Equation 13.17 shows that our selections for generalized stress-strain measures will correctly define the strain energy in the system. Next, we need to define the work done by the applied loads, P_i, or couples, C_i. The work done by a transverse force is the product of the force and the transverse displacement. Likewise, the work done by a couple is the product of the couple and rotation (slope) at its point of application. These contributions define a work term, W, given by

$$W = \int_L v(x)\ p(x)\, dx + \sum_i v(x_i)\ P_i + \sum_j v'(x_j)\ C_j\,.$$

The last two terms represent work done by concentrated point loads or couples. Thus, the total potential energy, $\Pi = U - W$ is

$$\Pi = \frac{1}{2} \int_L EI\,(\, v''(x)\,)^2\, dx - \int_L v(x)\ p(x)\, dx\ - \sum_i v_i\ P_i - \sum_j v'_j\ C_j\,. \tag{13.18}$$

To determine the displacement field, $v(x)$, that corresponds to the equilibrium state we must minimize Π and satisfy the boundary conditions on v and $v' = \theta$.

13.10 Hermite element matrices

To introduce our finite elements we select a series of line segments to make up the region L. There are numerous elements that could be selected. First we will select an element with two nodes. Next, it is necessary to assume a displacement approximation so we can evaluate the potential energy in Eq. 13.16. That equation contains second derivatives and thus we need to assume a solution for v that will at least have both the deflection, v, and the slope, v', continuous between elements. The most common assumption is to select the cubic Hermite polynomial presented in Fig. 3.6. The unknowns at each of the two element nodes are v and $v' = \theta$. These quantities will be called our *generalized displacements* or the generalized degrees of freedom. Thus, our element interpolation functions are the Hermite form in Fig. 3.6:

$$v(x) = \left[\ H_1^e(x)\ H_2^e(x)\ H_3^e(x)\ H_4^e(x)\ \right] \begin{Bmatrix} v_1 \\ v'_1 \\ v_2 \\ v'_2 \end{Bmatrix}^e$$

or $v(x) = \mathbf{H}^e(x)\ \boldsymbol{\delta}^e$, where $\boldsymbol{\delta}^e$ denotes the generalized displacements of the element. The derivatives of the displacements are

$$v'(x) = \theta(x) = \mathbf{H}^{e'}(x)\ \boldsymbol{\delta}^e\,, \qquad v''(x) = \mathbf{H}^{e''}(x)\ \boldsymbol{\delta}^e\,. \tag{13.19}$$

Since v'' and $\boldsymbol{\delta}^e$ have been selected as our generalized strains and generalized displacements we will use the notation of Eq. 13.19 and write Eq. 13.17 as

$$\varepsilon^e = \mathbf{B}^e\ \boldsymbol{\delta}^e$$

where $\varepsilon = v''$ in our present study. In the study of plates and shells additional curvature terms would be present in ε. Employing our generalized notation the stiffness matrix and distributed load vector can be written by inspection as

$$\mathbf{K}^e = \int_{L^e} \mathbf{B}^e(x)^T \ \mathbf{D}^e(x) \ \mathbf{B}^e(x) \, dx \ , \quad \mathbf{F}_p^e = \int_{L^e} \mathbf{H}^e(x)^T \ p^e(x) \ dx \ .$$

Here we will again use unit coordinates on the element and set $r = x / L^e$ so that $d(\ \ ) / dx = d(\ \ ) / dr \times 1 / L^e$. Thus,

$$\mathbf{B}^e = \mathbf{H}^{e''} = \frac{1}{L^2} \frac{d^2\mathbf{H}}{dr^2}$$

so for the cubic Hermite in Fig. 3.6 this becomes (with $L = L^e$)

$$\mathbf{B}^e = \frac{1}{L^2} \Big[\ (12r - 6) \quad L(6r - 4) \quad (6 - 12r) \quad L(6r - 2) \ \Big].$$

Recalling that

$$\int_L r^m \, dx = \frac{L}{(m + 1)}$$

and assuming that $\mathbf{E}^e$ is a constant then the stiffness (with $L = L^e$) is

$$\mathbf{K}^e = \frac{EI}{L^3} \begin{bmatrix} 12 & & & \text{sym.} \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}. \tag{13.20}$$

If the lateral load, p^e, is constant then

$$\mathbf{F}_p^e = p^e \int_{L^e} \mathbf{H}^{e^T} \, dx = p^e \ L^e \int_0^1 \mathbf{H}^{e^T}(r) \, dr = p^e \ L^e \begin{Bmatrix} 1/2 \\ L^{e^T}/12 \\ 1/2 \\ -L^{e^T}/12 \end{Bmatrix}. \tag{13.21}$$

Note that the distributed load puts half the resultant load at each end. It also causes equal and opposite nodal couples at each of the two nodes.

