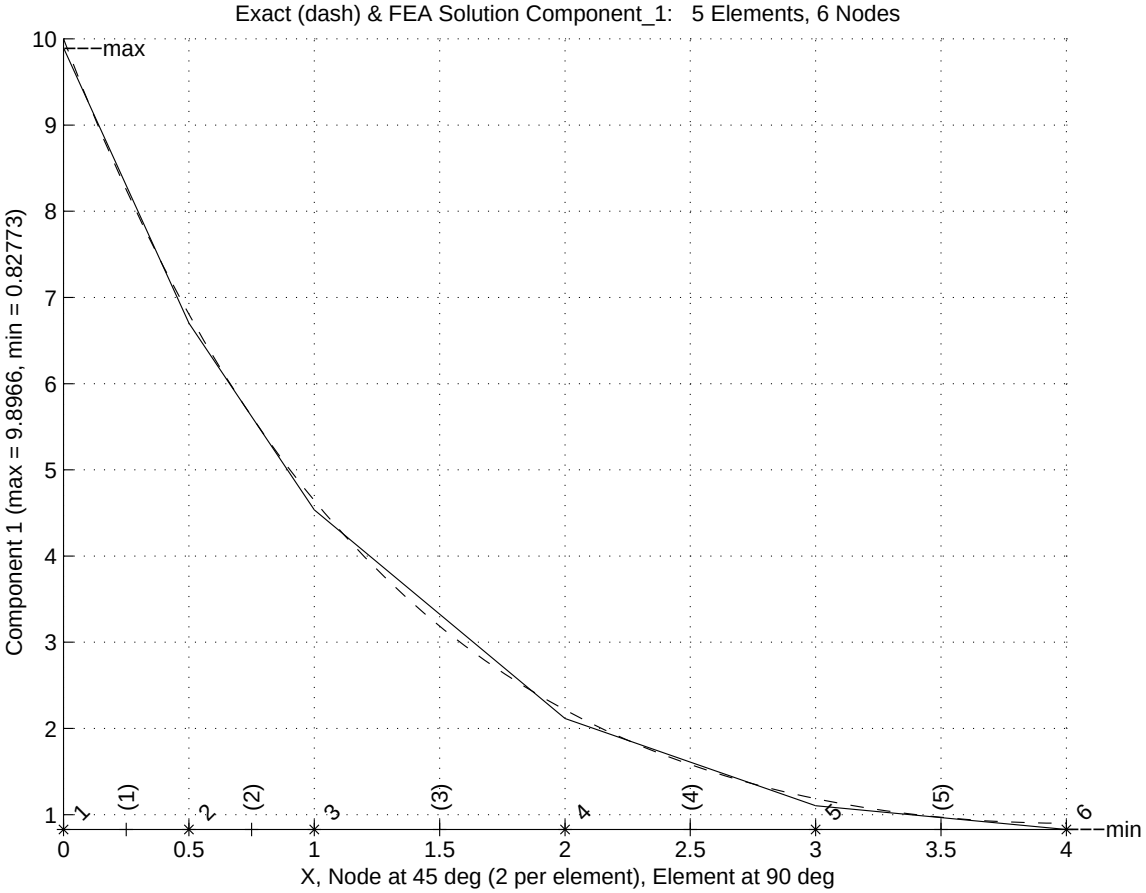
Problem P7.10b *Corrected (via MPC) L2 convection solution results*

```
title "1-D heat transfer example, MPC mesh fix, 3 L2"  ! 1
nodes    5 ! Number of nodes in the mesh                ! 2
elems    3 ! Number of elements in the system           ! 3
dof      1 ! Number of unknowns per node                ! 4
el_nodes 2 ! Maximum number of nodes per element        ! 5
space    1 ! Solution space dimension                   ! 6
b_rows   1 ! Number of rows in B (operator) matrix      ! 7
shape    1 ! Shape, 1=line, 2=tri, 3=quad, 4=hex        ! 8
gauss    2 ! Maximum number of quadrature point         ! 9
el_real  4 ! Number of real properties per element     !10
reals    3 ! Number of miscellaneous real properties   !11
el_homo    ! Element properties are homogeneous        !12
el_list    ! List results at each node of each element !13
post_1     ! Require post-processing, create n_tape1   !14
post_2     ! Require post-processing, create n_tape2   !15
example  101 ! Source library example number           !16
data_set   3 ! Data set for example (this file)        !17
exact_case 1 ! Analytic solution for list_exact, etc   !18
scp_neigh_el    ! Default SCP patch group type         !19
list_exact      ! List given exact answers at nodes    !20
list_exact_flux ! List given exact fluxes at nodes     !21
remarks     20  ! Number of user remarks               !22
quit ! keyword input, remarks follow                   !23
Combined heat conduction and convection from a bar:    !24
K*A*U,XX - h*P*(U-U_ext) = 0,  U(0)=U_0, dU/dx(L)=0     !25
For globally constant data the analytic solution is:   !26
U(x) = U_ext + (U_0-U_ext)*Cosh [m*(L-x)]/Cosh [mL]     !27
m^2 = h_e * P_e/(K_e * A_e), dimensionless             !28
Miscellaneous FE data:                                 !29
U_ext = GET_REAL_MISC (1)  external reference temp, F  !30
Miscellaneous data used ONLY for analytic solution:    !31
L     = GET_REAL_MISC (2)  exact length, ft            !32
U_0   = GET_REAL_MISC (3)  essential bc at x = 0, F    !33
Real FE problem properties are:                        !34
K_e = GET_REAL_LP (1) conductivity, BTU/ hr ft F       !35
A_e = GET_REAL_LP (2) area of bar, ft^2                !36
h_e = GET_REAL_LP (3) convection, BTU/ hr ft^2 F       !37
P_e = GET_REAL_LP (4) perimeter of area A_e, ft        !38
Elements 1 & 2 are NOT connected.  A correct solution  !39
can not be obtained without a mesh correction, or      !40
a "Multi-Point Constraint" that enforces the same      !41
answer at nodes 2 & 3. It is: 1*U_2 - 1*U_3 = 0.       !42
Flagged by setting the bc code to 2 at those nodes.    !43
 1   1 0.00        ! begin nodes                       !44
 2   2 1.33333333  ! correcting duplicate node         !45
 3   2 1.33333333  ! correcting duplicate node         !46
 4   0 2.66666667                                     !47
 5   0 4.00                                           !48
  1    1    2   ! begin elements                      !49
  2    3    4   ! not connected to element 1          !50
  3    4    5                                        !51
1 1 10.           ! essential BC                       !52
2 1 3 1 1. -1. 0. ! MPC data: 1*U_2 - 1*U_3 = 0.       !53
1 120. 0.01389 2. 0.5 ! elem: K_e, A_e, h_e, P_e       !54
20.0  4.0 10.0 ! misc: U_ext (L, U_0 for exact)        !55
```

Problem P7.10c *Invalid mesh corrected by MPC*

```
title "1-D heat transfer example, MPC mesh fix, Three L2"       ! 1
                                                                ! 2
*** CONSTRAINT EQUATION DATA ***                                ! 3
CONSTRAINT TYPE_1: (PAR_1 @ NODE_1) =  A_1.                     ! 4
EQ. NO.   NODE_1   PAR_1          A_1                           ! 5
     1      1      1          1.00000E+01                      ! 6
TYPE_2: A_1*(PAR_1 @ NODE_1) + A_2*(PAR_2 @ NODE_2) = A_3       ! 7
EQ. NODE_1  PAR_1  NODE_2  PAR_2    A_1       A_2       A_3     ! 8
 2    2     1      3       1    1.00E+0  -1.00E+0   0.00E+0     ! 9
                                                               !10
**  OUTPUT OF RESULTS AND EXACT VALUES IN NODAL ORDER  **       !11
    NODE,  X-Coord,    DOF_1,      EXACT1,                      !12
       1  0.0000E+00  1.0000E+01  1.0000E+01                    !13
       2  1.3333E+00  1.6582E+01  1.6390E+01                    !14
       3  1.3333E+00  1.6582E+01  1.6390E+01                    !15
       4  2.6667E+00  1.8730E+01  1.8575E+01                    !16
       5  4.0000E+00  1.9229E+01  1.9099E+01                    !17
```

Problem P7.10d *Selected MPC data and temperature results*



Problem P7.11a *Exact and L2 temperature for flux boundary condition*

processing. Apply a small model to the problem in Fig. 7.7.1, but replace the Dirichlet boundary condition at $x = 0$ with the corresponding exact flux $q(0) = 12.86$. Compare the temperature solution with that in Fig. 7.7.1. Why do we still get a solution when we no longer have a Dirichlet boundary condition? Solution: The definitions of the matrices are given in the text for the L2 element. Here we have simply multiplied the constant element coefficients times arrays defined with the F90 RESHAPE intrinsic to convert the vector data, stored by columns, into a matrix of the desired shape. One additional change is that we follow the usual practice here and assume that the external reference temperature can be different for each element. It is element real property number 5 here.

Selected results are compared in P7.11a. The element matrices are in P7.11b for both the element generation and post-processing phases. The latter involves the integral of the solution.

A set of sample data for this element and a flux boundary condition is given in P7.11c. The assigned flux (computed from the exact flux at $x = 0$) is noted in lines 10 and 56. The remarks, lines 38 to 40, address why we can get a unique solution without a Dirichlet boundary condition. Selected output from these data are in P7.11d. It is very unusual for a problem not to have at least one Dirichlet boundary condition so a warning is observed on line 3. The input flux data is echoed in line 7. As expected its value is equal and opposite to the convection heat loss listed in lines 50 to 57. The computed temperatures are reasonably accurate. Note that the value at $x = 0$ is in error by one percent compared to the Dirichlet boundary condition value used to compute the exact solution. That is mainly due to weak form implementation of the insulated condition at the right end of the bar ($x = 4$).

In lines 23 to 48 of P7.11d we see the element level reactions, which represent heat flows. These data were requested in line 9 of P7.11c. Since those reactions are not equal and opposite to each other there must have been an internal heat source per unit length and/or a surface convection heat loss. In this case their differences, cited in the rightmost SUMS column, should equal the heat flow lost from each element by convection. That is confirmed again bu looking at lines 50 to 57. Normally we do not look at such element reactions in detail but they can be important in applications like heat transfer and structural mechanics.

