14. Eigen-analysis

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14.1 Introduction: Another common type of analysis problem is where a coefficient in the
gevning differential equation, $\lambda$, is an unknown global constant to be determined. There is a
group of solutions to such a differential equation. This class of analysis is called an eigen-
problem from the German word eigen meaning belonging distinctly to a group. Eigen-problem
solutions require much more computation effort than solving a linear system of equations.

One application is acoustical vibrations where the pressure changes at a particular frequency
are determined in a closed volume. Today, automobile companies conduct extensive acoustical
studies for each automobile passenger compartment in order to design high quality sound
systems. Electromagnetic waveguides involve eigen-problem solutions to design radar systems,
microwave ovens and several other electrical components. The surface elevations of shallow
bodies of water, like harbors, are subjected to periodic excitations by the moon. Tidal studies are
called Seiche motion studies and are governed by eigen-problem solutions. Those problems, and others, are described by the scalar Helmholtz differential equation:

\[ \nabla^2 u(x, y, t) + \lambda u(x, y, t) = 0 \]  

which is solved a group for multiple eigen-values, \( \lambda_n \), and a corresponding set of solution eigenvectors to define the group of solutions. Another common application leading to an eigen-problem analysis is the vibration of elastic solids where the eigenvalues and their corresponding deflected mode shapes are to be determined. In theory, a continuous system has an infinite number of eigenvalues, and a finite element approximation of a system has as many eigenvalues as there are degrees of freedom (after enforcing the EBC). Most applications are interested in the first few or the last few eigenvalues and eigenvectors. There are efficient algorithms for both approaches.

For waveguides and other applications the eigenvalue can be a complex number, but for elastic vibrations it is a non-negative number. For elastic vibrations, a positive value corresponds to the square of the natural frequency of vibration, \( \lambda = \omega^2 \), and a zero value corresponds to a rigid body (non-vibrating) motion. There are at most six rigid body motions of a solid. Theoretically a vibration analysis should not yield a negative eigenvalue, \( \lambda < 0 \), but numerical errors can produce them as an approximation of a rigid body motion.

A practical finite element eigen-problem study can involve tens of thousands of unknowns after a few boundary conditions are imposed but the engineer is usually interested in less than ten of its eigenvalues and eigenvectors. The eigenvectors of interest are usually just examined as plotted mode shapes. For example, in linear buckling theory only the single smallest eigenvalue is useful and its plotted buckled mode shape can imply where additional supports will increase its buckling capacity.

Here, the examples are presented utilizing Matlab and some specific features of that environment need to be noted. Matlab provides the two functions \texttt{eig} and \texttt{eigs} for eigen-problem solutions. The finite element method provides two matrix arguments, say \( K \) and \( M \), that (after EBCs) are real, symmetric, and non-negative that define the problem as solving

\[ [K - \lambda_j M] \delta_j = 0, \ j = 1, 2, ... \]

For a ‘small’ numbers of degrees of freedom the function call \([V, \Lambda] = \text{eig} (K, M)\) returns two square matrices where each eigenvector, \( \delta_j \), as the j-th column of the first square matrix, \( V \), and the corresponding eigenvalue, \( \lambda_j \), is placed in the j-th row diagonal term of the matrix \( \Lambda \). In other words, the returned row number, j, of any eigenvector, \( \delta_j \), is also the column number of its corresponding eigenvector. The eigenvalues can be conveniently placed in a vector, with the same row numbers, by using the Matlab function \texttt{diag} as \( \lambda = \text{diag} (\Lambda) \).

In finite element applications the eigenvalues, \( \lambda \), can be complex numbers, but for common vibration problems they are positive real numbers that are the square of the natural frequency, or zero for rigid body motions (a maximum of six). Numerical round-off errors can make the theoretical positive number have a tiny complex part. Thus, the safe thing to do is to use the Matlab function \texttt{real} to transform \( \omega^2 = \lambda = \text{real}(\lambda) \) in vibration studies.

However, the eigenvalues, \( \lambda \), are NOT always in a sequential order either increasing or decreasing. For a buckling study, only the smallest eigenvalue is needed. That can be extracted
by using the Matlab function \( \text{min} \). But it is also necessary to extract its eigenvector that shows the buckled shape of the structure. That can be done using the extended \( \text{min} \) function that returns two arguments:

\[
[\lambda_1, \text{row}_1] = \text{min}(\lambda)
\]

where the smallest eigenvalue \( \lambda_1 \) was found in row \( \text{row}_1 \) of the vector \( \lambda \) and where the corresponding eigenvector column can be extracted by next using the row number as a column number and setting \( \delta_1 = V(:, \text{row}_1) \).

For a medium sized problem, say where you want three \( \lambda_j \) out of 20 degrees of freedom, you need to find those three smallest values in the vector \( \lambda \). That is done using the extended Matlab function \( \text{sort} \) as:

\[
[\lambda_{\text{new}}, \text{Order}] = \text{sort} (\lambda)
\]

where the vector \( \lambda_{\text{new}} \) contains all of the eigenvalue in ascending order such that the smallest eigenvalue is \( \lambda_{\text{new}}(1) = \lambda_{\text{small}} \), and where the vector subscript \( \text{Order} \) gives the row number where the new sorted value was found in the original random list of eigenvalues. The vector subscript \( \text{Order} \) is very important since it is the key to finding the eigenvector that corresponds to a particular eigenvalue. For example, the smallest eigenvalue and its eigenvector are \( \lambda_{\text{new}}(1), \delta_1 = V(:, \text{Order}(1)) \) and the fifth pair is \( \lambda_{\text{new}}(5) \) and \( \delta_5 = V(:, \text{Order}(5)) \).

For very large problems where the analysts need only a small number of the smallest eigenvalues, as in mechanical vibrations, or a small number of the largest eigenvalue then the call to the Matlab function \( \text{eigs} \) provides the much more efficient eigenvalue control with:

\[
[V, \Lambda] = \text{eigs} (K, M, n, 'sm').
\]

where the number \( n \) is the number of the eigenvalues required, and the string ‘sm’, requests the smallest eigenvalues. This choice requires the least dynamic memory since \( \Lambda \) is a small \( n \times n \) matrix and \( V \) is a rectangular matrix with only \( n \) columns. However, the eigenvalues on the diagonal of \( \Lambda \) are still usually in a random order and it is still necessary to sort them and to use the \( \text{Order} \) subscripts to extract the proper eigenvectors.

There are times when the largest eigenvalue of a finite element matrix is required. For example, in time history solutions the time step size limit for a stable solution depends on the inverse of the largest eigenvalue of the system. The \( \text{eigs} \) function defaults to giving the largest eigenvalues. The Iron’s Bound Theorem provides bounds on the extreme eigenvalues by using the corresponding element matrices (as they are built for assembly):

\[
\lambda_{\text{min}}^e \leq \lambda_{\text{min}} \leq \lambda_{\text{max}} \leq \lambda_{\text{max}}^e .
\]  \hspace{1cm} (14.1-2)

This means that the largest eigenvalue in the system is less that the largest eigenvalue of its smallest element.

**14.2 Finite element eigen-problems:** When a differential equation having an unknown global constant is solved by the finite element method the global constant factors out of all of the element matrices and appears in the assembled governing matrix system. The typical form of the matrix system becomes the ‘general eigen problem’:
\[ [K - \lambda M] \delta = 0 \]  \hspace{1cm} (14.2-1)

where \( K \) is typically a stiffness matrix or conduction matrix, \( M \) is typically a generalized mass matrix or a literal mass matrix for vibration problems, \( \delta \) corresponds to the nodal values of the primary unknowns (displacements, acoustical pressure, water elevation, etc.), and \( \lambda \) is the global unknown constant to be determined.

The consistent finite element theory creates a full symmetric element mass matrix. Based on prior finite difference methods which always create a diagonal mass matrix some users prefer to diagonalize the consistent mass matrix by using a diagonal matrix constructed from scaling up the original diagonal so the sum of the new diagonal equals the total mass. Some numerical experiments have shown improved numerical accuracy for both eigenvalue and time history solutions when the average of the consistent mass matrix and its diagonalized form are employed (the averaged mass matrix).

Recall that the integral form introduces the nonessential boundary conditions into the element matrices which are assembled into the system matrices. The essential boundary conditions must have been enforced such that \( \delta \) represents only the free unknowns in (14.2-1). Usually the EBC specify zero for the known nodal values.

Equation (14.2-1) represents a matrix set of linear homogeneous equations. For those equations to have a non-zero solution the determinant of the square matrix in the brackets must vanish. That is,

\[ \det[K - \lambda M] = |K - \lambda M| = 0 \]  \hspace{1cm} (14.2-2)

That condition, leads to a group of solutions (eigen-problems) equal in number to the number of free unknowns in \( \delta \) after the EBC have been enforced:

\[ [K - \lambda_j M] \delta_j = 0, \quad j = 1, 2, ... \]  \hspace{1cm} (14.2-3)

where the \( \lambda_j \) are the eigenvalue corresponding to the eigenvector \( \delta_j \). The usual convention is to normalize the eigenvector so that the absolute value of its largest term is unity.

For vibration studies the eigenvalue is the square of the natural frequency; \( \lambda = \omega^2 \). When calculated by a finite element approximation the frequency (in radians per second) \( \omega_n \) is more accurate than the next higher frequency, \( \omega_{n+1} \). As more degrees of freedom are added each natural frequency estimate becomes more accurate. Generally, engineers are interested in a small number (\( \leq 10 \)) of natural frequencies. Solutions should continue to use an increased number of degrees until the highest desired eigenvalue is unchanged to a desired number of significant figures after additional degrees of freedom are included in the model.