When we wrote Eq. 13.18 we assumed that point loads would only be applied at the node points. This may not always be true and we should consider such a load condition as a special case of a distributed load. In that case the length of the distributed load approaches zero and the magnitude of the force per unit length approaches infinity, but the resultant load P is constant. That is, we define the load to be $p(x) = P \ \delta(x - x_0)$ where δ is the Dirac Delta distribution. Then the generalized load vector is

$$\mathbf{F}_p^e = \int_{L^e} \mathbf{H}^{e^T}(x) P \ \delta(x - x_0) \, dx$$

which is integrated by inspection by using the integral property of the Dirac Delta to yield $\mathbf{F}_p^e = P \, \mathbf{H}^e(r_0)$ where $r_0 = x_0 / L$ is the point of application of the load. To check this concept, assume that the load is at Node 1. Then, $r_0 = 0$ so that

$$\mathbf{F}_p^{e^T} = P \, [\, 1 \quad 0 \quad 0 \quad 0 \,]$$

as expected. That is, all the force goes into Node 1 and no element nodal moments are generated. Other common loading conditions can be treated in the same way and a number have been tabulated by Akin [1] and others. As another example, if $p(x)$ varies linearly from p_1^e to p_2^e at the nodes of element e then

$$p(x) = (1 - x/L^{e)}\ p_1^e + x/L^e\ p_2^e = [\,(1-r) \quad r\,]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^e$$

and

$$\mathbf{F}_p^e = \int_{L^e} \begin{Bmatrix} 1-3r^2+2r^3 \\ L^e(r-2r^2+r^3) \\ 3r^2-2r^3 \\ L^e(r^3-r^2) \end{Bmatrix} p(x)\,dx = \frac{L^e}{20}\begin{bmatrix} 7 & 3 \\ L^e & 2L^e/3 \\ 3 & 7 \\ -2L^e/3 & -L^e \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^e. \quad (13.22)$$

If the load is constant so that $p_1^e = p_2^e = p_e$, then this reduces to Eq. 13.21, as expected. Likewise, if $p_1^e = 0$ and $p_2^e = p$, this defines a triangular load and

$$\mathbf{F}_p^{e^T} = \frac{PL}{20}[\,3 \quad 2L/3 \quad 7 \quad -L\,]. \quad (13.23)$$

It is common to tabulate such results in terms of an applied unit resultant load. That resultant is

$$R^e = \int_{L^e} p^e(x)\ dx\,.$$

For common load variations, such as constant, linear, parabolic, and cubic forms where p varies in proportion to r^n, the resultant loads are $R^e = pL/(n+1)$. The location, $\bar{x}$, of the resultant applied load is found from

$$\bar{x}\,R^e = \int_{L^e} px\ dx$$

and the corresponding results are $\bar{x} = L(n+1)/(n+2)$. Thus, if we normalize Eq. 13.23 by dividing the resultant load, $pL/2$, the result is

$$\mathbf{f}_p^{e^T} = [\,3/10 \quad L/15 \quad 7/10 \quad -L/10\,].$$

We can also check the unit load results by applying statics to the data in that figure. To check the load summary, we first take the sum of the moments about Node 1. This gives

$$+1 = 0 + (7/10)L + L/15 - L/10 = L(21+2-3)/30, \quad \text{OK}.$$

Similarly, the sum of the moments about Node 2 is verified.