1.8 Problems from chapter 8

Problem 8.1 A hollow coaxial cable is made from a hollow conducting core and an insulating outer layer with $\varepsilon_1 = 0.5$, $\varepsilon_2 = 2.0$ and charge densities of $\zeta_1 = 100$ and $\zeta_2 = 0$, respectively. The inner, interface, and outer radii are 5, 10, and 25 mm. The corresponding inner and outer potentials (boundary conditions) are $\phi = 500$ and $\phi = 0$, respectively. Use the element formulation in Fig. 8.2.1 to compute the interface potential. Solution: The element definition in Fig. 8.3.2 will work for all C^0 line elements in the library. We just need a new data file since the differential equation is the same. Data and results for 2 cubic line elements are in P8.1

```
!   ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS FOLLOW ***   ! 1
!.............................................................   ! 2
! Combined heat conduction through, convection from a bar:       ! 3
!   K*A*U,xx - h*P*(U-U_ext) = 0, hard coded L2 element          ! 4
!  Real element properties are:                                  ! 5
!  1) K_e   = conductivity, BTU/ hr ft F                         ! 6
!  2) A_e   = area of bar, ft^2                                  ! 7
!  3) h_e   = convection, BTU/ hr ft^2 F                         ! 8
!  4) P_e   = perimeter of area A_e, ft                          ! 9
!  5) Q_e   = source per unit length, BTU/ hr ft                 !10
!  6) U_ext = external reference temperature, F                  !11
 REAL(DP) :: DL                                ! Length         !12
 REAL(DP) :: K_e, A_e, h_e, P_e, Q_e, U_ext  ! properties       !13
                                                                !14
 DL    = COORD (LT_N, 1) - COORD (1, 1) ! Length                !15
 K_e   = GET_REAL_LP (1) ! thermal conductivity                 !16
 A_e   = GET_REAL_LP (2) ! area of bar                          !17
 h_e   = GET_REAL_LP (3) ! convection coefficient on perimeter  !18
 P_e   = GET_REAL_LP (4) ! perimeter of area A_e                !19
 Q_e   = GET_REAL_LP (5) ! source per unit length, BTU/ hr ft   !20
 U_ext = GET_REAL_LP (6) ! external temperature                 !21
                                                                !22
!         INTERNAL & CONVECTION SOURCES                        !23
 C = (Q_e + h_e * P_e * U_ext) * DL * (/ 1, 1 /) * 0.5d0       !24
                                                                !25
!         SQUARE MATRIX, CONDUCTION & CONVECTION               !26
 S = K_e * A_e / DL * RESHAPE ((/ 1, -1, -1, 1 /), (/2,2/))    &  !27
   + h_e * P_e * DL * RESHAPE ((/ 2,  1,  1, 2 /), (/2,2/))/6.d0 !28
!***  END ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS ***       !29
                                                                !30
! *** POST_PROCESS_ELEM PROBLEM DEPENDENT STATEMENTS FOLLOW *** !31
!H_INTG (LT_N) Integral of interpolation functions, H, available !32
!.............................................................   !33
!   K*A*U,xx - h*P*(U-U_ext) = 0, hard coded L2 element          !34
 REAL(DP) :: DL, h_e, P_e, U_ext    ! Length, properties        !35
 REAL(DP), SAVE :: Q_LOSS, TOTAL    ! Face and total heat loss  !36
 LOGICAL,  SAVE :: FIRST = .TRUE.   ! printing                  !37
                                                                !38
  IF ( FIRST ) THEN                             ! first call    !39
    TOTAL = 0 ; FIRST = .FALSE. ; WRITE (6, 5) ! print headings !40
    5 FORMAT ('*** CONVECTION HEAT LOSS ***', /, &              !41
    &         'ELEMENT    HEAT_LOST') ; END IF ! first call     !42
                                                                !43
!   GET LENGTH & CONSTANT PROPERTIES                           !44
  DL    = COORD (LT_N, 1) - COORD (1, 1) ! Length of bar       !45
  h_e   = GET_REAL_LP (3)   ! convection coefficient on perimeter !46
  P_e   = GET_REAL_LP (4)   ! perimeter of area                !47
  U_ext = GET_REAL_LP (5)   ! external temperature             !48
                                                               !49
!  HEAT LOST ON THIS FACE: Integral over face of h * (T - T_inf) !50
  H_INTG (1:2) = DL / 2             ! Integral of H array      !51
  D (1:2) = D(1:2) - U_ext          ! Temp difference at nodes !52
  Q_LOSS  = h_e * P_e * DOT_PRODUCT (H_INTG, D)  ! Face loss   !53
  TOTAL   = TOTAL + Q_LOSS           ! Running total           !54
                                                               !55
  PRINT '(I6, ES15.5)', THIS_EL, Q_LOSS                        !56
  IF ( THIS_EL == N_ELEMS ) PRINT *, 'TOTAL = ', TOTAL         !57
! *** END POST_PROCESS_ELEM PROBLEM DEPENDENT STATEMENTS ***   !58
```

Problem P7.11b *Exact element and post-processing matrices for L2*

```
title "1-D heat transfer, fluxes only, with 5 L2"       ! 1
nodes      6 ! Number of nodes in the mesh              ! 2
elems      5 ! Number of elements in the system         ! 3
dof        1 ! Number of unknowns per node              ! 4
el_nodes   2 ! Maximum number of nodes per element      ! 5
space      1 ! Solution space dimension                 ! 6
b_rows     1 ! Number of rows in B (operator) matrix    ! 7
shape      1 ! Shape, 1=line, 2=tri, 3=quad, 4=hex      ! 8
el_react     ! Compute & list element reactions         ! 9
loads        ! An initial source vector is input        !10
gauss      0 ! Maximum number of quadrature point       !11
el_real    6 ! Number of real properties per element    !12
reals      3 ! Number of miscellaneous real properties  !13
el_homo      ! Element properties are homogeneous       !14
post_el      ! Require post-processing, create n_tape1  !15
exact_case 1 ! Exact analytic solution                  !16
list_exact      ! List given exact answers at nodes     !17
list_exact_flux ! List given exact fluxes at nodes      !18
remarks    22 ! Number of user remarks                  !19
quit ! keyword input, remarks follow                    !20
Combined heat conduction, convection from a bar:        !21
K*A*U,xx - h*P*(U-U_ext) + Q_e = 0,  dU/dx(L)=0,        !22
K*A*dU/dx(0)= q = 12.86 (reaction for U(0) = 10)        !23
For globally constant data the analytic solution is:    !24
U(x) = U_ext - (U_0-U_ext)*Cosh [m*(L-x)]/Cosh [mL]      !25
m^2 = h_e * P_e/(K_e * A_e), dimensionless              !26
Real FE problem properties are:                         !27
K_e   = GET_REAL_LP (1) conductivity, BTU/ hr ft F      !28
A_e   = GET_REAL_LP (2) area of bar, ft^2               !29
h_e   = GET_REAL_LP (3) convection, BTU/ hr ft^2 F      !30
P_e   = GET_REAL_LP (4) perimeter of area A_e, ft       !31
Q_e   = GET_REAL_LP (5) source per unit len, BTU/ hr ft !32
U_ext = GET_REAL_LP (6) external reference temp, F      !33
Miscellaneous data used ONLY for analytic solution:     !34
U_ext = GET_REAL_MISC (1) external ref temperature, F   !35
L     = GET_REAL_MISC (2) exact length, ft              !36
U_0   = GET_REAL_MISC (3) essential bc at x = 0, F      !37
Note: Flux loadings only. Convection allows a unique    !38
solution (prevents "rigid body motion"), otherwise      !39
solution would be known within an arbitrary constant.   !40
Note: Input flux, 12.86 BTU (exact), offsets the sum of !41
      element convection losses, 12.86 BTU (phys chk)   !42
         1         0 0.00       ! begin nodes           !43
         2         0 0.5                                !44
         3         0 1.00                               !45
         4         0 2.00                               !46
         5         0 3.00                               !47
         6         0 4.00                               !48
    1    1    2    ! begin elements                     !49
    2    2    3                                         !50
    3    3    4                                         !51
    4    4    5                                         !52
    5    5    6                                         !53
1 120. 0.01389 2. 0.5 0. 0. ! homo el, K A h P Q U_ext  !54
0.0  4.0 10.0   ! misc properties for exact_case 1      !55
 1  1 12.86     ! point flux as input source (loads)    !56
 6  1 0.0       ! terminate source with last dof        !57
```

Problem P7.11c *Data for input flux condition version*

```
TITLE 1-D heat transfer example, flux only, with 5 L2      ! 1
                                                           ! 2
WARNING: PARAMETER 1 HAS NO BOUNDARY CONDITION             ! 3
                                                           ! 4
 *** INITIAL FORCING VECTOR DATA ***                       ! 5
      NODE   PARAMETER     VALUE    EQUATION              ! 6
          1         1   1.28600E+01       1               ! 7
          6         1   0.00000E+00       6               ! 8
                                                           ! 9
 *** EXTREME VALUES OF THE NODAL PARAMETERS ***            !10
 PARAMETER      MAXIMUM,   NODE      MINIMUM,   NODE       !11
  DOF_1,     9.8966E+00,      1   8.2773E-01,      6       !12
                                                           !13
** OUTPUT OF RESULTS & EXACT VALUES IN NODAL ORDER **      !14
     NODE,  X-Coord,    DOF_1,      EXACT1,               !15
        1  0.0000E+00  9.8966E+00  1.0000E+01              !16
        2  5.0000E-01  6.7012E+00  6.8051E+00              !17
        3  1.0000E+00  4.5367E+00  4.6438E+00              !18
        4  2.0000E+00  2.1152E+00  2.2157E+00              !19
        5  3.0000E+00  1.1036E+00  1.1847E+00              !20
        6  4.0000E+00  8.2773E-01  9.0072E-01              !21
                                                           !22
** ELEMENT REACTION, INTERNAL SOURCES AND SUMS **          !23
                ELEMENT  1                                 !24
  NODE  DOF   REACTION      ELEM_SOURCE    SUMS           !25
    1    1    1.28600E+1    0.00000E+0                    !26
    2    1   -8.71053E+0    0.00000E+0                    !27
 SUM:    1    4.14947E+0    0.00000E+0    4.14947E+0      !28
                ELEMENT  2                                 !29
  NODE  DOF   REACTION      ELEM_SOURCE    SUMS           !30
    2    1    8.71053E+0    0.00000E+0                    !31
    3    1   -5.90103E+0    0.00000E+0                    !32
 SUM:    1    2.80950E+0    0.00000E+0    2.80950E+0      !33
                ELEMENT  3                                 !34
  NODE  DOF   REACTION      ELEM_SOURCE    SUMS           !35
    3    1    5.90103E+0    0.00000E+0                    !36
    4    1   -2.57507E+0    0.00000E+0                    !37
 SUM:    1    3.32596E+0    0.00000E+0    3.32596E+0      !38
                ELEMENT  4                                 !39
  NODE  DOF   REACTION      ELEM_SOURCE    SUMS           !40
    4    1    2.57507E+0    0.00000E+0                    !41
    5    1   -9.65674E-1    0.00000E+0                    !42
 SUM:    1    1.60940E+0    0.00000E+0    1.60940E+0      !43
                ELEMENT  5                                 !44
  NODE  DOF   REACTION      ELEM_SOURCE    SUMS           !45
    5    1    9.65674E-1    0.00000E+0                    !46
    6    1    0.00000E+0    0.00000E+0                    !47
 SUM:    1    9.65674E-1    0.00000E+0    9.65674E-1      !48
                                                           !49
 *** CONVECTION HEAT LOSS ***                              !50
 ELEMENT    HEAT_LOST                                      !51
      1    4.14947E+00                                     !52
      2    2.80950E+00                                     !53
      3    3.32596E+00                                     !54
      4    1.60940E+00                                     !55
      5    9.65674E-01                                     !56
  TOTAL =  12.859999994                                    !57
```