For solid vibrations, a non-zero EBC or a non-zero forcing term must be enforced before reaching the form of (14.2-1). Those conditions are imposed on a static solution to determine the stress state they cause. That solid stress state is used in turn to calculate the additional element “geometric stiffness matrix” which is assembled into the system geometric stiffness matrix, say \( K_G \), (also known as the initial stress matrix) that is added to the original structural stiffness matrix to yield the net system stiffness matrix, \( K \), in (14.2-1). That process is known as including stress stiffening effects on the vibration problem.

For structural buckling problems the system geometric stiffness matrix is the second matrix in (14.2-1) and \( \lambda \) becomes the unknown ‘buckling load factor’, BLF, which can be positive or negative:
\[ \mathbf{K} - BLF \mathbf{K}_o \delta = 0 \]  

(14.2-4)

In that case, the system degrees of freedom, \( \delta \), represent the (normalized) structural displacements in the (linearized) buckled shape.

![Figure 14.3-1 A two DOF spring-mass system](image)

**14.3 Spring-mass systems:** Most engineers are introduced to vibrations through the simple harmonic motion of a system of massless linear springs jointed at their ends by point masses, as shown in Fig. 14.3-1. That system, before the EBC, has three displacement degrees of freedom, \( \delta \). Thus, the point masses have a system diagonal mass matrix of

\[
\mathbf{M} = \begin{bmatrix}
0 & 0 & 0 \\
0 & m_1 & 0 \\
0 & 0 & m_2 \\
\end{bmatrix}.
\]

Recall that each linear spring element has a stiffness matrix of \( \mathbf{k}^e = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \), so the assembled stiffness matrix in the equations of motion, \([\mathbf{K} - \omega^2 \mathbf{M}]\delta = 0\), before EBC is

\[
\mathbf{K} = \begin{bmatrix}
k_1 & -k_1 & 0 \\
-k_1 & (k_1 + k_2) & -k_2 \\
0 & -k_2 & k_2 \\
\end{bmatrix}.
\]

Enforcing the EBC that the displacement (and velocity and acceleration) of the first node is zero eliminates the first row and column of the two system matrices and yields the eigenproblem to be solved for the two vibration frequencies of the spring-mass system as

\[
\begin{bmatrix}
(k_1 + k_2) & -k_2 \\
-k_2 & -k_2 \\
\end{bmatrix} - \omega^2 \begin{bmatrix}
m_1 & 0 \\
0 & m_2 \\
\end{bmatrix} = 0.
\]

Of course, the simplest spring-mass system is that of a single massless spring and a single point mass. Then the determinant of the matrix system reduces to \( |k_1 - \omega^2 m_1| = 0 \), which gives the single degree of freedom result that its natural frequency (in radians per second) is

\[
\omega = \sqrt{k/m}.
\]

(14.3-1)

However, when continuous (continuum) solutions are modeled then the elastic body is no longer massless and its spatial distribution of mass must be considered as must the spatial distribution of its elastic properties. That requires using differential equations to represent the
equation of motion. Even then, occasional engineering approximations will introduce point masses and/or point springs into the resulting matrix system. The supplied function library provides tools to define (input) and assemble point masses and/or point stiffnesses (springs) into the matrix systems. Those data are stored in text file msh_mass_pt.txt and/or msh_stiff_pt.txt and are read by functions get_and_add_pt_mass.m, etc.

There are handbook analytic solutions for the natural frequencies for the most common homogeneous continuous bars, beams, shafts, membranes, plates, and shells for numerous boundary conditions and including point spring supports and local point masses. The finite element method allows for the fast modelling of the vibration of any elastic solid. Every user should validate any finite element result with a second calculation and handbooks provide one useful check for that task.

Example 14.3-1 Given: A vertical axial bar of length \( L \) and area \( A \) has an elastic modulus of \( E \) and a mass density of \( \rho \) is fixed at one end. The other end is connected to equipment considered to be a point mass of value \( M \). Approximate the natural frequency of this system including and neglecting the equipment mass. Solution: Use a two-node linear bar element with

\[
K^e = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad m^e = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad m^p = M \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

An elastic bar behaves the same as a linear spring. Denote the usual bar ‘axial stiffness’ as \( k = EA/L \) and its total bar mass as \( m = \rho AL \), and assemble the line element and point element matrices to form the system natural frequency relation that

\[
\begin{bmatrix} k & -\omega_j^2 \left( \frac{m}{6} [2 1] + M [0 0] \right) \end{bmatrix} = 0
\]

Applying the essential boundary condition reduces this to a single dof problem:

\[
\left| k[1] - \omega_1^2 \left( \frac{m}{6} [2] + M[1] \right) \right| = 0
\]

Thus, the first (and only available) natural frequency is \( \omega = \sqrt{k/(M + m/3)} \), rad/sec. This shows that increasing the end mass decreases the frequency. Had the elastic bar been treated as a massless spring \( (m = 0) \) this would be the exact frequency. For an elastic bar without an end mass this becomes \( \omega = \sqrt{3k/m} \). Compared to the exact frequency of \( \omega = \pi/2 \sqrt{k/m} \) the single linear element estimate is in error by about 10.35.

14.4 Vibrating string: The differential equation of the transverse motion of an elastic string, without bending resistance, damping or external transverse loads is

\[
T \frac{\partial v^2}{\partial x^2} - \rho \frac{\partial v^2}{\partial t^2} = 0 \quad (14.4-1)
\]

where \( v(x,t) \) is the transverse displacement of the string with a tension of \( T \), and a mass density per unit length of \( \rho \), and \( t \) denotes time. Later, when time histories are studied in detail this will
be referred to as a wave equation. This PDE is of the hyperbolic class. From physics, it is known that strings (and most elastic bodies) vibrate with simple harmonic motion (SHM). Applying that assumption here, define a separation of variables

\[ v(x, t) = v_0(x) \sin(\omega t) \]  

(14.4-2)

where \( v_0(x) \) is a “mode shape” defining the shape of the string along the x-direction as it changes between positive and negative values that change with a frequency of \( \omega \). The second time derivative of that assumption is

\[ \frac{\partial v^2}{\partial t^2} = -\omega^2 v_0(x) \sin(\omega t) = -\omega^2 v(x, t) \]

The assumption of SHM changes the hyperbolic PDE over time and space into an elliptic ODE in space:

\[ T \frac{d^2v}{dx^2} + \rho \omega^2 v = 0 \]

(14.4-3)

When the ODE has an unknown global constant \( (\omega^2) \) multiplying the solution value it is called a scalar Helmholtz equation. This has the usual EBC and NBC options to define a unique solution. For a guitar string, the usual EBCs are that both ends have zero transverse displacements.

Next, introduce a finite element mesh and the usual element interpolations that in each element

\[ v(x) = H(r) \mathbf{v}^e = (H(r) \mathbf{v}^e)^T = \mathbf{v}^e^T H(r)^T \]

Noting that the frequency, \( \omega \), is an unknown global constant it is pulled outside the spatial integral and will also be pulled outside the governing matrix system: \([K - \omega^2 M]\{v\} = \mathbf{c}_{\text{NBC}}\).

When the EBCs of \( v(0) = 0 = v(L) \) are enforced the rows in \( \mathbf{c}_{\text{NBC}} \) associated with the remaining free DOFs are zero and the reduced problem is the same as (14.2-2).

The typical element stiffness and consistent mass matrices are

\[ \mathbf{K}^e = \int_{L_e} H(r)^T \frac{dH(r)}{dx} \frac{dH(r)}{dx} dx, \quad \mathbf{M}^e = \int_{L_e} H(r)^T \rho \mathbf{L} H(r) dx \]

(14.4-5)

For constant properties and a constant Jacobian the closed form matrices for the two-node and three-node Lagrangian line elements are

\[ \mathbf{K}^e = \frac{T_e}{3L_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, \quad \mathbf{M}^e = \frac{\rho T_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \]

(14.4-7)
respectively. The resulting vibration shapes (mode shapes) come in sets that are either symmetric or anti-symmetric with respect to the center of the string.

Note that the assumptions imply that the string has slope continuity along its entire length (with unknown values at the support points). That observation suggests that using Hermite elements will give more accurate eigenvalues and more physically realistic mode shape plots. For the two-node cubic Hermite element the nodal unknowns are the string deflection and its slope $\nu^T = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2]$, where $\theta$ denotes the slope of the string. The element matrices are

$$K^e = \frac{T^e}{30 L^e} \begin{bmatrix} 36 & 3L^e & -36 & 3L^e \\ 3L^e & 4L^3 & -3L & -L^2 \\ -36 & -3L^2 & 36 & -3L^2 \\ 3L^e & -L^2 & -3L & 4L^2 \end{bmatrix},$$

$$M^e = \frac{\rho^e L^e}{420} \begin{bmatrix} 156 & 22L^e & 54 & -13L^e \\ 22L^e & 4L^3 & 13L^e & -3L^2 \\ 54 & 13L^2 & 156 & -22L^e \\ -13L^e & -3L^2 & -22L^e & 4L^2 \end{bmatrix}. \quad (14.4-8)$$

Similarly, if the three-noded quintic element is used to solve Eq. 14.4-1 then the matrices are:


$$M^e = \frac{\rho A L}{13,860} \begin{bmatrix} 2,092 & 114 L & 880 & -160 L & 262 & -29 L \\ 114 L & 8 L^2 & 88 L & -12 L^2 & 29 L & -3 L^2 \\ 880 & 88 L & 5,632 & 0 & 880 & -88 L \\ -160 L & -12 L^2 & 0 & 128 L^2 & 160 L & -12 L^2 \\ 262 & 29 L & 880 & 160 L & 2,092 & -114 L \\ -29 L & -3 L^2 & -88 L & -12 L^2 & -114 L & 8 L^2 \end{bmatrix}. \quad (14.4-9)$$

The disadvantage of that approach is that the eigen-problem is twice as large as the one using Lagrange interpolation and the same number of nodes. But it should give results more accurate than a Lagrange element solution with twice as many nodes (without interior supports).

