Examples 8.1-1 to -4 2/18/20 (see Downloads #7)

Checking the weak Neumann condition (NBC) of $\frac{\partial u(r)}{\partial x} = 0$ at x=1. For the quadratic element (L3)

$$u(r) = [(1 - 3r + 2r^2)\ \ (4r - 4r^2)\ \ (-r + 2r^2)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e, \qquad u(r) = [H_1(r)\ \ H_2(r)\ \ H_3(r)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e$$

$$\frac{\partial u}{\partial r}(r) = [(-3 + 4r)\ \ (4 - 8r)\ \ (-1 + 4r)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e, \qquad \frac{\partial u(r)}{\partial r} = \frac{\partial \boldsymbol{H}(r)}{\partial r} \boldsymbol{u}^e = \left[\frac{\partial H_1(r)}{\partial r}\ \ \frac{\partial H_2(r)}{\partial r}\ \ \frac{\partial H_3(r)}{\partial r}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e$$

$$\frac{\partial u(r)}{\partial x} = \frac{\partial u(r)}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u(r)}{\partial r}\left(\frac{\partial x}{\partial r}\right)^{-1}$$

$$x(r) = [(1 - 3r + 2r^2)\ \ (4r - 4r^2)\ \ (-r + 2r^2)] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^e, \qquad \frac{\partial x(r)}{\partial r} = \left[\frac{\partial H_1(r)}{\partial r}\ \ \frac{\partial H_2(r)}{\partial r}\ \ \frac{\partial H_3(r)}{\partial r}\right] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^e$$

In general, when the element nodes are not uniformly spaced like the parametric nodes then $\partial x/\partial r$ is a variable. In these examples the element length $L^e = L/n_e = 1/3$, and the nodes are uniformly spaced so

$$\frac{\partial x(r)}{\partial r} = [(-3 + 4r)\ \ (4 - 8r)\ \ (-1 + 4r)] \begin{Bmatrix} x_1 \\ x_1 + L^e/2 \\ x_1 + L^e \end{Bmatrix}^e$$

$$\frac{\partial x(r)}{\partial r} = \big(-3x_1 + 4(x_1 + L^e/2) - 1(x_1 + L^e)\big) + r\big(4x_1 - 8(x_1 + L^e/2) + 4(x_1 + L^e)\big) = (L^e) + r(0) = L^e$$

Thus the geometric Jacobian is constant and $\frac{\partial r}{\partial x} = \left(\frac{\partial x}{\partial r}\right)^{-1} = \frac{1}{L^e}$ and

$$\frac{\partial u(r)}{\partial x} = \frac{1}{L^e}\frac{\partial u(r)}{\partial r}$$

In the example where *u(0)=0* and *du/dx(L)=0*, with *L=1* and $L^e = 1/3$ the gathered u values at nodes 5, 6, 7 are

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e = \begin{Bmatrix} 0.3156 \\ 0.3429 \\ 0.3519 \end{Bmatrix}$$

and the physical end slope at the third element node (r = 1) is

$$\frac{\partial u(r=1)}{\partial x} = \frac{1}{3}[(-3 + 4)\ \ (4 - 8)\ \ (-1 + 4)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^e$$

$$\frac{\partial u(r=1)}{\partial x} = \frac{1}{3}\big((1)0.3156 + (-4)0.3429 + (3)0.3519\big) = -0.0001 \neq 0$$

In general, the finite element solution exactly satisfies the Dirichlet boundary conditions but only weakly satisfies the Neumann boundary conditions. The accuracy of the Neumann conditions can be enhanced by refining the mesh in the direction normal to the boundary.

Any boundary slope recovered from an element interpolation will always be less accurate than a reaction slope recovered from the system equilibrium equations, at the same point.

In general, a finite element solution is most accurate at the nodes and least accurate interior to an element. Conversely, the element gradient (slope) is least accurate at the nodes and most accurate in the element interior. Locations in the element interior where the element gradient is most accurate usually occur at the numerical integration points. In the literature, such points are often called 'super-convergent points'.