Problem P7.11d *Selected results from flux condition version*

```
title "Coaxial cable radial electrical potential"          ! 1
axisymmetric ! Problem is axisymmetric, x radius           ! 2
nodes      5 ! Number of nodes in the mesh                 ! 3
elems      2 ! Number of elements in the system            ! 4
dof        1 ! Number of unknowns per node                 ! 5
el_nodes   3 ! Maximum number of nodes per element         ! 6
b_rows     1 ! Number of rows in B (operator) matrix       ! 7
gauss      3 ! Maximum number of quadrature point          ! 8
el_real    2 ! Number of real properties per element       ! 9
bar_long     ! Use distance between bar chart nodes        !10
remarks    8 ! Number of user remarks                      !11
quit ! keyword input, remarks follow                       !12
! 1/R * d[R K_RR dT/dR]/dR + Q = 0, Example 109            !13
! Buchanan Example 2.19, electrical cable                  !14
! Real FE problem properties are:                          !15
! K_RR = GET_REAL_LP (1) permittivity: 0.5 and 2.0         !16
! Q   = GET_REAL_LP (2) charge density: 100. and 0.        !17
! [material]           [1]      [2]                        !18
! Mesh T=500, R=5 *----*----*----*----* R= 25, T=0         !19
! Nodes, (Elem)   1    2(1) 3    4(2) 5                    !20
1  1   5.     ! begin nodes, flag, x                       !21
2  0   7.5                                                 !22
3  0   10.    ! T_analytic = 918.29, FEA 910.76           !23
4  0   17.5                                                !24
5  1   25.0                                                !25
1 1 2 3 ! begin elements                                   !26
2 3 4 5                                                    !27
1 1 500.     ! essential bc (electrical potential)         !28
5 1 0.                                                     !29
 1 0.5 100.  ! el, permittivity, charge density           !30
 2 2.0 0.    ! el, permittivity, charge density           !31
                                                           !32
           Selected Output                                 !33
***  OUTPUT OF RESULTS IN NODAL ORDER  ***                 !34
NODE,  Radius r,   DOF_1,                                  !35
   1  5.0000E+00  5.0000E+02                               !36
   2  7.5000E+00  1.3646E+03                               !37
   3  1.0000E+01  9.1076E+02 ! T_analytic = 918.29         !38
   4  1.7500E+01  3.5780E+02                               !39
   5  2.5000E+01  0.0000E+00                               !40
                                                           !41
DOF_1, EVALUATED AT 5 NODE POINTS                          !42
RANGE ON GRAPH IS  0.00000E+00 TO  1.36461E+03             !43
PT  VALUE +---+---+---+---+---+---+---+---+---+---+       !44
1 5.00E+2 XXXXXXXXXXXXXXX    +                   +       !45
          +                   +                   +       !46
2 1.37E+3 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX       !47
          +                   +                   +       !48
3 9.11E+2 XXXXXXXXXXXXXXXXXXXXXXXXXXX             +       !49
          +                   +                   +       !50
          +                   +                   +       !51
          +                   +                   +       !52
4 3.58E+2 XXXXXXXXXXXX        +                   +       !53
          +                   +                   +       !54
          +                   +                   +       !55
          +                   +                   +       !56
5 0.00E+0 X                   +                   +       !57
PT  VALUE +---+---+---+---+---+---+---+---+---+---+       !58
```

Problem P8.1 *Coaxial cable data and results*

1.9 Problems from chapter 9

Problem 9.1 Use the subroutines in Fig. 4.5.2 to form similar functions for a C^1 rectangular element by taking a tensor product of the one-dimensional Hermite interpolation relations. This will be a 16 degree of freedom element since each node will have $u, \partial u/\partial x, \partial u/\partial y,$ and $\partial^2 u/\partial x \partial y$ as nodal unknowns. Solution: Taking the products of the 1-D functions yields the interpolations and first derivatives shown in P9.1a and P9.1b. Such and element would usually be needed for a fourth order differential equation which would have the second derivatives in the integral form. Thus a similar subroutine for the second derivatives would be needed too.

Problem 9.5 Verify that if the above Q4 element maps onto a rectangle, with is sides parallel to the global axes, of length L_x and height L_y then the Jacobian is constant at all points in the element. Solution: Shown in Section 11.6.

```
SUBROUTINE SHAPE_16_R (R, S, A, B, H)                    ! 1
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*        ! 2
! C1 RECTANGULAR ELEMENT IN UNIT COORDINATES             ! 3
! USING TENSOR PRODUCTS OF 1D HERMITE BASIS              ! 4
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-*        ! 5
Use Precision_Module                                     ! 6
 IMPLICIT NONE                                           ! 7
 REAL(DP), INTENT(IN)  :: R, S, A, B                     ! 8
 REAL(DP), INTENT(OUT) :: H (16)                         ! 9
 REAL(DP)              :: HR (4), HS (4)                 !10
                                                         !11
! DOF ARE W W,X W,Y W,XY AT EACH NODE (N_G_DOF=4)        !12
! X // R, Y // S.               S                        !13
! A = PHYSICAL LENGTH IN X      4 -------- 3             !14
! B = PHYSICAL LENGTH IN Y      I          I             !15
! R,S = LOCAL UNIT COORDS       I          I             !16
! 1 at (0,0), 3 at (1,1)        1 -------- 2 ->R         !17
                                                         !18
! Evaluate the 1D Hermite interpolations                 !19
 CALL SHAPE_C1_L (R, A, HR) ; CALL SHAPE_C1_L (S, B, HS) !20
                                                         !21
! Form tensor products                                   !22
  H (1)  = HR (1) * HS (1) ; H (2)  = HR (2) * HS (1)    !23
  H (3)  = HR (1) * HS (2) ; H (4)  = HR (2) * HS (2)    !24
  H (5)  = HR (3) * HS (1) ; H (6)  = HR (4) * HS (1)    !25
  H (7)  = HR (3) * HS (2) ; H (8)  = HR (4) * HS (2)    !26
  H (9)  = HR (3) * HS (3) ; H (10) = HR (4) * HS (3)    !27
  H (11) = HR (3) * HS (4) ; H (12) = HR (4) * HS (4)    !28
  H (13) = HR (1) * HS (3) ; H (14) = HR (2) * HS (3)    !29
  H (15) = HR (1) * HS (4) ; H (16) = HR (2) * HS (4)    !30
END SUBROUTINE SHAPE_16_R                                !31
```

Problem P9.1a *Shape functions from 1-D C^1 products*

```
SUBROUTINE DERIV_16_R (R, S, A, B, DH)                                  ! 1
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *                 ! 2
! FIRST DERIVATIVES OF A C1 RECTANGULAR ELEMENT IN                      ! 3
! UNIT COORDINATES USING TENSOR PRODUCTS OF 1D BASIS                    ! 4
! *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *-* *                 ! 5
Use Precision_Module                                                    ! 6
 IMPLICIT NONE                                                          ! 7
 REAL(DP), INTENT(IN)  :: R, S, A, B                                    ! 8
 REAL(DP), INTENT(OUT) :: DH (2, 16)                                    ! 9
 REAL(DP)              :: HR (4), DR (4), HS (4), DS (4)                !10
                                                                        !11
! DOF ARE W W,X W,Y W,XY AT EACH NODE (N_G_DOF=4)                       !12
! X // R, Y // S.              S                                        !13
! A = PHYSICAL LENGTH IN X     4 -------- 3                             !14
! B = PHYSICAL LENGTH IN Y     I          I                             !15
! R,S = LOCAL UNIT COORDS      I          I                             !16
! 1 at (0,0), 3 at (1,1)       1 -------- 2 ->R                         !17
                                                                        !18
! Evaluate the 1D Hermite interpolations                                !19
  CALL SHAPE_C1_L (R, A, HR) ; CALL SHAPE_C1_L (S, B, HS)               !20
  CALL DERIV_C1_L (R, A, DR) ; CALL DERIV_C1_L (S, B, DS)               !21
                                                                        !22
! Form tensor products for R direction                                  !23
  DH (1, 1)  = DR (1) * HS (1) ; DH (1, 2)  = DR (2) * HS (1)           !24
  DH (1, 3)  = DR (1) * HS (2) ; DH (1, 4)  = DR (2) * HS (2)           !25
  DH (1, 5)  = DR (3) * HS (1) ; DH (1, 6)  = DR (4) * HS (1)           !26
  DH (1, 7)  = DR (3) * HS (2) ; DH (1, 8)  = DR (4) * HS (2)           !27
  DH (1, 9)  = DR (3) * HS (3) ; DH (1, 10) = DR (4) * HS (3)           !28
  DH (1, 11) = DR (3) * HS (4) ; DH (1, 12) = DR (4) * HS (4)           !29
  DH (1, 13) = DR (1) * HS (3) ; DH (1, 14) = DR (2) * HS (3)           !30
  DH (1, 15) = DR (1) * HS (4) ; DH (1, 16) = DR (2) * HS (4)           !31
                                                                        !32
! Form tensor products for S direction                                  !33
  DH (2, 1)  = HR (1) * DS (1) ; DH (2, 2)  = HR (2) * DS (1)           !34
  DH (2, 3)  = HR (1) * DS (2) ; DH (2, 4)  = HR (2) * DS (2)           !35
  DH (2, 5)  = HR (3) * DS (1) ; DH (2, 6)  = HR (4) * DS (1)           !36
  DH (2, 7)  = HR (3) * DS (2) ; DH (2, 8)  = HR (4) * DS (2)           !37
  DH (2, 9)  = HR (3) * DS (3) ; DH (2, 10) = HR (4) * DS (3)           !38
  DH (2, 11) = HR (3) * DS (4) ; DH (2, 12) = HR (4) * DS (4)           !39
  DH (2, 13) = HR (1) * DS (3) ; DH (2, 14) = HR (2) * DS (3)           !40
  DH (2, 15) = HR (1) * DS (4) ; DH (2, 16) = HR (2) * DS (4)           !41
END SUBROUTINE DERIV_16_R                                               !42
```

Problem P9.1b *Gradients from 1-D C^1 products*

1.10 Problems from chapter 10

Problem 10.1: the sum of the weights is always the measure (generalized volume) of the non-dimensional parametric space in which they are tabulated. The first measure is for a unit triangle so its measure is half the base times the height or $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$, and the last is a cube that is two units long on each edge so its measure is $2 \times 2 \times 2 \times 2 = 8$.

1.11 Problems from chapter 11

Problem 11.1-B The two-dimensional Laplace equation $\partial^2 u \,/\, \partial x^2 + \partial^2 u \,/\, \partial y^2 = 0$ is satisfied by the cubic $u(x, y) = -x^3 - y^3 + 3xy^2 + 3x^2 y$. It can be used to define the exact essential (Dirichlet) boundary conditions on the edges of any two-dimensional shape. Obtain a finite element solution and sketch it along with the exact values using a

domain of: a) the above unit square, or b) a rectangle over $0 \le x \le 3$ and $0 \le y \le 2$, or c) a quarter circle of radius 2 centered at the origin.

Clearly selecting a cubic element would give the exact solution everywhere, so we want to use lower order elements to illustrate the error estimator. Here we show an irregular mesh of 482 linear (T3) triangular elements and 273 nodes of which the 62 on the boundary have had values assigned from the exact solution. That was done by including the keyword controls of *exact_case* 20 and *use_exact_bc* to override the null input values. We also turned on controls *list_exact*, *list_exact_flux* so that the printed output (omitted here) could be used to contrast the FEA results. Since we have given formulations for both general numerically integrated elements and the T3 elements we have to existing approaches that just need data. Here we have used the hard codes T3 form, selected with *example* 202. The mesh was obtained by starting with a crude mesh and running two solutions through the error estimator to adapt to the input mesh shown in P11.1Ba. The computed solution is given in P11.1Bb.