**Example 14.4-1 Given:** A guitar string is needed for a musical instrument. Determine how the tension of the string affects the first natural frequency of the string. **Solution:** The first symmetric mode is the fundamental one. Approximate it using a single quadratic element, with both ends fixed. Then $L^e = L$ and the system matrices are

$$\begin{pmatrix} \frac{T}{3L} & 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} - \omega^2 \frac{\rho L}{30} \begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -c(0)_{NBC} \\ 0 \\ c(L)_{NBC} \end{pmatrix}$$
Enforcing the two EBC gives

\[
\left( \frac{T}{L} [16] - \omega^2 \frac{\rho L}{10} [16] \right) \{v_2\} = \{0\}
\]

For a non-trivial solution, \(v_2 \neq 0\), the determinant of the square matrix in the brackets must vanish. That happens only for specific values of \(\omega_k\), \(1 \leq k \leq n_d\) that are equal in number to the number of free DOF in the mesh. Here, there is only one so

\[
\omega_1^2 = 10 T/\rho L^2, \quad \omega_1 = 3.1623 \sqrt{T/\rho L^2}
\]

The exact solution for the \(k\)-th mode is \(\omega_k = k \pi \sqrt{T/\rho L^2}\), rad/sec where odd values of \(k\) are symmetric modes and even values are non-symmetric. The answer to the given question is that the frequency of the string vibration increases with the square root of its tension.

Here, the single quadratic element has only 0.66 % error in the first natural frequency. The first mode shape is a half-sine curve, the amplitude of which is normalized to unity. The half-sine mode shape is approximated spatially by a parabolic segment in a single element model.

**Example 14.4-2 Given:** Write a Matlab script to find the first three frequencies of the tensioned string using a uniform mesh with two quadratic Lagrangian line elements where \(T = 6e5 N\), \(L = 2 m\), and \(\rho = 0.0234 kg/m\). **Solution:** Figure 14.4-1 details all of the required calculations (but it does not list the computed mode shapes. The coordinates and properties are set manually and (14.4-7) is inserted to build the element matrices once since they are the same for each element. The loop over each element defines the element connection list manually and uses it to assemble the two elements. The EBCs eliminate the first and fifth DOF, so only rows and columns 2 through 4 of the two square matrices are passed to the Matlab function `eig.m` which returns the mode shapes (as columns of a square matrix, and the square of the natural frequencies \(\omega_k^2\)) as the diagonal elements of a second square matrix. Then the square root gives the three actual frequencies, which are compared to the exact values. (The script `String_vib_L3.m` is included in the Applications Library.) The first frequency is accurate to 0.4%, the second is accurate to 0.6%, and the third has significantly increased to 20%. The execution of the script lists the mode shapes as

<table>
<thead>
<tr>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.7068</td>
<td>-1.0000</td>
<td>0.4068</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.0000</td>
<td>-1.0000</td>
</tr>
<tr>
<td>0.7068</td>
<td>1.0000</td>
<td>0.4068</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

where the zero boundary displacements were added to the two ends so the complete (normalized) mode shapes can be plotted, as shown in Figure 14.4-2. There, the dashed line is the exact mode shape. Note that the first two modes with low frequency error have small changes in slope between the two elements. However, the third mode has a large change in slope between the
elements. The odd mode numbers are symmetric and should have a zero slope at the center point. A refined mesh of several small elements would reduce the change in slope between elements. Even though the differential equation order does not require the use of C1 elements it is known from the analytic solutions that the slope of the tensioned string should be continuous. Thus, Hermite elements would give more accurate frequencies. This fact is illustrated in Fig. 14.4-3 where the quintic Hermite polynomial was used to create the stiffness and mass matrices in (14.4-9) and to solve for the first four natural frequencies by using a single three noded element. Figure 14.4-4 shows the increased accuracy in the fourth mode shape when two quintic elements are employed.

When a structure has symmetric geometry, materials, and boundary conditions it is common to be able to use only half the structure. Using the zero slope natural boundary condition at the center point yields only the odd numbered symmetric modes. Repeating the half model with a zero displacement at the center point yields only the even numbered anti-symmetric modes. Combining the two solution sets gives the full range of symmetric and anti-symmetric modes.
function String_vib_2_L3 % vibration of a string
T U_xx = rho U_tt, SHM: T U_xx = -omega^2 rho U

%%%%%%%%%%%%%%%%%%%%%%%%%
% x_1=0 x_2 x_3 x_4 x_5=L
% T <==(1)-------(2)-----> T
% Fixed U_1 U_2 U_3 U_4 U_5 Fixed
% Connectivity : e i j k
% 1 1 2 3 % L3
% 2 3 4 5 % L3
%%%%%%%%%%%%%%%%%%%%%%%%%
n_e = 2 ; n_g = 1 ; n_n = 3 ; n_m = 5 ; % constants
n_i = n_n*n_g ; n_d = n_g*n_m ; % elem & system DOFs
L = 2. ; T = 6e5 ; rho = 0.0234 ; % given m, N, kg/m
X = [0 0.25 0.5 0.75 1]*L ; % node coordinates
L_e = L / n_e ; % element length

% Constant quadratic L3 element square matrices
K_e = T *[ 7, -8, 1, ; \ldots \text{ stiffness L3} 
  -8, 16, -8, ; \ldots 
  1, -8, 7 ] / (3*L_e) ;
M_e = rho *L_e*[4, 2, -1 ; \ldots \text{ mass L3} 
  2, 16, 2 ; \ldots 
  -1, 2, 4 ] / 30 ;

% Assemble two 3x3 element sq matrices into a 5x5
S=zeros (n_d, n_d) ; M=zeros (n_d, n_d) ; % allocate
for k = 1:n_e % loop over elements
  rows = [1:n_n] + (k - 1)*(n_n - 1) ; % connectivity
  S (rows, rows) = S (rows, rows) + K_e ; % add to stiff
  M (rows, rows) = M (rows, rows) + M_e ; % add to mass
end % for element k

% Solve for eigenvalues and eigenvectors, general form
[Vec, Diag] = eig(S(2:4, 2:4), M(2:4, 2:4)) ; % partition
% Vec = eigenvector cols, Diag = eigenvalue^2 on diag

fprintf ('String natural frequencies \n')  % heading
exact = pi() * sqrt(T/(rho*L^2)) ; % exact constant
for k = 1:n_d-2 % loop over DOFs less two EBCs
  omega = sqrt(real(Diag (k, k))) ; % natural freq
  true = k * exact ; % exact natural freq
  fprintf ('%i, FEA = %8.3e, Exact = % 8.3e \n', k, omega, true) % pretty print
end % for all active DOF
% end of String_vib_2_L3 % Running gives:

% String natural frequencies
% 1, FEA = 7.984e+03, Exact = 7.954e+03
% 2, FEA = 1.601e+04, Exact = 1.591e+04
% 3, FEA = 2.873e+04, Exact = 2.386e+04

Figure 14.4-1 Tensioned string eigenvalue-eigenvector calculations
Figure 14.4-2 Tensioned string exact (dashed) and L3 approximations
Figure 14.4.3 Three modes of tensioned string from one quintic element
Figure 14.4-4 Fourth string mode with one (top) and two quintic elements
Example 14.4-3 Given: Use a quadratic line element fixed at one end, with and averaged mass matrix, to approximate the first two frequencies of axial vibration, and compare them to the exact values of $\pi/2$ and $3\pi/2$ times $\sqrt{EA/mL}$. Solution: The stiffness and averaged mass matrices are given in the summary. The natural frequency problem is

$$
\left[ E^e A^e \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} - \lambda_j \frac{m^e}{60} \begin{bmatrix} 9 & 2 & -1 \\ 2 & 36 & 2 \\ -1 & 2 & 9 \end{bmatrix} \right] \begin{bmatrix} u_j^e \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c(0)_{NBC} \\ 0 \end{bmatrix}
$$

Enforcing the EBC at node 1, reduces the problem to two degrees of freedom. The general eigenproblem is

$$
\left| E^e A^e \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \lambda_j \frac{m^e}{60} \begin{bmatrix} 36 & 2 \\ 2 & 9 \end{bmatrix} \right| = 0.
$$

Setting the determinant to zero gives the characteristic equation

$$
0 = [240 \ EA^2 - 107 \ EA \ mL + 4L^2m^2 \lambda^2 ]/45 \ mL
$$

Calculating the two roots of the quadratic equation gives

$$
\lambda_1 = \omega_1^2 = (107 - \sqrt{7609}) \ EA/(8 \ mL) \rightarrow \omega_1 = 1.5720\sqrt{EA/mL} \ \text{rad/sec}
$$

$$
\lambda_2 = \omega_2^2 = (107 + \sqrt{7609}) \ EA/(8 \ mL) \rightarrow \omega_2 = 4.9273\sqrt{EA/mL}
$$

This gives the first two frequency errors of 0.08% and 4.56%, respectively. Changing the interpretation of the coefficients to torsional shafts, this corresponds to the free vibration of a fixed-free shaft.

