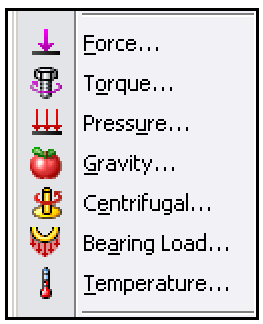
Stress Analysis Summary (from online notes Chapter 3)

Introduction

Finite element static stress analysis satisfies the differential equations for force and moment equilibrium (Newton's Laws). However, it does so by a mathematically equivalent method by minimizing the total potential energy of the system (a scalar quantity): $\Pi = U - W \ \rightarrow minimum$. Here, *U* is the strain energy (potential energy) stored in the structure and *W* is the external mechanical work done by forces acting on the structure. Detailed finite element analysis theory is covered in Mech 417.

These quantities are defined in terms of the displacement vector, $\vec{\delta} = u\vec{\imath} + v\vec{\jmath} + w\vec{k}$, at points within the solid. The displacement components (u, v, w) are calculated first and are the most accurate result from a stress study. Boundary conditions are applied to the displacement. Users must try to visualize how the (omitted) surroundings define the interface displacements.

Force vectors, $\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$, can be associated with each displacement vector. Forces can be applied at points, lines, area, and volumes. The *coefficient of thermal expansion* material property will cause forces (and strains) when there is a temperature change from a stress free condition. Cylindrical bearing joints have special load distribution shapes (Mech 401). The *mass density* material property is used in any gravity load calculations. The static force selections are:



Available force conditions in SWS

Displacements for Equilibrium

As an example of static equilibrium, consider a linear spring covered in introductory physics. The spring has one end fixed against displacement (the boundary condition), has a stiffness of *K*, and the free end is subjected to an axial force of *F* that causes the free end to develop an axial displacement of *u* when the spring is in equilibrium with the applied force. Recall that the elastic potential energy stored in the spring and the mechanical work done by the force are:

$$U = \frac{1}{2}Ku^2 \ \ and \ \ W = Fu$$

respectively, so the total potential energy is $\Pi = U - W$. For equilibrium we require that the unknown displacement, *u*, minimize the total potential energy. Thus, we take the derivative of Π with respect to *u* and set it to zero; and that is the algebraic equation of equilibrium:

$$\frac{\partial U}{\partial u} = 0 = \frac{2}{2} Ku - F.$$

Therefore, the unique displacement that corresponds to the state of equilibrium has the value

$$u = F/K$$

as expected from physics. For three-dimensional structures this process gives a matrix equation

$$\boldsymbol{Ku} = \boldsymbol{F}$$

which easily involves tens of thousands of displacement components, $\boldsymbol{u}$. The solution for the displacements (after the boundary conditions on $\boldsymbol{u}$ are enforced) requires numerical methods to solve the matrix equations. The symbolic solution is simple:

$$\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{F}.$$

Enough displacements must be prescribed to *prevent rigid body motion* of the system (three translations and three rotations). Otherwise, the solver will fail or give infinite displacements. Once all the displacements are known, the mechanical strains are calculated (below), the material properties are used to calculate the stresses, and finally the stresses or strains are used to calculate a material failure criterion.