The FEA and exact solution contours along with their corresponding flux vectors (using Fourier law signs) are comparable in P11.1Bc. Since the exact solution is known we can compute the exact error in the solution at each node and contour them as in P11.1Bd. Of course, the SCP error estimator uses the different energy norm values to guide the adaptivity. Each element energy norm error estimate is constant. Next we average the element energy norm error estimates at each node and contour that distribution to compare it to the exact energy norm error estimates available from the given analytic solution. These two sets of contours are compared in P11.1Be and their agreement is quite good.

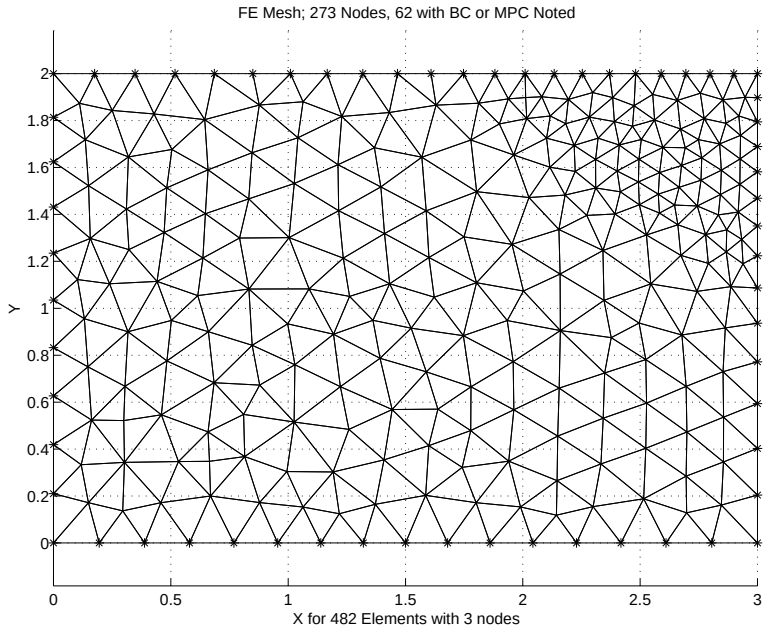
Problem 11.6 Solve the fin example of Fig. 11.5.14 by hand: a) with insulated edges, b) with edge convection. Solution: The insulated edge case is given in detail in the text by Allaire. The edge convection case requires three edge convection elements but will give very similar results because the edges are so thin. In other words, one just assembles the edge effects onto the previous algebraic equations. The new data are indicated in Fig. P11.6Ba, the mesh and contours in P11.6Bb, and selected corresponding results are shown in Fig. P11.6Bc. The net convection loss changes little.

Problem 11.8c For the conduction mesh in Figs.11.9 and 11 prepare a combined surface plot of the element and SCP flux for: c) the RMS values.

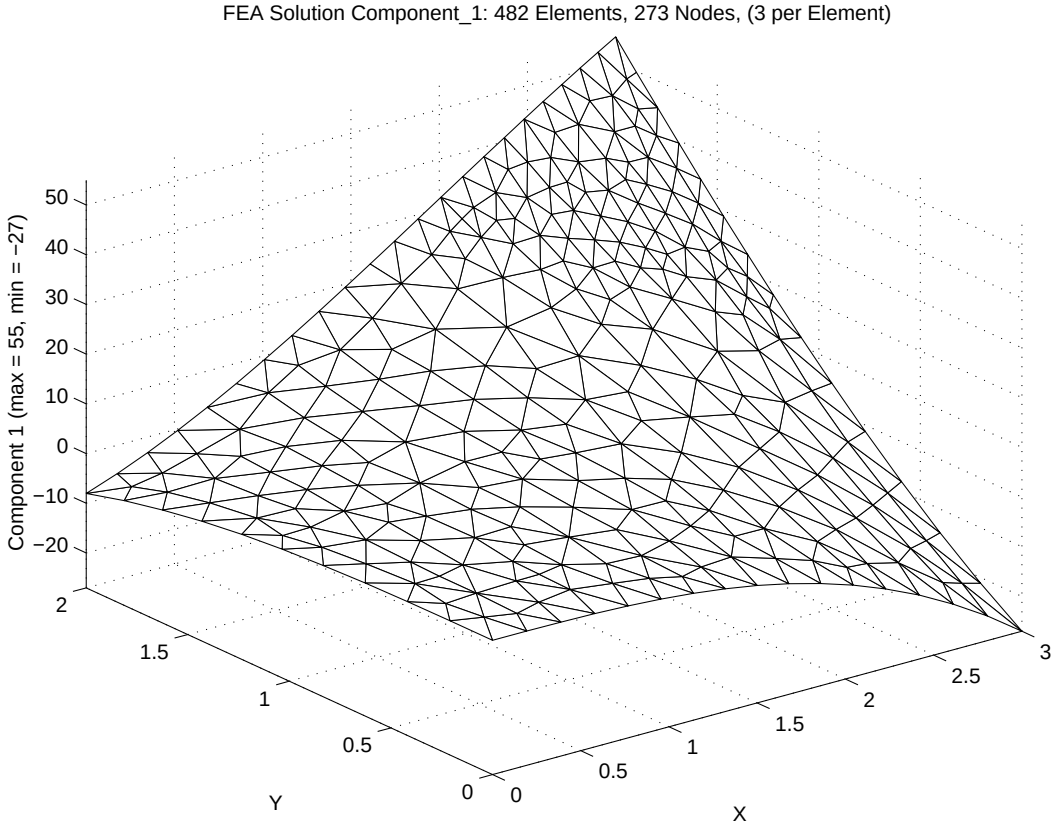
Solution: Post processing each element gives the constant flux values of:

```
*** FLUX COMPONENTS AT ELEMENT INTEGRATION POINTS ***
ELEMENT, PT,  X-Coord,    Y-Coord,     FLUX_1,      FLUX_2
      1  1  1.3333E+00  6.6667E-01 -4.0000E+00 -2.5000E+00
      2  1  3.3333E+00  6.6667E-01 -1.1000E+01  0.0000E+00
      3  1  2.6667E+00  1.3333E+00 -8.5000E+00 -2.5000E+00
      4  1  3.3333E+00  2.6667E+00 -8.5000E+00  0.0000E+00
```

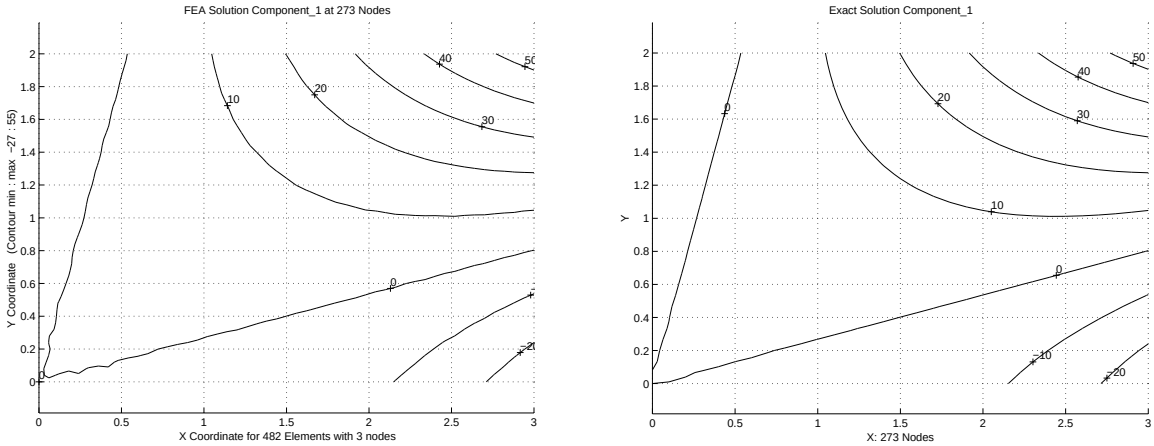
Averaging them via the patches gave the values in Fig. 11.11. The RMS values of the two results are combined in Problem P11.8.

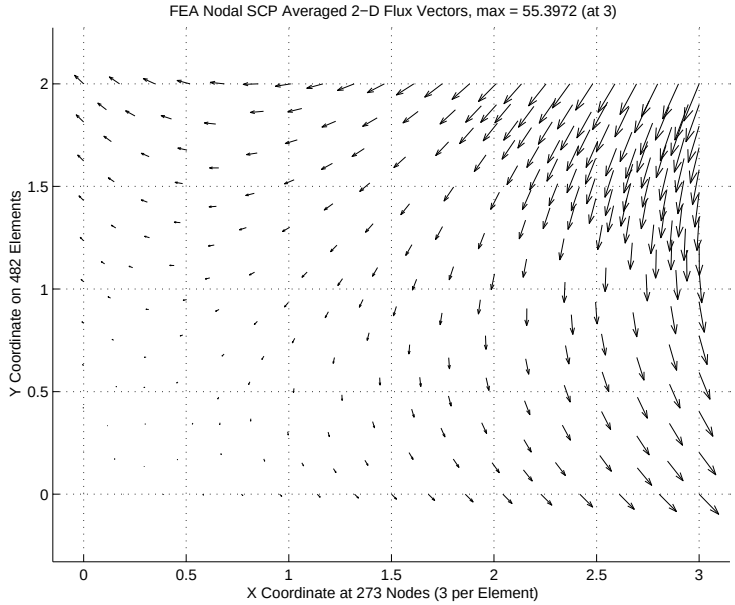


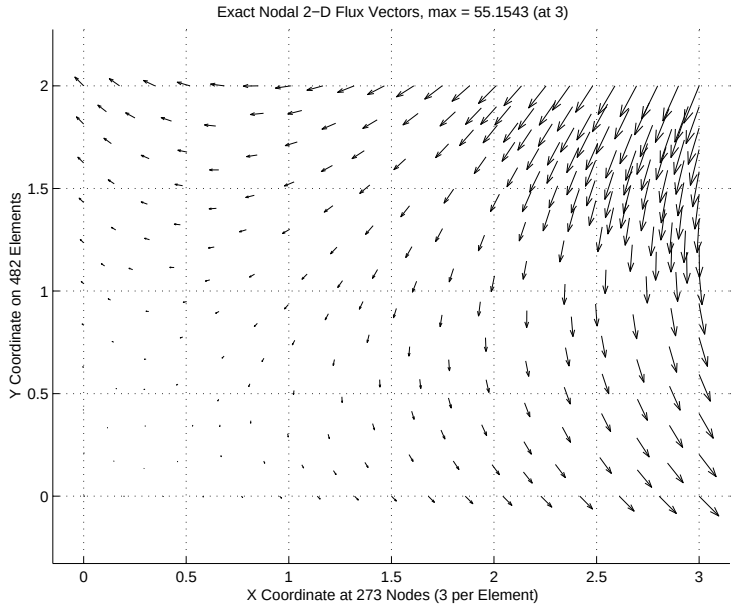
Problem P11.1Ba *Mesh and Dirichlet condition points*