14.5 Torsional vibrations: The equation of motion of a vibrating torsional shaft or drill string is of the same form as the transvers string vibration:

$$
GJ \frac{\partial^2 \theta}{\partial x^2} - \rho J \frac{\partial^2 \theta}{\partial t^2} = 0
$$

(14.5-1)

where $G$ is the material shear modulus, $J$ is the cross-section polar moment of inertia, $\rho$ is the mass density per unit length, and $\theta$ is the (small) angle of twist of the shaft. The rotational inertia of the shaft cross-section is defined as $l = \rho J$. By analogy to all of the stiffness matrices in Section 14.4 the torsional stiffness matrix always includes the term $GJ/L \equiv k_t$ (divided by a number). That term is known as the torsional stiffness of the shaft and many problems supply that number to describe a shaft segment of length $L$.

It is common for equipment to be attached to a vibrating system. If the mass, or rotational inertia, of the equipment is large then the usual practice is to treat the equipment as a point mass and attach it to a node at an element interface. That is, the point mass (or inertia) is added to the diagonal of the system mass matrix at the node where the equipment is located.
For a very long oil well drill string the end ‘bottom hole assembly’ (BHA) is relatively short but has a very large rotational inertia. In the spring-mass simplified models this type of vibration is called the rotational pendulum. Thus, it is often treated as a point source of rotational inertia, say $I_L$, that is placed on the diagonal of the rotational inertia matrix at the free end node.

That physical argument is justified by considering the secondary boundary condition (from 14.4-4) at $L$:

$$\left[ \theta \left( GJ \frac{d\theta}{dx} \right) \right]_L = \theta_L \tau_L$$

where $\tau_L$ is the external torque applied to the end of the shaft. The BHA is being considered as a rotating planar rigid body. From Newton’s law the torque applied to a rotating rigid disk at its center of mass is

$$\tau_{disk} = \rho J_{disk} \frac{\partial^2 \theta}{\partial t^2} = I_{disk} \frac{\partial^2 \theta}{\partial t^2}$$

But, for SHM at a frequency $\omega$ Newton’s law gives: $\tau_{disk} = -\omega^2 I_{disk} \theta_{disk}$. An equal and opposite torque is applied to the shaft from the disk, $\tau_L = -\tau_{disk}$. Therefore, the system analogous to (14.3-4) becomes

$$\theta_{disk} (\omega^2 I_{disk} \theta_{disk}) - 0 - \theta^T K \theta + \omega^2 \theta^T M \theta = 0$$

which shows that the planar disk inertia, $I_{disk}$, is simply placed on the diagonal of the inertia matrix, $M$, at the (end) node where the disk is attached to the shaft, as expected.

**Example 14.5-1 Given:** Create a Matlab script to determine the natural frequencies of torsional vibration of a circular vertical shaft, fixed at the top, and having a large point inertia at its end.

**Solution:** The script, `Torsional_Vib_BHA_L3.m`, is shown in Fig. 14.5-1 where it manually sets data from a published study that has an exact analytic solution. It utilizes two three-node quadratic line elements in a mesh with five nodes. Again, the Matlab `eig` function is used to solve the eigen-problem equations. The first frequency is overestimated by only 0.4%. 


function Torsion_Vib_BHA_L3 % Revised 4/5/17
% Drill string torsional vibration with end BHA inertia
% v = Angle of twist per unit length, G = shear modulus,
% rho = mass density, t=time, J_e= shaft polar mass inertia
% x1=0 x2 x3 x4 x5=L Exact_freq1 = 2.41 rad/sec
% Fixed *----(1)-----*----(2)-----* BHA_lumped J
% v1 v2 v3 v4 v5 Exact_freq2 = 7.93 rad/sec
% Connectivity list: [1 2 3; 3 4 5] for L3_C0
%

n_e = 2; % number of elements
n_n = 3; % number of nodes per element
n_i = 3; % number of DOF per element
n_d = 5; % system degrees of freedom (DOF)

% specific problem: Thomson Ex 56.2 drill string & BHA
L = 5000.; L_e = L / n_e; % domain, element length, ft
G = 1.728e9; % shear modulus, lb/ft^2
rho = 15.22; % mass density, slug/ft^3
J_e = 9.4e-4; J_BHA = 29.3; % string and BHA inertia
Free = [ 2 3 4 5] % vector subscript after EBC

% Constant quadratic element square matrices
K_e = G * J_e * [ 7, -8, 1, ... 
     -8, 16, -8, ... 
     1, -8, 7 ] / (3*L_e); % stiffness L3_C0
M_e = J_e*rho*L_e*[4, 2, -1; ... 
     2, 16, 2; ... 
     -1, 2, 4 ] / 30 % inertia L3_C0

% Assemble two n_i by n_i square matrix terms
S = zeros (n_d, n_d); M = zeros (n_d, n_d); % allocate
for k = 1:n_e % loop over elements
    rows = [1:n_n] + (k - 1)*(n_n - 1); % get connectivity
    S (rows, rows) = S (rows, rows) + K_e; % add stiffness
    M (rows, rows) = M (rows, rows) + M_e; % add inertia
end % for element k

M (5, 5) = M (5,5) + J_BHA; % add point inertia on diagonal

% Solve for free eigenvalues and eigenvectors, general form
[Vec, Diag] = eig(S(Free, Free), M(Free, Free)); % solve
fprintf ('Diagonal of computed eigenvalues (rad/sec): \n');
Diag = sqrt (real (Diag)); disp(Diag); % convert to freq
fprintf ('Columns of eigenvectors (less EBC) \n');
disp(Vec)
% Gives freq1 = 2.42, freq2 = 7.99 rad/sec

Figure 14.5-1 Torsional frequencies for a shaft with end-point inertia
14.6 Beam vibrations: The transverse vibration, $v$, of a beam yields the same matrix eigen-problem even though it begins a differential equation of motion containing fourth order spatial derivatives:

$$E I \frac{d^4 v}{dx^4} - \rho A \frac{d^2 v}{dt^2} = 0$$  \hspace{1cm} (14.6-1)

where $A$ and $I$ are the area and moment of inertia of the cross-section, $E$ and $\rho$ are the elastic modulus and mass density of the material so $\rho A = m$ is the mass per unit length. The bending stiffness matrix (from 9.4-2) and consistent mass matrix (9.4-5) are

$$K^e = \int_L E \frac{d^2 H(x)}{dx^2}^T \frac{d^2 H(x)}{dx^2} \, dx, \quad M^e = \int_L H(x)^T \rho A e^e H(x) \, dx.$$  \hspace{1cm} (14.6-2)

In Chapter 9 these matrices for the classic two-node cubic beam are

$$K^e = \frac{E I}{L^3} \begin{bmatrix} 14 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \quad M^e = \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$  \hspace{1cm} (14.6-3)

and for the three-node quintic beam they are

$$K^e = \frac{E I}{35L^3} \begin{bmatrix} 5,092 & 1,138L & -3,584 & 1,920L & -1,508 & 242L \\ 1,138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3,584 & -896L & 7,168 & 0 & -3,584 & 896L \\ 1,920L & 320L^2 & 0 & 1,280L^2 & -1,920L & 320L^2 \\ -1,508 & -242L & -3,584 & -1,920L & 5,092 & -1,138L \\ 242L & 38L^2 & 896L & 320L^2 & -1,138L & 332L^2 \end{bmatrix}$$  \hspace{1cm} (14.6-4)

$$M^e = \frac{\rho A L}{13,860} \begin{bmatrix} 2,092 & 114L & 880 & -160L & 262 & -29L \\ 114L & 8L^2 & 88L & -12L^2 & 29L & -3L^2 \\ 880 & 88L & 5,632 & 0 & 880 & -88L \\ -160L & -12L^2 & 0 & 128L^2 & 160L & -12L^2 \\ 262 & 29L & 880 & 160L & 2,092 & -114L \\ -29L & -3L^2 & -88L & -12L^2 & -114L & 8L^2 \end{bmatrix}$$

Figure 14.6-1 shows a cantilever beam with an elastic spring support at the end point. If the point spring stiffness is zero then the system becomes a standard cantilever and if the spring stiffness is infinite the system becomes a propped cantilever. For all intermediate values of the stiffness this is called an elastic support. Such a system could also have an end point rotational spring to control the end rotation of the beam. Recall that the ratio $k_b \equiv E I / L^3$ is known as the bending stiffness of a beam of span $L$. The analytic solution for the system in Fig. 14.6-1 depends on the ratio of the spring stiffness to the bending stiffness, say $R \equiv k / k_b$. A Matlab script to find the frequencies of the beam in Fig. 14.6-1 is given in Fig. 14.6-2, and the results of the script are in Fig. 14.6-3. That figure shows that as the spring value gets very large the frequency approaches that of a propped cantilever, as expected.
function Cantilever_beam_spr_freq_L3  % w/wo spring
% -d2/dx2 (E I d2u/dx2) = rho d2u/dt2
% Four beam natural frequencies and mode shapes
% for quintic L3_C1 element with vertical end spring
% NODES: 1----2----3 --> r
% DOF: 1, 2, 3, 6, 7, 8
Free = [3 4 5 6]  ; % free DOF after fixed EBC
EI = 1.; rho = 1.; L = 1.; L_2 = L^2; m = rho*L^2;
k = EI / L^3; k_5 = 500*k;  % beam & spring stiffness