Strains

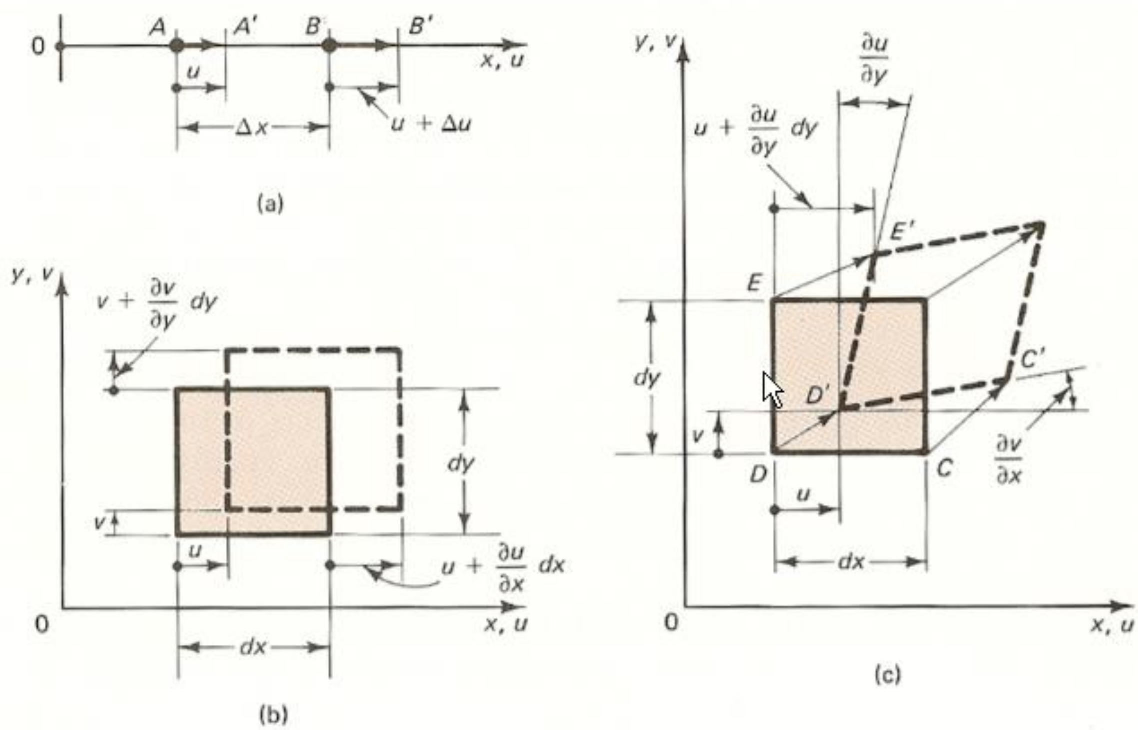
From the Strain-Displacement Relations (Mech 311), the gradient components of the displacements are used to create the six components of the symmetrical strain tensor, $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yy}, \varepsilon_{yz}, \varepsilon_{zz}$. The strain components define the geometrical distortions of a differential volume of material (see figure on the next page). The strains are dimensionless. The three normal strains (volume changes) have repeated subscripts, while the other three (angle changes) are shear strains.

Stress

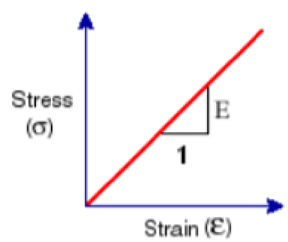
A stress, like a pressure, is a distributed force per unit area. A material law (Hooke's law) gives a constitutive tensor to define stresses in terms of strain. The material law uses the material properties of *modulus of elasticity*, E, and *Poisson's ratio*, ν. Hooke's Law defines the stresses as $\boldsymbol{\sigma} = \boldsymbol{E}\ \boldsymbol{\varepsilon}.$ The corresponding six stress components are $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz}$. The stresses are the least accurate quantity in a stress analysis. The modulus of elasticity, *E*, is usually tabulated to only three significant figures. Therefore, the stresses are generally only accurate to the same number of significant figures. The stresses or the strains are used to define a material failure criterion.

Failure Criterion

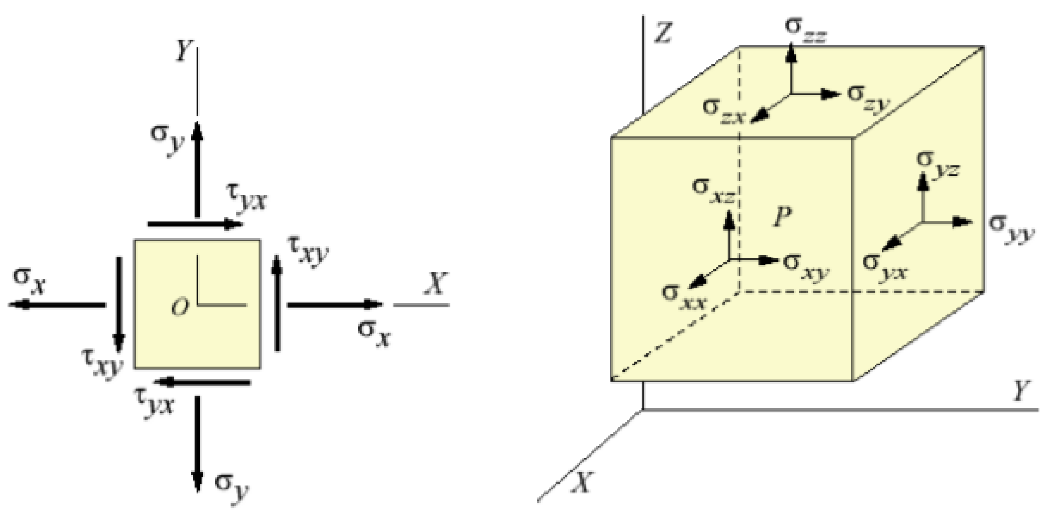
A course on materials science (Msne 301) will teach you which failure criteria to check to predict a material failure. For brittle materials (like ceramic) it is usually the maximum tension stress (P1); while for ductile materials (like steel) it is usually the von Mises distortional energy criterion (VON).