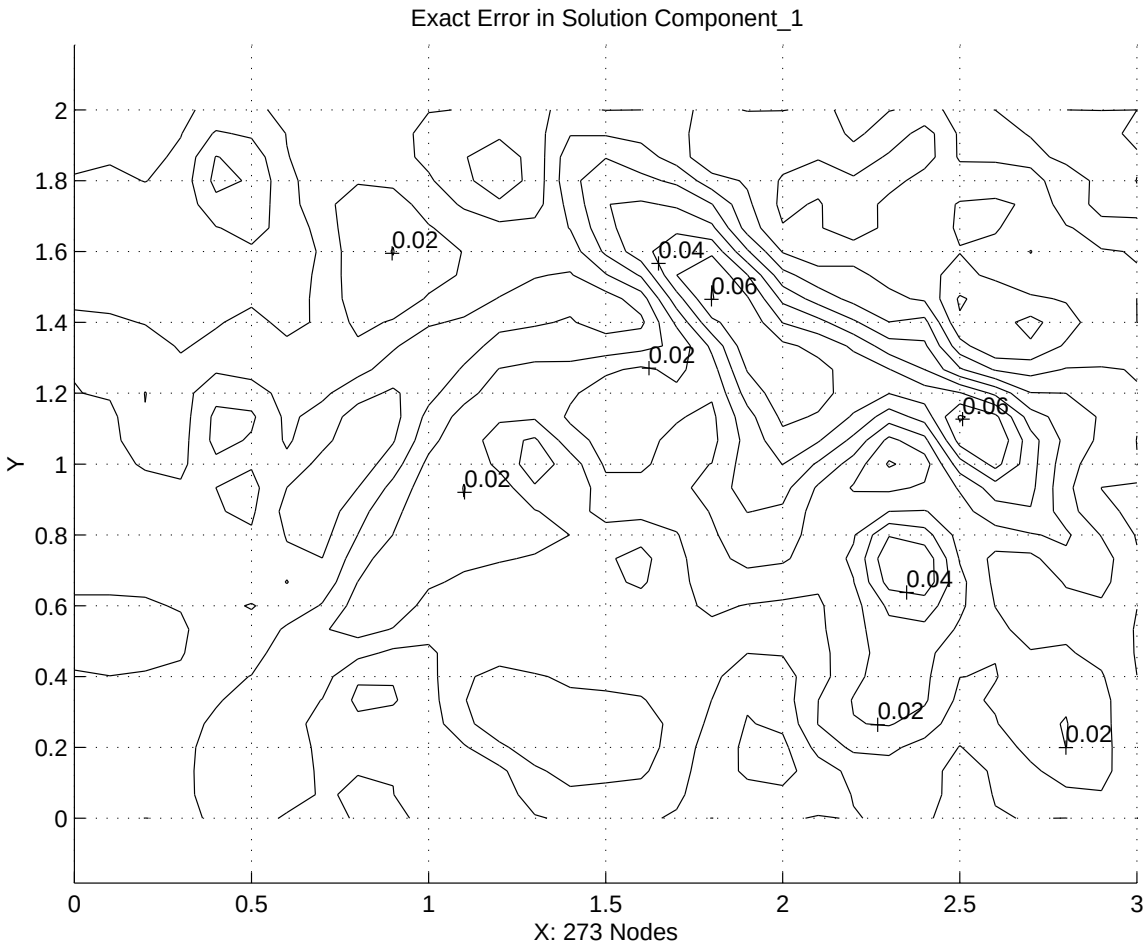
Problem P11.1Bb *Result carpet plot for linear triangles*



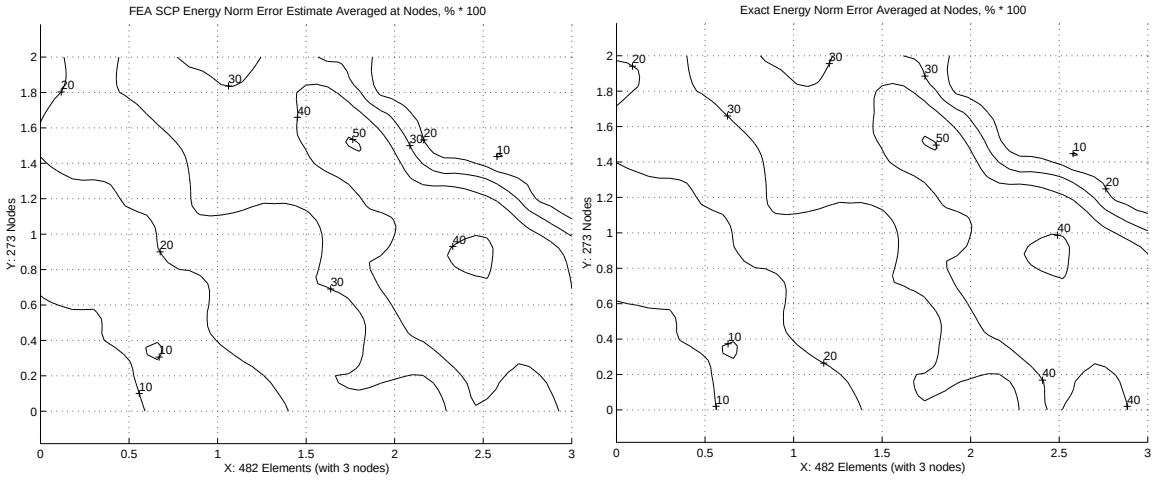




Problem P11.1Bc *FEA and exact solution contours and flux vectors*



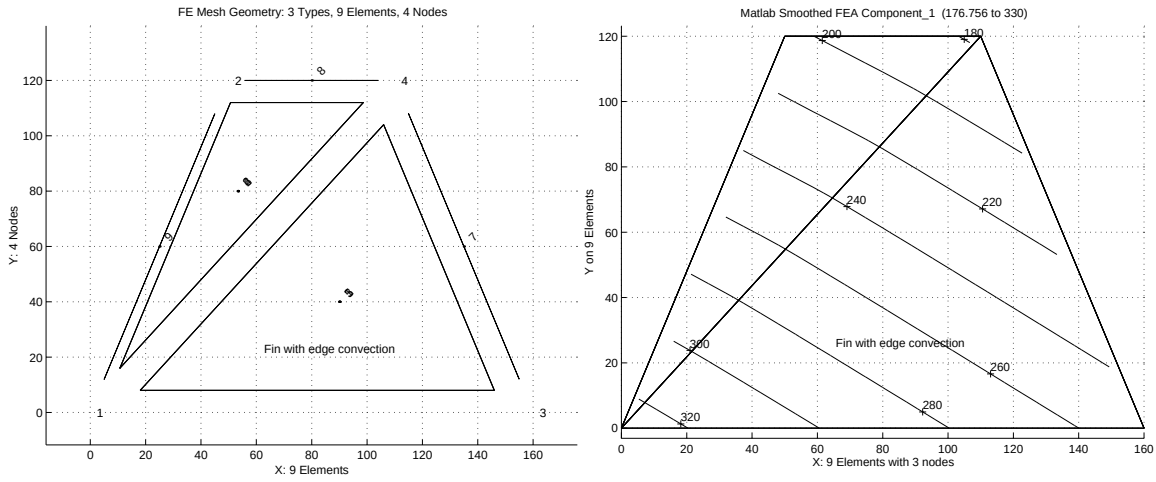
Problem P11.1Bd *Exact error at the mesh nodes*



Problem P11.1Be *SCP and exact averaged energy norm error*

```
title "Fin face, edge convection, Allaire p343 2T3, 3L2"    ! 1
example  209 ! Application source code library number       ! 2
elems      2 ! Number of elements in the system             ! 3
nodes      4 ! Number of nodes in the mesh                  ! 4
b_rows     2 ! Number of rows in the B (operator) matrix    ! 5
dof        1 ! Number of unknowns per node                  ! 6
el_homo      ! Element properties are homogeneous           ! 7
el_real    5 ! Number of real properties per element        ! 8
convect_coef 1.e-5 ! Coefficient on all mixed segments      ! 9
convect_temp    30 ! Temperature on all mixed segments      !10
convect_thick 1.25 ! Thickness for edge convections  <= new !11
el_segment   3 ! Maximum nodes on boundary segment          !12
mixed_segs   7 ! Number of mixed boundary segments   <= new !13
post_mixed     ! Compute convection loss                    !14
el_types     3 ! Number of different types of elements      !15
type_shape 2 2 1 ! Shape code of each element type   <= new !16
type_nodes 3 3 2 ! Number of nodes on each type      <= new !17
type_gauss 1 4 2 ! Number of Gauss points in types   <= new !18
type_parm  2 2 1 ! Parametric space of element type  <= new !19
end ! Terminate the keyword control, remarks follow         !20
 1    1    0.0    0.0 ! node, bc-flag,x, y                  !21
 3    1  160.0    0.0                                       !22
 2    0   50.0  120.0                                       !23
 4    0  110.0  120.0                                       !24
1 1    1 3 4 ! elem, el_type, 3 nodes. conducting          !25
2 1    2 1 4 ! elem, el_type, 3 nodes. conducting          !26
1 2    1 3 4 ! face, el_type, 3 nodes. convecting          !27
2 2    2 1 4 ! face, el_type, 3 nodes. convecting          !28
3 2    1 3 4 ! face, el_type, 3 nodes. convecting          !29
4 2    2 1 4 ! face, el_type, 3 nodes. convecting          !30
5 3    3 4   ! edge, el_type, 2 nodes. convecting   <= new  !31
6 3    4 2   ! edge, el_type, 2 nodes. convecting   <= new  !32
7 3    2 1   ! edge, el_type, 2 nodes. convecting   <= new  !33
  1  1  330. ! node, dof, value                            !34
  3  1  250. ! node, dof, value                            !35
 1 0.2 0.2 0.0 0.0 1.25 ! el, k_x, k_y, K_xy, Q, thick     !36
```

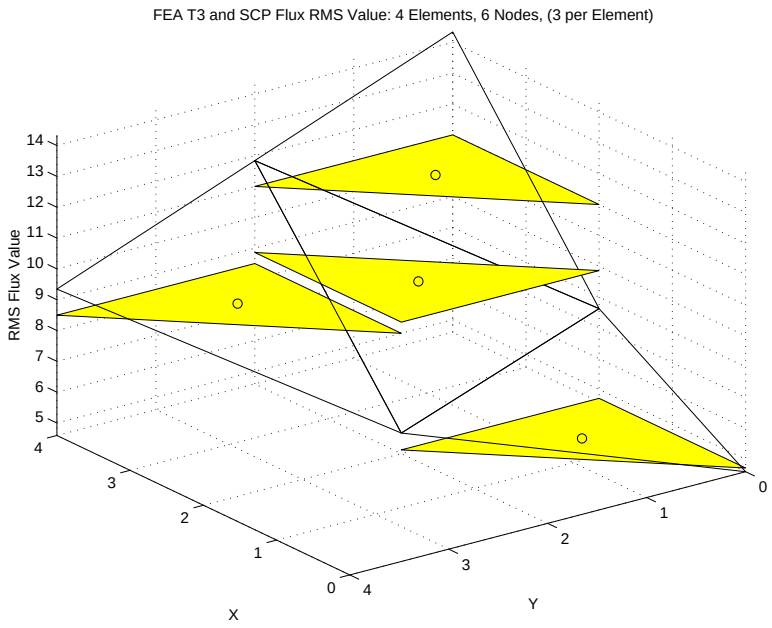
Problem P11.6Ba *New edge convection data*