S = (k / 35) * ...  % beam bending stiffness L3_CO
[5092 1180^L -3584 1920^L -1508 242^L ;
  1180^L 332^L 0 -896^L 320^L -242^L 38^L 2 ;
  -3584 -896^L 7168 0 -3584 896^L 320^L -242^L 38^L 2 ;
  1920^L 320^L 0 -896^L 320^L -242^L -1920^L 320^L 2 ;
  -1508 -242^L -3584 -1920^L 5092 -1138^L 0 ;
  242^L 38^L 2 896^L 320^L -1138^L 332^L 2 ;

M = (m / 13860) * ...  % beam mass matrix L3_CO
[2092 114^L 880 -160^L 262 -29^L ;
  114^L 880 -12^L 29^L -3^L 2 ;
  880 88^L 5632 0 880 -88^L ;
  -160^L -12^L 0 -128^L 2 160^L -12^L 2 ;
  262 -29^L 880 160^L 2092 -114^L 2 ;
  -29^L -3^L 2 -88^L -12^L -114^L 8^L 2 ;

stiff_pt = 1 ;  % turn on(1)/off vert spring at DOF 5
if (stiff_pt == 1); S(5,5) = S(5,5) + k_5 ; % add k_5
    fprintf('Beam frequencies with end spring rad/s \n');
else;
    fprintf('Beam frequencies with free end rad/s \n');
end ;  % now solve system after EBCs
[Vec_eig, D] = eigs(S(Free,Free),M(Free,Free),4,'sm');
RadPS = sqrt(real(diag(D)));  % freq in rad/sec

disp(RadPS); fprintf('Modes 1-4 \n'); DOF=zeros(6,4);
Big = max(abs(Vec_eig(:,1:2:3, :)));  % scale factors
for k =1:4; Vec_eig(:,k) = Vec_eig(:,k)/Big(k); end;
 DOF (Free, 1:4) = Vec_eig  % show values to plot

Figure 14.6-1 Beam with a vertical spring support

Figure 14.6-2 Natural frequencies of a cantilever with a transverse spring
14.7 Membrane vibration: There are many applications that use thin membranes of various shapes. A vibrating or pressurized membrane with zero displacement on its boundary, of any shape, corresponds to the shape of a soap bubble. The Matlab logo is one example that is very similar to the amplitude of the vibration of an L-shaped membrane (with the displacements enlarged). A membrane is a thin tensioned material undergoing transverse small displacements.

Many membranes require a vibration analysis. For a global uniform membrane tension per unit length on the boundary, $T$, the equation of motion is

$$T \nabla^2 v(x, y) + \rho h \frac{\partial^2 v(x, y)}{\partial t^2} = 0$$

(14.7-1)

Where $v(x, y)$ is the transverse displacement, $\rho$ is the material mass density, $h$ is the membrane thickness, so $\rho h$ is the mass per unit area. The first term is the same as that for isotropic heat conduction and gives a membrane stiffness matrix, Section 10.7, of:

$$K^e = \int_{A^e} B^e \nabla T B^e \, dA = \int_{A^e} \left[ \frac{\partial H}{\partial x} \right]^e \nabla T \left[ \frac{\partial H}{\partial y} \right]^e \, dA$$

(14.7-2)

The membrane of thickness, $h$, also has a mass per unit area, $m = \rho h$, and the corresponding consistent mass matrix of:

$$M^e = \int_{A^e} H^e \rho h H^e \, dA$$

(14.7-3)

which assemble into the vibration eigen-problem $[K - \omega^2 J M] \delta_j = 0$. The form of those element matrices were developed in Chapter 10.
The vibration of a flat L-shaped membrane to date has no analytic solution and requires a numerical solution. Very accurate theoretical upper and lower bounds for the natural frequencies of the first ten modes have been published. Consider an L-shaped membrane made up of three unit square sub-regions. That membrane is very similar to the one that serves as the Matlab logo.

Note that any L-shaped membrane has a singular point at the re-entrant corner. For static solutions, the radial gradient of the deflected shape at that corner is theoretically infinite and no finite element solution with a uniform mesh will ever reach the theoretical solution. Not only does the asymptotic analytic solution have an infinite gradient, but that gradient changes rapidly with the angle from the first exterior edge to the second edge. These two facts mean that ideally the mesh should be manually controlled to make the elements smaller as they radially approach any singular point, but the mesh should have several elements sharing one vertex at the singular point.

In the early days of finite element solutions, when the available memory was very very small, a special type of ‘singularity elements’ were developed to include in the mesh at singular points. Today, with adaptive mesh generation it is up to the user to create a reasonable mesh refinement at any re-entrant corner. (For adaptive error analysis solutions the user must also limit the size of the smallest element at a re-entrant corner to prevent all new elements being placed there in the attempt to calculate an infinite gradient.)

For a regular shape, like an L-shaped membrane, symmetric and anti-symmetric modes may negate the presence of a re-entrant corner. For example, it will be seen that the third mode of the L-shaped membrane has the same mode shape (with +, -, + signs) in each of the three square sub-regions. Thus, the interior vertical and horizontal lines passing through the corner have zero displacement amplitudes. In other words, it is exactly like computing the first mode of a square with no re-entrant corner.

Figure 14.7-1 shows a mesh of nine-noded quadrilaterals and 1,977 degrees of freedom (nodes) that were used to mode the L-shaped membrane and the anti-symmetric modes of a U-shaped membrane. The bottom portion shows the mode 1 shape obtained with four-node quadrilateral elements and 252 DOF. The two first mode frequency estimates differ by less than 4%. That view angle, like the one of the Matlab logo, hides the reentrant corner.

The calculations were done by the script Membrane_vibration.m which is in the Application Library. It accepts a membrane of any shape meshed with any element in the provided library. The elements are numerically integrated to alloy for curved membranes and variable Jacobian elements. To produce the mode shape plots invokes the script mode_shape_surface.m which is in the general function library.
Figure 14.7-1 L-shaped membrane first mode of vibration with Q9 and Q4 elements
Figure 14.7-2 First three mode shapes of the L-shaped membrane
Figure 14.7-3 First three symmetric modes of half of a U-shaped membrane
14.8 Structural buckling: There are two major categories leading to the sudden failure of a mechanical component: structural instability, which is often called buckling, and brittle material failure. Buckling failure is primarily characterized by a sudden, and usually catastrophic, loss of structural stiffness that usually renders the structure unusable. The buckling estimate is computed from a finite element eigen-problem solution. Slender or thin-walled components under compressive stress are susceptible to buckling. Buckling studies are much more sensitive to the component restraints that a normal stress analysis. A buckling mode describes the deformed shape the structure assumes (but not the direction) when it buckles, but (like a vibration mode) says nothing about the numerical values of the displacements or stresses. The numerical values may be displayed, but are only relative and are usually scaled to have a maximum absolute value of one.

Undergraduate linear buckling studies usually focus on the buckling of columns (beams) using the Euler theory. The ideal estimates depend heavily on the EBCs (end constraints) and handbooks give the solution for ideal beams with various end, and mid-span, constraints. Many of those critical column buckling loads can be written as $P_{cr} = k \frac{\pi^2 EI}{L^3}$ where $k$ represents the effects of the EBCs and where $EI/L^3$ is typically called the beam stiffness. The five most common EBC constants for an ideal beam are shown in Fig. 14.8-1.

This section addresses the finite element implementation of the common linear theory of buckling, which like the Euler theory of column buckling, tends to under estimate the load case necessary to cause buckling. An accurate buckling estimate depends heavily on the geometry of the model. For example, here a beam is assumed to be a perfect straight line when the axial load is applied. Furthermore, it is also assumed that the axial load acts exactly through the centroid of the cross-section of the beam. Otherwise, the loading becomes eccentric causing a transverse moment to be applied as well as the axial load. If the part has small irregularities that have been omitted, they can drastically reduce the actual buckling load.

![Figure 14.8-1 Restraints influence the critical buckling load, $P_{cr} = k \frac{\pi^2 EI}{L^3}$](image)

The linear theory of buckling of structures, like the Euler theory of column buckling, tends to under estimate the load case necessary to cause buckling. An accurate buckling estimate depends heavily on the geometry of the model. For example, here a beam is assumed to be a perfect straight line when the axial load is applied. Furthermore, it is also assumed that the axial load acts exactly through the centroid of the cross-section of the beam. Otherwise, the loading becomes eccentric causing a transverse moment to be applied as well as the axial load. If the part
has small irregularities that have been omitted, they can drastically reduce the actual buckling load.

When a constant axial load acts along the entire beam, say \( N_B \), it factors out of the element initial stress (geometric stiffness) matrices and the assembly process to appear as a constant in the system matrices. The goal is to find the scaling up or down of the applied load that causes linear buckling. That scale factor is called the “buckling load factor (BLF)” and replacing \( N_B \) with \( BLF \times N_B \) leads to a buckling eigen-problem to determine the value of \( BLF \).

For a general structure with a set of loads (called a load case) the system geometric stiffness matrix depends on the stress in each element. That means that it depends on all of the assembled loads, say \( F_{\text{ref}} \). In general, a linear static analysis is completed and the deflections are determined from the system equilibrium using the elastic stiffness matrix and any foundation stiffness matrix:

\[
[K_E + K_k] \{ v \} = \{ F_{\text{ref}} \}.
\]

Those displacements are post-processed to determine the stresses in each element. Then each element geometric stiffness matrix is calculated from those stresses. The element stresses, and thus its geometric stiffness matrix, is directly proportion to the resultant system load, so as the loads are scaled by the buckling factor the system geometric stiffness matrix increases by the same amount

\[
F \rightarrow BLF F_{\text{ref}}, \quad K_N \rightarrow BLF K_{\text{ref}}
\]

The scaling value, \( BLF \), that renders the combined system stiffness to have a zero determinant (that is to become unstable) is the increase (or decrease) of all of the applied loads that will cause the structure to buckle. That critical value is calculated from a buckling eigen-problem using the previous matrices plus the geometric stiffness matrix based on the current load case:

\[
|K_E + K_k - BLF K_{\text{ref}}| = 0 \tag{14.8-1}
\]

That equation is solved for the value of the \( BLF \). Then the system load case that would theoretically cause buckling is \( BLF F_{\text{ref}} \). The solution of the eigen-problem also yields the relative buckled mode shape (eigen-vector), \( v_{BF} \). The magnitude of the buckling mode shape displacements are arbitrary and most commercial software normalizes them to range from 0 to 1.