13.11 Cantilever with triangular load

To present an analytic example of this element consider a single element solution of the cantilever beam shown in Fig. 13.24 to determine the deflection and slope at the free end. Usually the deflected shape of a beam is defined by a fourth or fifth order polynomial in x. Thus, our cubic element solution will usually give only an approximate solution. For a single element the system equations are

$$\frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} v_1 \\ \theta_1 \\ \hline v_2 \\ \theta_2 \end{array}\right\} = \frac{WL}{2}\left\{\begin{array}{c} 3/10 \\ L/15 \\ \hline 7/10 \\ -L/10 \end{array}\right\} + \begin{Bmatrix} 0 \\ 0 \\ F_2 \\ M_2 \end{Bmatrix}.$$

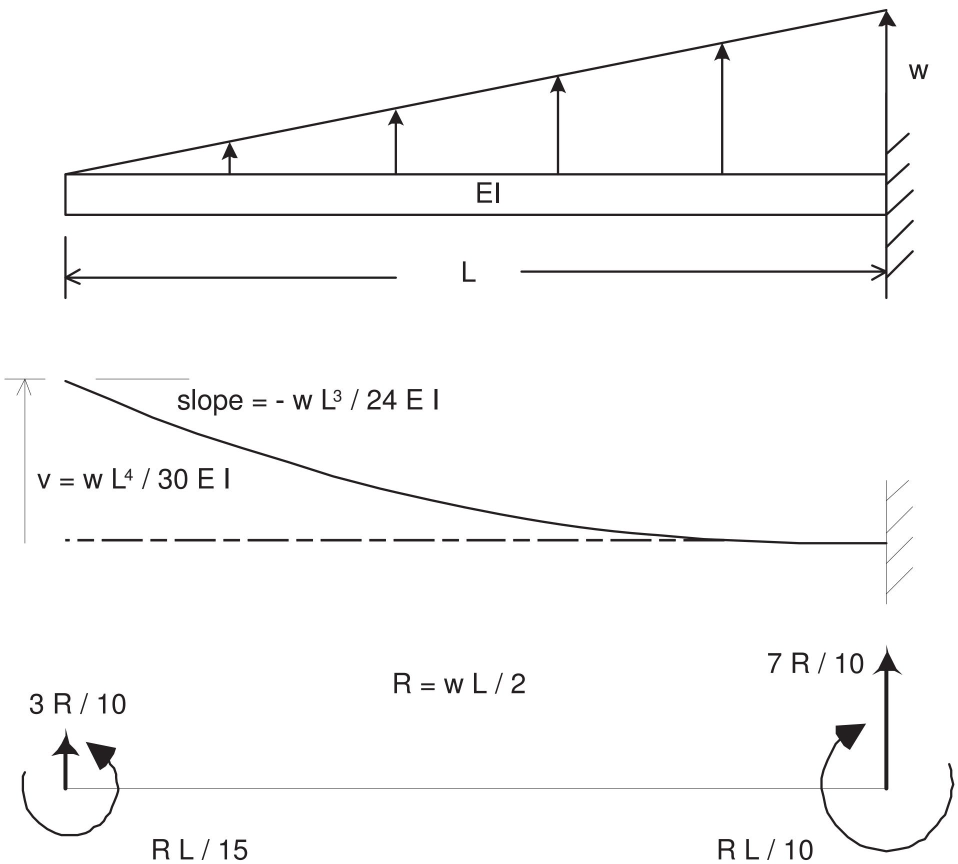

Figure 13.24 *A single element approximate solution*

The right side support requires that $v_2 = 0 = \theta_2$. The reduced equations become

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \frac{WL}{2}\begin{Bmatrix} 3/10 \\ L/15 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

so that

$$\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \frac{L^3}{12\,EIL^2}\begin{bmatrix} 4L^2 & -6L \\ -6L & 12 \end{bmatrix}\begin{Bmatrix} 3/10 \\ L/15 \end{Bmatrix}\frac{WL}{2} = \frac{WL^3}{EI}\begin{Bmatrix} L/30 \\ -1/24 \end{Bmatrix}.$$

The exact solution is $120\,EI\,v = wL^4\,[\,4 - 5x/L + (x/L)^5\,]$ so that the exact values of the maximum deflection and slope are $v = WL/(30\,EI)$ and $\theta = -\,WL^3/(24\,EI)$, respectively. Thus, our single element solution gives the exact values of both v and θ at the nodes, but is only approximate in the interior of the element. The last two equations give the exact reactions.

In practical analysis one can often utilize a partial model to reduce the data preparation and more importantly the analysis cost. One must be alert for planes where the geometry, material property, supports and loads are symmetric mirror images. Often the loading conditions occur in an anti-symmetric, or negative mirror image, fashion so that one can still use a half portion model and simply recognize that the deflections and