Problem P11.6Bb *New mesh and temperature contours*

```
*** MIXED BOUNDARY CONDITION SEGMENTS ***                          ! 1
 NOTE: CONSTANT CONVECTION ON ALL MIXED SEGMENTS USING             ! 2
   CONVECTION COEFFICIENT    =    1.00E-05                         ! 3
   CONVECTION TEMPERATURE    =   30.000                            ! 4
   LINE CONVECTION THICKNESS =    1.250   <== new                  ! 5
SEGMENT, TYPE,  3 MAX NODES ON THE SEGMENT                         ! 6
      1     2       1    3    4                                   ! 7
      2     2       2    1    4                                   ! 8
      3     2       1    3    4                                   ! 9
      4     2       2    1    4                                   !10
      5     3       3    4   <== new                              !11
      6     3       4    2   <== new                              !12
      7     3       2    1   <== new                              !13
                                                                   !14
*** REACTION RECOVERY ***                                          !15
   NODE, PARAMETER,     REACTION, EQUATION                         !16
      1,  DOF_1,       4.0271E+01     1  <== new                  !17
      3,  DOF_1,       1.8104E+01     3  <== new                  !18
REACTION RESULTANTS                                                !19
PARAMETER,    SUM          POSITIVE     NEGATIVE                  !20
 DOF_1,      5.8375E+01   5.8375E+01   0.0000E+00                  !21
                                                                   !22
***  OUTPUT OF RESULTS IN NODAL ORDER  ***                         !23
NODE,  X-Coord,    Y-Coord,    DOF_1,                              !24
   1  0.0000E+00  0.0000E+00  3.3000E+02                           !25
   2  5.0000E+01  1.2000E+02  2.0404E+02 <== new                   !26
   3  1.6000E+02  0.0000E+00  2.5000E+02                           !27
   4  1.1000E+02  1.2000E+02  1.7676E+02 <== new                   !28
                                                                   !29
** BEGIN APPLICATION BOUNDARY SEGMENTS POST PROCESSING **         !30
*** CONVECTION HEAT LOSS ***                                       !31
ELEMENT    HEAT_LOST                                               !32
     1    2.13362E+01                                             !33
     2    7.44956E+00                                             !34
     3    2.13362E+01                                             !35
     4    7.44956E+00                                             !36
     5    2.97989E-01   <== new                                   !37
     6    1.20299E-01   <== new                                   !38
     7    3.85158E-01   <== new                                   !39
     TOTAL =   58.37493 <== new                                    !40
```

Problem P11.6Bc *New edge convection results*



Problem P11.8c *Constant and SCP RMS flux surfaces*

Problem 11.7 For the solid conducting bar of Fig. 11.2.2 assume a length $L = 8$ cm, a width $2W = 4$ cm, and a thickness of $2H = 1$ cm. The material has a thermal conductivity of $k = 3\,W/cm\,C$, is surrounded by a fluid at a temperature of $\Theta_\infty = 20\,C$, and has a surface convection coefficient of $h = 0.1\,W/cm^2\,C$. Employ two equal length conduction elements and associated face, line, or point convection elements to obtain a solution for the temperature distribution and the convection heat loss. Use conduction elements that are: a) linear line elements, b) bilinear rectangular elements, c) trilinear brick elements.

Solution: We have developed most of the necessary linear element matrices to formulate a hand solution and could produce the few missing forms easily since the Jacobian matrix for the conduction elements has a constant diagonal form. Instead we will use the numerically integrated general routines in Fig. 11.8.1, for all conduction effects, Fig. 11.8.3 to form the convection matrices, and finally Fig. 11.8.4 to post-process the convection elements to determine the convection heat loss, by recovering the integral of $\mathbf{H}^b$ evaluated in Fig. 11.8.3. These routines could be used with any of the linear, quadratic, or cubic elements in the current library.

1.12 Problems from chapter 12

Problem 12.1 Implement the isotropic $\mathbf{E}^e$ matrix for the plane strain assumption with the stress components in the xx, yy, xy order. Solution: the listing in Problem P12.1.

Problem 12.2 Implement the isotropic $\mathbf{E}^e$ matrix for the general solid with the stress components in the xx, yy, xy, zz, xz, yz order. Solution: see listing Problem P12.2.

```
SUBROUTINE E_PLANE_STRAIN (E)                                              ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 2
!     PLANE_STRAIN CONSTITUTIVE MATRIX DEFINITION                          ! 3
! STRESS & STRAIN COMPONENT ORDER: XX, YY, XY, SO N_R_B = 3                ! 4
! PROPERTY ORDER: 1-YOUNG'S MODULUS, 2-POISSON'S RATIO                     ! 5
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 6
Use System_Constants    ! for DP, N_R_B                                    ! 7
Use Sys_Properties_Data ! for function GET_REAL_LP                         ! 8
 IMPLICIT NONE                                                             ! 9
 REAL(DP), INTENT(OUT) :: E (N_R_B, N_R_B) ! Constitutive                  !10
 REAL(DP) :: C_1, C_2, P_R, S_M, Y_M       ! Miscellaneous                 !11
                                                                           !12
! N_R_B = NUMBER OF ROWS IN B AND E MATRICES                               !13
! P_R   = POISSON'S RATIO,  S_M   = SHEAR MODULUS                          !14
! Y_M   = YOUNG'S MODULUS OF ELASTICITY                                    !15
                                                                           !16
  IF ( EL_REAL < 2 ) STOP 'KEYWORD el_real<2 IN E_PLANE_STRAIN'            !17
  Y_M = GET_REAL_LP (1) ; P_R = GET_REAL_LP (2)                            !18
                                                                           !19
  C_1 = Y_M * (1 - P_R) / ((1 + P_R)*(1 - P_R - P_R))                      !20
  C_2 = Y_M *  P_R / ((1 + P_R)*(1 - P_R - P_R))                           !21
  S_M = 0.5d0 * Y_M / ( 1 + P_R)                                           !22
  E (1, 1) = C_1     ; E (2, 1) = C_2     ; E (3, 1) = 0.d0                !23
  E (1, 2) = C_2     ; E (2, 2) = C_1     ; E (3, 2) = 0.d0                !24
  E (1, 3) = 0.d0    ; E (2, 3) = 0.d0    ; E (3, 3) = S_M                 !25
END SUBROUTINE E_PLANE_STRAIN                                              !26
```

Problem P12.1 *Plane strain assumption isotropic constitutive law*

```
SUBROUTINE E_SOLID_STRESS (E)                                              ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 2
!     SOLID_STRESS CONSTITUTIVE MATRIX DEFINITION                          ! 3
! STRESS COMPONENT ORDER: XX, YY, XY, ZZ, XZ, YZ, SO N_R_B = 6             ! 4
! PROPERTY ORDER: 1-YOUNG'S MODULUS, 2-POISSON'S RATIO                     ! 5
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *     ! 6
Use System_Constants    ! for DP, N_R_B                                    ! 7
Use Sys_Properties_Data ! for function GET_REAL_LP                         ! 8
 IMPLICIT NONE                                                             ! 9
 REAL(DP), INTENT(OUT) :: E (N_R_B, N_R_B) ! Constitutive                  !10
 REAL(DP) :: C_1, C_2, P_R, S_M, Y_M       ! Miscellaneous                 !11
                                                                           !12
! N_R_B = NUMBER OF ROWS IN B AND E MATRICES                               !13
! P_R   = POISSON'S RATIO,  S_M   = SHEAR MODULUS                          !14
! Y_M   = YOUNG'S MODULUS OF ELASTICITY                                    !15
                                                                           !16
  IF ( N_R_B /= 6 ) STOP 'INVALID N_R_B IN E_SOLID_STRESS'                 !17
  IF (EL_REAL < 2) STOP 'KEY el_real < 2 IN E_SOLID_STRESS'                !18
  Y_M = GET_REAL_LP (1) ; P_R = GET_REAL_LP (2)                            !19
                                                                           !20
  C_1  = Y_M * (1 - P_R) / (1 + P_R) / (1 - P_R - P_R)                     !21
  C_2  = C_1 * P_R / (1 - P_R); S_M  = 0.5d0 * Y_M / (1 + P_R)             !22
  E = 0.d0 ! Mostly                                                        !23
  E (1, 1) = C_1 ; E (2, 1) = C_2 ; E (4, 1) = C_2  ! normals              !24
  E (1, 2) = C_2 ; E (2, 2) = C_1 ; E (4, 2) = C_2  ! normals              !25
  E (1, 4) = C_2 ; E (2, 4) = C_2 ; E (4, 4) = C_1  ! normals              !26
  E (3, 3) = S_M ; E (5, 5) = S_M ; E (6, 6) = S_M  ! shears               !27
END SUBROUTINE E_SOLID_STRESS                                              !28
```

Problem P12.2 *Isotropic solid constitutive matrix*

```
SUBROUTINE ELASTIC_B_SOLID (DGH, B)                                    ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *   ! 2
! 3-D ELASTICITY STRAIN-DISPLACEMENT RELATIONS (B)                     ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *   ! 4
Use System_Constants ! for DP, N_R_B, N_G_DOF, N_SPACE               ! 5
Use Elem_Type_Data   ! for LT_FREE, LT_N                             ! 6
 IMPLICIT NONE                                                       ! 7
 REAL(DP), INTENT(IN)  :: DGH (N_SPACE, LT_N)                        ! 8
 REAL(DP), INTENT(OUT) :: B   (N_R_B, LT_N * N_G_DOF)                ! 9
 INTEGER  :: J, K, L, M                                              !10
                                                                     !11
! B       = STRAIN-DISPLACEMENT MATRIX (RETURNED)                    !12
! DGH     = GLOBAL DERIVATIVES OF ELEM INTERPOLATIONS                !13
! LT_N    = NUMBER OF NODES PER ELEMENT                              !14
! N_G_DOF = NUMBER OF PARAMETERS PER NODE                            !15
! N_R_B   = NUMBER OF STRAINS (IN B) XX, YY, XY, ZZ, XZ, YZ          !16
! N_SPACE = DIMENSION OF SPACE                                       !17
                                                                     !18
  B = 0.d0                                                           !19
  DO J = 1, LT_N               ! ROW NUMBER                          !20
    K = N_G_DOF * (J - 1) + 1 ! FIRST COLUMN,  U TERMS               !21
    L = K + 1                  ! SECOND COLUMN, V TERMS              !22
    M = L + 1                  ! THIRD COLUMN,  W TERMS              !23
    B (1, K) = DGH (1, J)      ! DU/DX  FOR XX NORMAL                !24
    B (3, K) = DGH (2, J)      ! DU/DY  FOR XY SHEAR                 !25
    B (5, K) = DGH (3, J)      ! DU/DZ  FOR XZ SHEAR                 !26
    B (2, L) = DGH (2, J)      ! DV/DY  FOR YY NORMAL                !27
    B (3, L) = DGH (1, J)      ! DV/DX  FOR XY SHEAR                 !28
    B (6, L) = DGH (3, J)      ! DV/DZ  FOR YZ SHEAR                 !29
    B (4, M) = DGH (3, J)      ! DW/DZ  FOR ZZ NORMAL                !30
    B (5, M) = DGH (1, J)      ! DW/DX  FOR XZ SHEAR                 !31
    B (6, M) = DGH (2, J)      ! DW/DY  FOR YZ SHEAR                 !32
  END DO                                                             !33
END SUBROUTINE ELASTIC_B_SOLID                                       !34
```