For straight beams with an axial load several of the above steps are skipped because each element’s axial stress and its geometric stiffness matrix are known by inspection. The eigenvalue \( BLF \) is the ratio of the buckling load case to the applied load case. The following table gives the interpretation of possible values for the buckling factor.

### Table 14.8-1 Interpretation of the buckling load factor

<table>
<thead>
<tr>
<th>Value</th>
<th>Status</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>[</td>
<td>BLF</td>
<td>\gt 1]</td>
</tr>
<tr>
<td>[</td>
<td>BLF</td>
<td>\leq 1]</td>
</tr>
<tr>
<td>(BFL &lt; 0)</td>
<td>Buckling possible</td>
<td>Buckling occurs if the directions of the applied loads are reversed</td>
</tr>
</tbody>
</table>
In theory, there are as many buckling modes as there are DOF in a structure. Due to the limitations of linear buckling theory, only the first buckling mode is of practical importance. Structural buckling is usually instantaneous and catastrophic. A Factor of Safety of four is often applied to linear buckling estimates. There are commercial finite element systems that can solve the more accurate nonlinear post-buckling behavior of structures.

The estimated buckling force can be unrealistically large. A linear buckling calculation implies that the compression load-deflection relation, and thus the material compression stress-strain relation is linear. That is true only up to where the stress reaches the material compressive yield stress, say \( S_{yc} \). In other words, the critical compressive stress, \( \sigma_{cr} = P_{cr}/A \), caused by the critical force, \( P_{cr} \), must also be considered. To do that the critical for relation must be re-written in terms of the cross-sectional area. That is done by using the geometric ‘radius of gyration’, \( r_g \), defined as \( I \equiv A r_g^2 \). Then for typical boundary conditions

\[
P_{cr} = k\pi^2 EI/L^2 = k\pi^2 E(A r_g^2)/L^2 = k\pi^2 EA/(L/r_g)^2
\]

where the ratio \( L/r_g \) is known as the ‘slenderness ratio’ of the cross-section. It is evaluated at the location of the smallest \( I \) found on any plane perpendicular to the axis of the load. The critical compressive stress is \( \sigma_{cr} = P_{cr}/A = k\pi^2 E/(L/r_g)^2 \leq S_{yc} \).

![Figure 14.8-2 Simple truss with a potential buckling load](image)

**Example 14.8-1 Given:** The constant EA truss in Fig. 14.8-2 has the connection list \([1\ 2\ 3\ 1]\) and a previous analysis showed that the inclined member is force (and stress) free and the vertical (first) carries an axial force of \( N = -P \). Repeat the truss analysis with the addition of the initial stress (geometric) stiffnesses to determine the value of \( P \) required to cause the truss to buckle (in its original plane). **Solution:** The elastic stiffness matrix of a truss member must be rotated from its horizontal position. The direction angles of the first (vertical) and second (inclined) members are \( C_x = 90^\circ, C_y = 0 \), and \( C_x = 45^\circ, C_y = 45^\circ \), respectively. Thus, their respective transformation sub-matrices are

\[
e^{e=1}(\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad e^{e=2}(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}, \quad [T(\theta)] = \begin{bmatrix} t(\theta) & 0 \\ 0 & t(\theta) \end{bmatrix}
\]
and the original and transformed elastic stiffness matrices, \( S_E^e = \{ T(\theta) \}^T S_L^e \{ T(\theta) \} \), are

<table>
<thead>
<tr>
<th>DOF</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_E^e = \frac{E A L}{L} \begin{bmatrix} 1 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \ -1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
<td>( S_E^{e=1} = \frac{E A}{L} \begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; -1 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 1 \end{bmatrix} )</td>
<td>( S_E^{e=2} = \frac{E A}{(\sqrt{2}L)^2} \begin{bmatrix} 1 &amp; 1 &amp; -1 &amp; -1 \ 1 &amp; 1 &amp; -1 &amp; -1 \ -1 &amp; -1 &amp; 1 &amp; 1 \ -1 &amp; -1 &amp; 1 &amp; 1 \end{bmatrix} )</td>
<td></td>
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</tr>
</tbody>
</table>

The initial stress (geometric) stiffness matrix transforms in the same manner:

<table>
<thead>
<tr>
<th>DOF</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_i^e = N_L \begin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; -1 \ 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; -1 &amp; 0 &amp; 1 \end{bmatrix} )</td>
<td>( S_i^{e=1} = \frac{-P}{L} \begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
<td>( S_i^{e=2} = 0 )</td>
</tr>
</tbody>
</table>

The structure has six degrees of freedom, but only the first two are free, so only those partitions need to be assembled:

\[
\left( \frac{E A}{L} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \frac{E A}{2\sqrt{2}L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) - P \left( \frac{1}{L} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \{u_1\} = \{0\}
\]

The critical buckling force causes the determinant to vanish:

\[
\begin{bmatrix}
\frac{E A}{(2\sqrt{2}L)} - \frac{P_{\text{crit}}}{L} & \frac{E A}{2\sqrt{2}L} \\
\frac{E A}{2\sqrt{2}L} & \frac{E A}{2\sqrt{2}L}
\end{bmatrix} = 0 = \frac{E A}{2\sqrt{2}L} \left[ \begin{bmatrix} 1 - 2\sqrt{2} \frac{P_{\text{crit}}}{E A} & 1 \\ 1 & (1 + 2\sqrt{2}) \end{bmatrix} \right]
\]

Evaluating the determinant gives the critical load of \( P_{\text{crit}} = \frac{E A}{(2\sqrt{2} - 1)/7} = 0.261 E A \), which is the same value obtained from the mechanics of materials.

**Example 14.8-2 Given:** A fixed–pinned beam (case 4 of Fig. 14.8-1) has an axial compression load of \( P \). Determine the approximate value of \( P \) that causes the beam to buckle. **Solution:** A single cubic beam has only one DOF, but could give an upper bound on the buckling load (try it). A three degree of freedom model will give a better estimate. That can be a three-node mesh with either two cubic beam elements or a single quintic element. They would both have the middle and end slope as DOF, along with the middle transverse deflection to define the mode shape.

The elastic stiffness, geometric stiffness and mass matrices for the quintic beam (L3_C1) were given in (9.5-2). For a single quintic element model the two stiffnesses matrices are
The determinant to zero yields a cubic characteristic equation

\[ -81 \lambda^3 + 1575 \lambda^2 - 6020 \lambda + 4900 = 0. \]

Computing the roots of that polynomial gives: \( \lambda_3 = 14.6546 \), \( \lambda_2 = 3.6628 \), and \( \lambda_1 = 1.1270 \). The smallest eigenvalue corresponds to the first, and most critical, buckling load. Therefore, the estimated buckling load is \( P_1 = 18 \times \lambda_1 EI/L^2 = 20.286 EI/L^2 \), where \( EI/L^2 \) is known as the beam stiffness. The exact coefficient is 20.187. The quintic beam-column element gave an error of 0.5%. Repeating this calculation with two cubic beam elements gives a 2.5% error.

**Example 14.8-3 Given:** Describe how the Ex. 14.8-2 changes if a transverse spring is present as in Fig. 14.8-2. **Solution:** Only the active elastic stiffness matrix changes by adding the spring stiffness to the diagonal location in \( K_E \) corresponding to the node to which the spring is attached:

\[
K_E = \frac{EI}{35 L^3} \begin{bmatrix}
5,092 & 1,138 L & -3,584 & 1,920 L & -1,508 & 242 L \\
1,138 L & 332 L^2 & -896 L & 320 L^2 & -242 L & 38 L^2 \\
-3,584 & -896 L & 7,168 & 0 & -3,584 & 896 L \\
1,920 L & 320 L^2 & 0 & 1,280 L^2 & -1,920 L & 320 L^2 \\
-1,508 & -242 L & -3,584 & -1,920 L & 5,092 & -1,138 L \\
242 L & 38 L^2 & 896 L & 320 L^2 & -1,138 L & 332 L^2 \\
\end{bmatrix}
\]

Defining the eigenvalue as \( \lambda = P_{cr} L^2 / 18 EI \), setting the determinant to zero yields a cubic characteristic equation

\[ -81 \lambda^3 + 1575 \lambda^2 - 6020 \lambda + 4900 = 0. \]

Computing the roots of that polynomial gives: \( \lambda_3 = 14.6546 \), \( \lambda_2 = 3.6628 \), and \( \lambda_1 = 1.1270 \). The smallest eigenvalue corresponds to the first, and most critical, buckling load. Therefore, the estimated buckling load is \( P_1 = 18 \times \lambda_1 EI/L^2 = 20.286 EI/L^2 \), where \( EI/L^2 \) is known as the beam stiffness. The exact coefficient is 20.187. The quintic beam-column element gave an error of 0.5%. Repeating this calculation with two cubic beam elements gives a 2.5% error.