Geometry of normal strain (a) 1D, (b) 2D, and (c) 2D shear strain



Hooke's Law



Stress components in 2D (left) and 3D

For a 1D problem there is only one displacement, u, one force F_x, one normal strain $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ and one normal stress $\sigma_{xx} = E\varepsilon_{xx}$. For a 3D solid SolidWorks reports all six stresses as well as important combinations of them that define common material failure criteria (see below). Components P1, P2, P3 are the principle normal stresses (eigenvalues of stress), INT=(P3-P1) is twice the maximum shear stress, and VON is the distortional energy failure prediction for ductile materials (VON is a scalar that has the units of stress). The latter two are compared to the *yield stress* material property.

Nodal and element stress results

Symbol	Label	Item	Symbol	Label	Item
σ_x	SX	Normal stress parallel to x-axis	σ_1	P1	1st principal normal stress
σ_y	SY	Normal stress parallel to y-axis	σ_2	P2	2nd principal normal stress
σ_z	SZ	Normal stress parallel to z-axis	σ_3	P3	3rd principal normal stress
τ_{xy}	TXY	Shear in Y direction on plane normal to x-axis	τ_I	INT	Stress intensity (P1-P3), twice the maximum shear stress
τ_{xz}	TXZ	Shear in Z direction on plane normal to x-axis			
τ_{yz}	TYZ	Shear in Z direction on plane normal to z-axis	σ_{vm}	VON	von Mises stress (distortional energy failure criterion)

Impact Loads

A static simulation can be used for suddenly applied loads if the magnitude of the loads is increased by an Impact Factor (Mech 311) which always has a minimum value of two. If loads are specified with a long varying time history, then a linear dynamic study (Mech 420) is applied instead of a static forcing model.

Dynamic Loads

If you require a dynamic (time varying) load history, or a vibration study, the energy formulation is expanded to include the Kinetic Energy. Recall, for a particle $KE = \frac{1}{2}mv^2$ where *m* is the mass and *v* is the velocity. For a solid body the KE becomes an integral over the total volume, *V*: $KE = \frac{1}{2}\int_M v^2 dm = \frac{1}{2}\int_V \rho v^2 dV$. Thus, the *mass density* material property, ρ, of each component is utilized.

Data Reliability

Geometry: the solid model is generally the most accurate data, but they are often changed to generate the mesh.
Material: measured properties are fairly accurate for standard materials, but are known only to a few (<=4) significant figures. Custom materials may have a wide range of properties.
Loads: are less accurate since they require assumptions about their magnitudes and where they are applied (e.g., at a point or over a small area)
Fixtures (Known Displacement Values): displacement restraints are the least accurate data. All parts and assemblies are surrounded by the rest of the world. Fixtures are assumptions used to approximate

the effect of the rest of the world on the displacements of a part or assembly. They are essential for creating a valid physical model and drastically affect the results. Thus, several reasonable assumptions about how a part or assembly is restrained should be investigated for each design. The most common error is to assume that air is preventing displacements of a part at some surface region. I see this error in at least two design projects each year!

Result Reliability

Displacements are most accurate. Stresses and failure criterion are least accurate.

Alternate Element Types

Solid bodies are often approximated by reduced geometrical and mathematical models.

Solids (Mech 417)→Surface model → Shell (Ceve 516) or plate or plane stress (Mech 311) or plane strain

Solids (Mech 417) → Line model → Beam (Mech 311) or bar or truss (Mech 211)

Beams, plates, and shells also introduce infinitesimal rotations as unknowns in the analysis, and allow for additional boundary conditions on those rotations.