Problem P12.3 *Strain-displacement matrix for a general solid*

```
! ..............................................................  ! 1
!    ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS FOLLOW ***  ! 2
! ..............................................................  ! 3
! Note: BODY (N_SPACE) is available through the interface         ! 4
!          Linear Triangle, T3, Plane stress only                 ! 5
! STRESS AND STRAIN COMPONENT ORDER: XX, YY, XY, SO N_R_B = 3     ! 6
 REAL(DP):: X_I, X_J, X_K, Y_I, Y_J, Y_K     ! Global coordinates ! 7
 REAL(DP):: A_I, A_J, A_K, B_I, B_J, B_K     ! Standard geometry  ! 8
 REAL(DP):: C_I, C_J, C_K, X_CG, Y_CG, TWO_A ! Standard geometry  ! 9
 REAL(DP):: THICK                            ! Element thickness  !10
                                                                  !11
!   DEFINE NODAL COORDINATES, CCW: I, J, K                       !12
   X_I = COORD (1,1) ;  X_J = COORD (2,1) ;  X_K = COORD (3,1)    !13
   Y_I = COORD (1,2) ;  Y_J = COORD (2,2) ;  Y_K = COORD (3,2)    !14
                                                                  !15
!   DEFINE CENTROID COORDINATES (QUADRATURE POINT)               !16
   X_CG = (X_I + X_J + X_K)/3.d0 ; Y_CG = (Y_I + Y_J + Y_K)/3.d0  !17
                                                                  !18
!   GEOMETRIC PARAMETERS H_I (X,Y) = (A_I + B_I*X + C_I*Y)/TWO_A  !19
   A_I = X_J * Y_K - X_K * Y_J; B_I = Y_J - Y_K; C_I = X_K - X_J  !20
   A_J = X_K * Y_I - X_I * Y_K; B_J = Y_K - Y_I; C_J = X_I - X_K  !21
   A_K = X_I * Y_J - X_J * Y_I; B_K = Y_I - Y_J; C_K = X_J - X_I  !22
                                                                  !23
!   CALCULATE TWICE ELEMENT AREA                                 !24
   TWO_A = A_I + A_J + A_K  ! = B_J*C_K - B_K*C_J also           !25
                                                                  !26
!   DEFINE 3 BY 6 STRAIN-DISPLACEMENT MATRIX, B                  !27
   B (1, 1:6) = (/ B_I,  0.d0, B_J,  0.d0, B_K,  0.d0 /)/TWO_A    !28
   B (2, 1:6) = (/ 0.d0, C_I,  0.d0, C_J,  0.d0, C_K  /)/TWO_A    !29
   B (3, 1:6) = (/ C_I,  B_I,  C_J,  B_J,  C_K,  B_K  /)/TWO_A    !30
                                                                  !31
   CALL E_PLANE_STRESS (E)        ! THE CONSTITUTIVE MATRIX       !32
   THICK = 1 ; IF ( AREA_THICK /= 1.d0 ) THICK = AREA_THICK       !33
   TWO_A = TWO_A * THICK                      ! true volume       !34
                                                                  !35
!   STIFFNESS MATRIX, WITH CONSTANT JACOBIAN                     !36
   S = MATMUL ( TRANSPOSE (B), MATMUL (E, B) ) * TWO_A * 0.5d0    !37
                                                                  !38
!   BODY FORCE PER UNIT VOLUME                                   !39
   IF ( HAS_BODY_FORCE ) THEN ! values in Body_Force array       !40
     C (1:5:2) = BODY_FORCE (1) * TWO_A / 6.d0  ! X component    !41
     C (2:6:2) = BODY_FORCE (2) * TWO_A / 6.d0  ! Y component    !42
   END IF ! or set up properties for body force                   !43
                                                                  !44
!  SAVE ONE POINT RULE TO AVERAGING, OR ERROR ESTIMATOR          !45
   LT_QP = 1 ; CALL STORE_FLUX_POINT_COUNT ! Save LT_QP          !46
   CALL STORE_FLUX_POINT_DATA ( (/ X_CG, Y_CG /), E, B )         !47
                                                                  !48
   if ( DEBUG_EL_SQ ) then                                        !49
     print *,'S matrix:' ; call rprint (S, LT_FREE, LT_FREE, 1)   !50
     print *,'C matrix:' ; call rprint (C,       1, LT_FREE, 1)   !51
   end if                                                         !52
!  End of application dependent code                             !53
!    ***  END ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS ***     !54
```

Problem P12.4 *Stiffness for isotropic plane stress CST*

```
SUBROUTINE ELASTICITY_E_MATRIX (E)                                  ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 2
!      ELASTICITY CONSTITUTIVE MATRIX DEFINITION                     ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 4
Use System_Constants    ! for DP, N_R_B, N_SPACE,                   ! 5
                        ! AXISYMMETRIC, PLANE_STRAIN                ! 6
 IMPLICIT NONE                                                      ! 7
 REAL(DP), INTENT(OUT) :: E (N_R_B, N_R_B)  ! CONSTITUTIVE          ! 8
! N_R_B = NUMBER OF ROWS IN B AND E MATRICES                        ! 9
! E     = SYMMETRIC CONSTITUTIVE MATRIX                            !10
                                                                    !11
IF ( ANISOTROPIC ) THEN                                             !12
  STOP 'ELASTICITY_E_MATRIX: ANISOTROPIC E UNAVAILABLE'             !13
! SELECT CASE (N_SPACE)                                             !14
!   CASE (1) ! 1-D or cylinder                                      !15
!     IF ( AXISYMMETRIC ) THEN      ! Components RR & TT            !16
!       CALL ANISOTROPIC_CYL_STRESS (E)                           !17
!     ELSE                          ! Component XX                  !18
!       CALL E_BAR_STRESS (E)       ; END IF                        !19
!   CASE (2) ! 2-D or axisymmetric                                  !20
!     IF ( AXISYMMETRIC ) THEN     ! Components RR ZZ RZ TT         !21
!       CALL ANISOTROPIC_AXI_STRESS (E)                           !22
!     ELSE IF ( PLANE_STRAIN ) THEN ! Components XX YY XY           !23
!       CALL ANISOTROPIC_2D_STRAIN (E)                            !24
!     ELSE ! plane stress            Components XX YY XY            !25
!       CALL ANISOTROPIC_2D_STRESS (E)      ; END IF                !26
!   CASE (3) ! 3-D                                                 !27
!     CALL ANISOTROPIC_3D_STRESS (E) ! XX YY XY ZZ ZX ZY            !28
! END SELECT                                                        !29
ELSE ! Isotropic                                                    !30
                                                                    !31
  SELECT CASE (N_SPACE)                                             !32
                                                                    !33
    CASE (1) ! 1-D or cylinder                                      !34
      IF ( AXISYMMETRIC ) THEN      ! Components RR & TT            !35
      ! CALL E_CYLINDER_STRESS (E)                                 !36
      ELSE                          ! Component XX                  !37
      ! CALL E_BAR_STRESS (E)                                      !38
      END IF                                                        !39
                                                                    !40
    CASE (2) ! 2-D or axisymmetric                                  !41
      IF ( AXISYMMETRIC ) THEN     ! Components RR ZZ RZ TT         !42
        CALL E_AXISYMMETRIC_STRESS (E)                              !43
      ELSE IF ( PLANE_STRAIN ) THEN ! Components XX YY XY           !44
        CALL E_PLANE_STRAIN (E)                                     !45
      ELSE ! plane stress            Components XX YY XY            !46
        CALL E_PLANE_STRESS (E)      ; END IF                       !47
                                                                    !38
    CASE (3) ! 3-D                                                  !49
      CALL E_SOLID_STRESS (E)        ! XX YY XY ZZ ZX ZY            !50
                                                                    !51
  END SELECT                                                        !52
END IF ! anisotropic material                                       !53
END SUBROUTINE ELASTICITY_E_MATRIX                                  !54
```

Problem P12.5 *A general elasticity constitutive matrix control*

```
SUBROUTINE ELASTICITY_B_MATRIX (DGH, XYZ, B)                    ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 2
!   GENERAL ELASTICITY STRAIN-DISPLACEMENT RELATIONS  (B)       ! 3
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ! 4
Use System_Constants ! for DP, N_R_B, N_G_DOF, N_SPACE,         ! 5
                     ! AXISYMMETRIC                            ! 6
Use Elem_Type_Data   ! for LT_FREE, LT_N, H (LT_N)              ! 7
 IMPLICIT NONE                                                  ! 8
 REAL(DP), INTENT(IN)  :: DGH (N_SPACE, LT_N) ! Gradients       ! 9
 REAL(DP), INTENT(IN)  :: XYZ (N_SPACE)       ! Coordinates     !10
 REAL(DP), INTENT(OUT) :: B   (N_R_B, LT_N * N_G_DOF)           !11
                                                                !12
! B       = STRAIN-DISPLACEMENT MATRIX (RETURNED)               !13
! DGH     = GLOBAL DERIVATIVES OF H                             !14
! H       = ELEMENT INTERPOLATION FUNCTIONS                     !15
! LT_N    = NUMBER OF NODES PER ELEMENT                         !16
! N_G_DOF = NUMBER OF PARAMETERS PER NODE                       !17
! N_R_B   = NUMBER OF STRAINS (ROWS IN B)                       !18
! N_SPACE = DIMENSION OF SPACE                                  !19
! XYZ     = CURRENT POINT COORDINATES, RADIUS = XYZ (1)         !20
                                                                !21
  SELECT CASE (N_SPACE)                                         !22
    CASE (1) ! 1-D or cylinder                                  !23
      IF ( AXISYMMETRIC ) THEN      ! Components RR & TT        !24
        CALL ELASTIC_B_CYLINDER (DGH, XYZ (1), B)               !25
      ELSE                          ! Component XX              !26
        CALL ELASTIC_B_BAR (DGH, B) ; END IF                    !27
    CASE (2) ! 2-D or axisymmetric                              !28
      IF ( AXISYMMETRIC ) THEN     ! Components RR ZZ RZ TT     !29
        CALL ELASTIC_B_AXISYMMETRIC (DGH, XYZ (1), B)           !30
      ELSE ! plane stress or strain. Components XX YY XY        !31
        CALL ELASTIC_B_PLANAR (DGH, B); END IF                  !32
    CASE (3) ! 3-D                                              !33
      CALL ELASTIC_B_SOLID (DGH, B)  ! XX YY XY ZZ ZX ZY        !34
  END SELECT                                                    !35
END SUBROUTINE ELASTICITY_B_MATRIX                              !36
```