**Figure 14.8-3** Beam-column with axial load and vertical spring
which makes the first coefficient in the left matrix, after defining $\lambda$, depend on the ratio of the spring axial stiffness to the transverse beam stiffness, $EI/L^3$:

$$(7.168 + 35 k L^3 /EI) = (7.168 + 35 \, k/k_{beam})$$

Increasing the spring stiffness will slightly increase the critical buckling force. The spring makes the algebra get much worse, but has no noticeable effect on how the numerical calculations are done after the spring stiffness is scattered to the proper diagonal location.

Note that the limit of $k \to \infty$ makes the spring act as a second pin support at the mid-span. That also cuts the free span length in half, significantly increases the bending stiffness, and then the spring significantly increases the force required to buckle the column.
% Beam buckling load and mode shape for a single quintic
% fixed-pinned L3 C1 element
% FIXED: 1,2 3,4 5,6 ROLLER PIN
% Theory 20.19 EI/L^2
L = 10.0; E = 70.e9; I = 61.3e-6; Free = [3 4 6] ; % model data
EIL2 = E*I/L^2; K_e = EIL2/(35*L) ; % beam & L3 stiffness
K = [7168, 0, 896*L ]; % bending matrix
0, 1280*L^2, 320*L^2 ;
896*L, 320*L^2, 332*L^2] * K_e ; % free DOF only
G = [3072, 0, 48*L ]; % geometric matrix
0, 256*L^2, -8*L^2 ;
48*L, -8*L^2, 28*L^2] / (630*L) ; % free DOF only

[Modes, BFsq] = eig(K, G) ; % solve for buckling load
BFs = diag(BFsq) ; % Buckling Factors (NOT in order)
[B_n, L_n] = min([abs(BFs)]) ; % min force value & where
fprintf('Buckling force estimate %7.2e \n', B_n) ; % eigenvalue
fprintf('Use force %7.2e \n', B_n/4) ; % safer

DOF(1:6)=0; DOF(Free) = Modes(:, L_n) ; % buckling mode to plot
Big = max(abs(DOF(1:2:5))) ; DOF = DOF/Big ; % scale mode
x = zeros (100,1); y = zeros (100,1) ; % allocate plot pts
for k = 1:101; % beam points in parametric space --> --> --> -->
    r = (k-1)/100 ; % 0 <= r <= 1 from Hermite_1D_C1_library
    r2 = r^2; r3 = r^3 ; r4 = r^4 ; r5 = r^5 ; % constants
    H=[(1-23*r2+66*r3-68*r4+24*r5) (r-6*r2+13*r3-12*r4+4*r5)*L ... 
        (16*r2-32*r3+16*r4 ) (-8*r2+32*r3-40*r4+16*r5)*L ... 
        (7*r2-34*r3+52*r4-24*r5) (-2*r2+5*r3-8*r4+4*r5)*L];
    y(k) = H * DOF' ; x(k) = r ; % mode amplitude & x/L
end ; % for k points on beam ,-- -- -- -- -- -- -- -- -- -- -- -- -- -- --
cif; hold on; grid on; axis([0, 1, -0.1, 1.1]); % clear plot
xlabel('x/L'); ylabel('Amplitude'); ratio=B_n/EIL2; % plot labels
title(['First buckling mode force ' num2str(B_n,'%7.2e')... 
' (', num2str(ratio,'%7.2e'), ' EI/L^2')]); % title
plot(x, y, 'b-'); xlb=[0,0.5,1]; ylb=[0,0,0]; % modes & beam info
plot(xb, yb, 'k-') ; % show beam line & nodes
p_text = 'Beam_buckle_shape'; print('-dpng', p_text); % save plot
n_text = ['Created png file ', p_text] ; % add extension
fprintf(1,'%s', n_text); fprintf(1, ' \n'); % tell user

Example 14.8-4 Given: Write a Matlab script to implement the calculations given in Ex. 14.8-1, and to plot the buckled mode shape. Solution: Such a script is above in Fig. 14.8-3. The mechanics calculations take about a dozen lines. After the EBCs are enforced only the mid-span displacement, mid-span rotation, and the end-point rotation remain free (DOFs 3, 4, 6). Those three rows and columns from the elastic stiffness and the geometric matrices are passed as input arguments to the Matlab function eig which returns three critical BLF values and their corresponding buckled mode shapes. The three BLF are extracted from the returned square matrix into a vector using the Matlab function diag. That was done because eig does NOT
always return the eigenvalues in increasing order. The Matlab $\text{abs} \ min$ function combination was used to find which eigenvalue was the smallest and to grab that value. The location of the smallest eigenvalue was then used to extract the column from the eigenvector square matrix which corresponds to the lowest buckling mode shape. (Note, the higher buckling modes are not important so an algorithm that finds and returns only the lowest eigenvalue is the most efficient way to do buckling studies.) The lowest BLF is printed, along with that value reduced by a Factor of Safety (FOS) of four. Then the retained buckling mode is plotted and saved.

The larger lower section of the figure primarily addresses the scaled magnitude of the fifth-degree polynomial used to calculate the critical load. To list or graph the mode shape the full system displacement vector must be restored by inserting the three free buckled displacement components in with the prior three EBC values. The magnitudes of the components of the buckling mode displacement are only relative, so they are scaled to have a maximum non-dimensional value of one. Then the six element degrees of freedom are interpolated at many points along the fifth degree polynomial to display the exact curve of the approximate buckled mode shape. The location of the three element nodes, and a dashed line representing the original column centerline are optionally added to provide a more informative graph (see Fig. 14.8-4).

![Figure 14.8-4 Linear buckled mode shape estimate of fixed-pinned column](image)

### 14.9 Beam frequency with an axial load:
To determine the natural frequencies of a structure subjected to an axial load the eigen-problem is $|(K_E + K_N) - \omega^2 M| = 0$, where $K_E$ is the usual elastic stiffness matrix, $K_N$ is the geometric stiffness matrix associated with the axial load, $N$, and $M$ is the usual mass matrix. When the axial load is an constant, say $N_B$, then it factors out of each element geometric stiffness matrix and becomes a global constant after assembly:

$$|(K_E + N_B K_n) - \omega^2 M| = 0 \quad (14.9-1)$$
where the unit initial stress matrix, $K_n$, is formed with a unit positive load. This equation shows that any axial load, $N_B$, has an influence on the natural frequency, $\omega$. In general, a tensile force (+) increases the natural frequency while a compression force (-) lowers the natural frequency.

One of the most common applications of this state is finding the natural frequencies of turbine blades in jet engines. Their high rotational speed imposes a tension centripetal force on the blades that increases with radial position. A static solution at a fixed rotational speed defines the element stresses that in turn define the element geometric (initial stress) matrix. Then the eigen-problem is solved to compute the blade frequencies at each rotational speed of the engine.

This eigen-problem is easily solved numerically. There are several published analytic solutions for a beam-column with various essential boundary conditions. Still, a single finite element can give an important insight into how the primary variables impact the resulting natural frequency. Consider finding the natural frequency of a fixed-pinned column (case 4) subjected to an axial tension force, $P$. A single cubic beam element model has only the single end rotation, $\theta_2$, as its one free degree of freedom. Substituting that term from the cubic beam matrices, from (9.5-1), and (9.5-3), into the third row of the system matrix (14.9-1) gives:

\[
\frac{EI}{L^3} [4L^2] + \frac{P}{L} \left[ \frac{2L^2}{15} \right] - \omega^2 \rho AL \left[ \frac{4L^2}{420} \right] = 0
\]

and dividing all terms by $EI/L$ gives

\[
4 + \frac{PL^2}{EI} \frac{2}{15} - \omega^2 \frac{\rho AL^4}{EI} = 0
\]

so, the natural frequency squared is:

\[
\omega^2 = 420 \frac{EI}{\rho AL^5} \left[ 1 + \frac{1}{30} \frac{PL^2}{EI} \right] = 420 \frac{EI}{m L^3} \left[ 1 + \frac{1}{30} \frac{P L^3}{L E I} \right]
\]

where $m = \rho AL$ is the mass of the beam, $EI/L^3$ is the beam elastic stiffness, and $P/L$ is the beam geometric stiffness. The final frequency becomes

\[
\omega = 20.49 \sqrt{\frac{EI}{m L^3}} \sqrt{1 + \frac{1}{30} \frac{P L^3}{L EI}}, \quad \text{or} \quad \omega = n_1 \sqrt{\frac{EI}{m L^3}} \sqrt{1 + \frac{1}{n_2} \frac{P L^3}{L EI}} \tag{14.9-2}
\]

The last re-arranged term is the general form that the exact analytic solution takes for various essential boundary conditions (that generate their corresponding $n_1$ and $n_2$ values). For example, a pinned-pinned single span beam the $i$-th frequency has $n_1 = \pi i^2 / 2, n_2 = \pi^2 i^2$. That general form shows that when the axial load is tension ($P > 0$) the natural frequency is increased. Conversely, when the load is compressive ($P < 0$) the natural frequency is reduced. The lower limit is when the frequency goes to zero, which corresponds to the column buckling (losing its stiffness): $P_{cr} = -n_2 EI/L^2$. The theoretical relation between the square of the frequency and the axial load is linear and is shown for compression loads in Fig. 14.9-1.

If the above single cubic element is changed to a pinned-pinned pair of EBCs, then it has two dof and thus two natural frequencies. Their values are

\[
\omega_1 = 10.954 \sqrt{\frac{EI}{m L^3}} \sqrt{1 + \frac{1}{12} \frac{P L^3}{L EI}}, \quad \omega_2 = 50.200 \sqrt{\frac{EI}{m L^3}} \sqrt{1 + \frac{1}{60} \frac{P L^3}{L EI}}
\]

and setting $\omega_1 = 0$ gives a buckling load estimate of $P_1 = -12 EI/L^2$ which is about 27% higher than the exact value of $-\pi^2 EI/L^2$.  