Problem P12.6 *A general elasticity strain-displacement matrix*

```
!  ***  ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS ***          ! 1
!    STATIC PLANE_STRAIN ANALYSIS, NON-ISOPARAMETRIC            ! 2
!    with BODY_FORCE (N_SPACE) via keyword control and/or       ! 3
!    DYNAMIC with mass matrix EL_M (LT_FREE, LT_FREE)           ! 4
! ----------------------------------------------------------    ! 5
! STRESS AND STRAIN COMPONENT ORDER: XX, YY, XY, SO N_R_B = 3   ! 6
!    LATER PLUS ZZ & EFFECTIVE AS N_R_B + 1 & 2                 ! 7
                                                                ! 8
INTEGER  :: IP                          ! loops                 ! 9
REAL(DP) :: DET, DET_WT                 ! volume                !10
REAL(DP) :: N_e (N_G_DOF, LT_FREE)      ! interpolation         !11
REAL(DP), SAVE :: RHO=1.d0              ! data                  !12
                                                                !13
! N_G_DOF = NUMBER OF GENERALIZED PARAMETERS (DOF) PER NODE     !14
! N_e     = GENERALIZED INTERPOLATION ARRAY                     !15
! PROPERTIES: 1-YOUNG'S MODULUS, 2-POISSON'S RATIO, AND         !16
!   3-YIELD STRESS, IF PRESENT, 4-MASS DENSITY, IF PRESENT      !17
                                                                !18
 IF ( EL_REAL > 3 ) RHO = GET_REAL_LP (4)  ! Mass density       !19
 N_e = 0.d0                                ! Save zeros         !20
 CALL STORE_FLUX_POINT_COUNT               ! Save LT_QP         !21
                                                                !22
!  FORM THE CONSTITUTIVE MATRIX (OR GET_APPLICATION_E_MATRIX)   !23
 CALL E_PLANE_STRAIN (E)                                        !24
                                                                !25
 DO IP = 1, LT_QP     !      NUMERICAL INTEGRATION LOOP         !26
   G        = GET_G_AT_QP   (IP)    ! GEOMETRY INTERPOLATIONS   !27
   GEOMETRY = COORD (1:LT_GEOM,:)   ! GEOMETRY NODES            !28
   XYZ      = MATMUL (G, GEOMETRY)  ! COORDINATES OF POINT      !29
                                                                !30
   DLG      = GET_DLG_AT_QP (IP)      ! GEOMETRIC DERIVATIVES   !31
   AJ       = GEOMETRIC_JACOBIAN ()   ! JACOBIAN                !32
   CALL INVERT_2BY2 (AJ, AJ_INV, DET) ! INVERSE, DET            !33
   DET_WT   = DET * WT (IP)                                     !34
                                                                !35
   H   = GET_H_AT_QP   (IP)    ! SCALAR INTERPOLATIONS          !36
   N_e (1, 1:LT_FREE:2) = H    ! Fill vector interpolations     !37
   N_e (2, 2:LT_FREE:2) = H    ! Fill vector interpolations     !38
   DLH = GET_DLH_AT_QP (IP)    ! SCALAR DERIVATIVES             !39
   DGH = MATMUL (AJ_INV, DLH)  ! PHYSICAL DERIVATIVES           !40
                                                                !41
!--->  FORM STRAIN DISPLACEMENT, B (OR GET_APPLICATION_B_MATRIX) !42
   CALL ELASTIC_B_PLANAR (DGH, B)                               !43
                                                                !44
!    EVALUATE ELEMENT STIFFNESS, MASS MATRICES                  !45
   S    = S + DET_WT * MATMUL (TRANSPOSE(B), MATMUL (E, B))     !46
   EL_M = EL_M + MATMUL (TRANSPOSE(N_e), N_e) * DET_WT * RHO    !47
                                                                !48
!    EVALUATE ANY BODY FORCE RESULTANTS                         !49
   IF ( HAS_BODY_FORCE ) THEN                                   !50
     C = C + MATMUL (TRANSPOSE(N_e), BODY_FORCE(1:2)) * DET_WT  !51
   END IF                                                       !52
                                                                !53
!    SAVE PT, CONSTITUTIVE & STRAIN_DISP FOR POST_PROCESS & SCP !54
   CALL STORE_FLUX_POINT_DATA (XYZ, E, B)                       !55
 END DO  ! Over quadrature points                               !56
!    ***  END ELEM_SQ_MATRIX PROBLEM DEPENDENT STATEMENTS ***   !57
```

Problem P12.7a *Plane strain analysis with body force and mass matrix*

```
! *** POST_PROCESS_ELEM PROBLEM DEPENDENT STATEMENTS FOLLOW ***    ! 1
!          STATIC PLANE_STRESS OR PLANE_STRAIN ANALYSIS            ! 2
! ------------------------------------------------------------    ! 3
! STRESS AND STRAIN COMPONENT ORDER: XX, YY, XY, SO N_R_B = 3      ! 4
!    PLUS ZZ & EFFECTIVE ARE N_R_B + 1 & 2                         ! 5
                                                                   ! 6
! PROPERTIES: 1-YOUNG'S MODULUS, 2-POISSON'S RATIO, AND           ! 7
!             3-YIELD STRESS, IF PRESENT                          ! 8
!             4-MASS DENSITY, IF PRESENT                          ! 9
                                                                   !10
INTEGER  :: J, N_IP                              ! loops           !11
REAL(DP), SAVE :: YIELD = HUGE (1.d0)            ! failure         !12
REAL(DP), SAVE :: PR = 0.25d0, RHO=1.d0          ! data            !13
                                                                   !14
 IF ( THIS_EL == 1 ) THEN ! PRINT TITLES & INITIALIZE              !15
   STRAIN = 0.d0 ; STRAIN_0 = 0.d0  ! INITIALIZE ALL "STRESS"      !16
   IF ( PLANE_STRAIN ) PRINT *, 'NOTE: USING PLANE STRAIN STATE'   !17
                                                                   !18
   WRITE (6, 50) ; 50 FORMAT ( /,                              & !19
   '*** STRESSES AT INTEGRATION POINTS ***',                /, & !20
   '       COORDINATES              STRESSES',              /, & !21
   'POINT  X       Y          XX           YY           XY', /, & !22
   'POINT                     ZZ           EFFECTIVE')           !23
 END IF ! NEW HEADINGS                                             !24
                                                                   !25
 IF ( EL_REAL > 1 ) PR    = GET_REAL_LP (2) ! set Poisson ratio    !26
 IF ( EL_REAL > 2 ) YIELD = GET_REAL_LP (3) ! set yield            !27
                                                                   !28
 WRITE (6,  * ) '        ELEMENT NUMBER ', IE                      !29
                                                                   !30
 CALL READ_FLUX_POINT_COUNT (N_IP) ! NUMBER OF QUADRATURE POINTS   !31
 DO J = 1, N_IP                    ! QUADRATURE POINT LOOP         !32
   CALL READ_FLUX_POINT_DATA (XYZ, E, B) ! PT, PROP, STRAIN_DISP   !33
                                                                   !34
!     MECHANICAL STRAINS & STRESSES                               !35
   STRAIN (1:N_R_B) = MATMUL (B, D)    ! STRAINS AT THE POINT     !36
   STRESS (1:N_R_B) = MATMUL (E, STRAIN (1:N_R_B)) ! STRESSES     !37
                                                                   !38
!     Z-STRESS INDUCED, ELSE ZERO IN PLANE STRESS                 !39
   IF ( PLANE_STRAIN ) STRESS (4) = PR * (STRESS (1) + STRESS (2)) !40
                                                                   !41
!     VON_MISES FAILURE CRITERION (EFFECTIVE STRESS, ADD TO END)  !42
   STRESS (5) = SQRT ( (STRESS (1) - STRESS (2) ) **2     &        !43
                     + (STRESS (2) - STRESS (4) ) **2     &        !44
                     + (STRESS (4) - STRESS (1) ) **2     &        !45
                     + 6 * STRESS (3) **2 ) * 0.707106812d0       !46
   IF ( STRESS (5) >= YIELD ) PRINT *,                   &        !47
     'WARNING: FAILURE CRITERION EXCEEDED IN ELEMENT =', IE        !48
                                                                   !49
!        LIST STRESSES AND FAILURE CRITERION AT POINT             !50
   WRITE (6, 52) J, XYZ (1:2), STRESS (1:3)                        !51
   WRITE (6, 51) J,            STRESS (4:5)                        !52
   52 FORMAT ( I3, 2(1PE11.3), 5(1PE14.5) )                        !53
   51 FORMAT ( I3, 22X,        5(1PE14.5) )                        !54
                                                                   !55
 END DO ! AT QUADRATURE POINTS                                     !56
!  *** END POST_PROCESS_ELEM PROBLEM DEPENDENT STATEMENTS ***      !57
```

Problem P12.7b *General planar stress post-processing*

Problem 12.3 Give an implementation of the $\mathbf{B}^e$ matrix for a general solid. Assume the strain components are in the *xx*, *yy*, *xy*, *zz*, *xz*, *yz* order. Solution: listing Problem P12.3.

Problem 12.4 Review the scalar implementation for the T3 element, given in Fig. 10.4.1, and extend it to the case of a two-dimensional stress model. Solution: an explicit hard code version of this element is given in Problem P12.4.

Problem 12.5 Implement a general evaluation of the elasticity $\mathbf{E}^e$ matrix that will handle all the 1-, 2-, and 3-dimensional cases. Solution: see listing Problem P12.5.

Problem 12.6 Implement a general evaluation of the elasticity $\mathbf{B}^e$ matrix that will handle all the 1-, 2-, and 3-dimensional cases. Solution: see listing Problem P12.6.

Problem 12.7 Modify the plane stress implementation shown in Figs. 12.4.5 and 6 to solve a plane strain problem. Remember that an additional normal stress must be considered in the post-processing stage. Validate the results with a patch test. Solution: The square matrix formulation is shown in P.12.7a which has been generalized to include the element mass matrix, *EL_M*, for possible use in a dynamic solution. If only static problems are of interest then line 47 should be omitted. Also of interest are lines 37 and 38 where the scalar interpolation array $\mathbf{H}^e$ has been inserted into the non-zero terms of the generalized interpolation array **N** via an implied loop in each row. The mass matrix is twice as big as in the scalar cases of Chapter 11 because we must account for mass distribution in both directions. Also note that $\mathbf{N}^e$ is used in computing the body force resultant vector in line 51. The necessary input, if globally constant, can be supplied by a control keyword *body_force* followed by a value for each spatial dimension. That also makes the logical variable *HAS_BODY_FORCE* true.

The post-processing is slightly generalized in P12.7b. It defaults to a plane stress post-processing ($\sigma_z = 0$) unless the keyword *plane_strain* appears in the control data. Line 40 shows the σ_z recovery if it is not zero. Likewise, the effective stress of line 43 is valid for either planar result. Since we had to recover two extra stresses at this stage the system allocates (*n_r_b* + 2) storage locations for the stress and strain arrays since one often wants to keep actual failure results with the stress or strain on which they are based. However, this sometimes requires the specification of range sizes, as in lines 36 and 37 that would not be necessary if recovered items were stored in separate arrays. The element formulation is valid for any element in the library, and the results and post-processing were validated using a patch test.