33
Example 14.9-1 Given: Write a Matlab script to determine the natural frequencies of the quintic fixed–pinned beam-column (case 4 of Fig. 14.8-1) in Example 14.8-1 with an axial tension load of P. Solution: Only the quintic beam element mass matrix is needed in addition to the elastic stiffness and geometric stiffness given in that example.

\[
\begin{bmatrix}
EI & 7,168 & 0 & 896 L \\
0 & 1,280 L^2 & 320 L^2 & 332 L^2 \\
896 L & 320 L^2 & 332 L^2 & 28 L^2 \\
\end{bmatrix} + \frac{P}{630 L} \begin{bmatrix}
3,072 & 0 & 48 L \\
0 & 256 L^2 & -8 L^2 \\
48 L & -8 L^2 & 28 L^2 \\
\end{bmatrix}
\]

\[-\omega^2 \frac{\rho A L^4}{EI} \begin{bmatrix}
5632 & 0 & -88 L \\
0 & 128 L^2 & -12 L^2 \\
-88 L & -12 L^2 & 8 L^2 \\
\end{bmatrix} = 0\]
The Matlab solution script, *Beam_Col_Freq_L3.m*, is shown in Fig. 14.9-2, with the omission of the trailing mode shape plot details for a unit property beam. Figure 14.9-3 shows the first vibrational mode shape approximation and the exact sine curve shape.

```
% Natural frequency of a beam column via one quintic
% L3_C1 pinned-pinned element
% NODES:  1-------2-------3 <=== P
% DOF:  1,2  3,4  5,6  ; % Four frequencies
% Free = [2  3  4  6]  ; % free after 0 EBC
EI = 1. ; rho = 1. ; L = 1. ; L_2 = L^2 ; m = rho*L ;
k = EI / L^3 ; P = -1  ; % beam stiffness, axial load
S = (k / 35) * ...  % beam bending stiffness L3_C0
[ 5092 1138*L -3584 1920*L -1508  242*L ;
  1138*L 320*L -896*L 320*L -242*L  38*L_2 ;
-3584  7168  0  -3584  896*L ;
  1920*L  320*L -242*L -1920*L  320*L ;
-1508 -242*L  -3584  -1920*L  5092 -1138*L ;
  242*L  38*L_2  896*L 320*L_2 -1138*L  332*L_2];
M = (m / 13860) * ...  % beam mass matrix L3_C0
[ 2092 114*L  880 -160*L  262  -29*L ;
  114*L  8*L_2  88*L -12*L_2  29*L  -3*L_2 ;
  880  88*L  5632  0  880  -88*L ;
-160*L  -12*L_2  0  128*L_2  160*L -12*L_2 ;
  262  29*L  880  160*L  2092 -114*L ;
-29*L  -3*L_2  -88*L -12*L_2  -114*L  8*L_2];
G = (P / 630) * ...  % beam geometric stiffness L3_C0
[ 1668/L  39  -1536/L  240 -132/L -9 ;
  39  28*L  -48  -8*L  9  -5*L ;
-1536/L  48  3072/L  0  -1536/L  48 ;
  240  -8*L  0  256*L -240  -8*L ;
-132/L  9  -1536/L -240  1668/L  -39 ;
-9  -5*L  48  -8*L  -39  28*L];
K = S(Free,Free) + G(Free,Free)  ; % total stiffness

[Vec_eig, D_vals] = eig (K, M(Free,Free))  ; % solve
RadPS = sqrt(real(diag(D_vals)))  ; % freq in rad/sec
fprintf('Mode 1-4 frequencies \n')  ; % print header
disp(RadPS)  ; % show first four freq
Big = max(abs(Vec_eig(2, 1)))  ; % scale mid displ
DOF = zeros(6, 1)  ; % set up plot storage
DOF (Free) = Vec_eig(:, 1) / Big  ; % mode 1 to plot
```

Figure 14.9-2 Frequencies of beam-column with axial load
14.10 Plane-Frame modes and frequencies: Chapter 10 presented the planar stiffness and mass matrices for the classic cubic beam element and the more useful quintic beam element. Those matrices are also repeated in the summary of this chapter. The vibration analysis of a frame basically just replaces the linear equation solver with a call to the eigenvalue solver. Since the planar frame has three degrees of freedom per node the main new effort is plotting the true cubic of quintic vibration modes (or the lowest buckling mode). An example natural frequency analysis will use the upper left quarter of the plane frame shown in Fig. 10.7-3.

14.11 Modes and frequencies of 2-D continua: The natural frequency calculations for plane-stress, plane-strain, and axisymmetric solids are almost identical. Thus, the plane-stress case will be demonstrated here. The definitions elastic stiffness matrix and the consistent mass matrices were given in (11.11-1) and (11.14-4), respectively:

\[
S^e = \int_{V^e} B^e T^e B^e dV, \quad M^e = \int_{V^e} N^T \rho^e N dV \quad (14.11-1)
\]

Note that the mass array used vector interpolation $N$ instead of the scalar interpolation $H$ used for the generalized mass matrices for scalar one-dimensional arrays. The total mass (and its generalized mass matrix from $H$) occur in each space-dimension. In other words, the provided script for plane-stress or plane-strain integrates a generalized mass matrix and then scatters a copy to the x-direction degrees of freedom, and the y-direction rather than integrate the same terms twice.

Plane-stress and plane-strain applications have the same strain-displacement matrices, $B^e$, but their constitutive matrices, $E^e$, have different entries. The axisymmetric solids have one more row in their $B^e$ and $E^e$ matrices. For planar solids the differential volume is $dV = t^e dA$ for an element thickness of $t^e$, and for the axisymmetric solid it is $dV = 2\pi R dA$. In the example
script used here numerical integration is used to formulate the element matrices for curved quadratic triangular (T6) elements.

Empirical studies have shown that alternate forms of the mass matrix are available. Special exact integration rules that use the nodes as quadrature points produce a diagonal mass matrix. The same diagonal matrix, $M_{\text{diag}} = \begin{bmatrix} M \end{bmatrix}$, can be obtained by scaling up the diagonal of the consistent mass matrix:

$$M_{\text{diag}} = \text{diag}(M) \times \frac{\text{sum}(M)}{\text{sum} \left( \text{diag} \left( M \right) \right)}$$

(14.11-2)

to preserve the total of mass present. For elements including rotary inertia, like beams, the sums are split to retain the proper units for different degrees of freedom. Numerical studies of natural frequency calculations have shown that the average of the above two forms improve the rate of convergence of the frequencies and their mode shapes.

For planar problems the first few mode shapes tend to show a bending response in the longest direction, then an extension in the longest direction or a shear distortion in the shortest direction. Real structures usually have hundreds of mode shapes and they are difficult to interpret unless you do that daily. The provided application script Plane_Stress_T6_freq_sm.m illustrates most of the finite element concepts covered in this book. It uses numerically integrated isoparametric (curved) triangles to build the model and the Matlab function eigs to extract a small number of modes. It also uses the Matlab function sort to place them in increasing order. The requested numbers of eigenvalues are listed. The corresponding eigenvectors can optionally be listed, but usually they are optionally processed into a plot file.

There are few analytical benchmarks for planar shapes. The planar vibration script was compared against one listed in Fig. 14.11-1 and the numerical eigenvalue results in Fig. 14.11-2 compare better than usual. The system has 882 degrees of freedom (before the EBCs and 840 after). Therefore, the system had 840 natural frequencies but only the first five were extracted. The errors in the first four frequencies are only 0.05%, 0.04%, 0.20%, and 0.72%, respectively. The first five mode shapes (eigenvectors) of the first few modes are shown in Fig. 14.11-3.

<table>
<thead>
<tr>
<th>In-plane vibrations of a deep cantilever shear beam (sq. plate)</th>
<th>Mesh 20 x 20 = 400 T6 numerically integrated elements. Plate is 10 x 10 x 0.01 m, E = 2e11 N/m^2, nu = 0.3, rho = 8e3 Kg/m^3</th>
<th>Theoretical freq., Hz: 1-52.404 2-125.69 3-140.78 4-222.54</th>
</tr>
</thead>
</table>

Figure 14.11-1 Description of planar vibration validation problem
NOTE: Rigid body motions give zero frequencies

Eigenvalue 1 = 1.0853e+05
Natural frequency, Hz (rad/sec) 1 5.24309e+01 (3.29433e+02)

Eigenvalue 2 = 6.2416e+05
Natural frequency, Hz (rad/sec) 2 1.25738e+02 (7.90037e+02)

Eigenvalue 3 = 7.8570e+05
Natural frequency, Hz (rad/sec) 3 1.41074e+02 (8.86396e+02)

Eigenvalue 4 = 1.9834e+06
Natural frequency, Hz (rad/sec) 4 2.24143e+02 (1.40833e+03)

Eigenvalue 5 = 2.3093e+06
Natural frequency, Hz (rad/sec) 5 2.41860e+02 (1.51965e+03)

Created png file mode_shape_1_2d
Created png file mode_shape_2_2d
Created png file mode_shape_3_2d
Created png file mode_shape_4_2d
Created png file mode_shape_5_2d

Figure 14.11 Planar vibration numerical validation results

Figure 14.11-5 First five modes of a planar deep cantilever vibration validation problem
14.12 Acoustical vibrations: xxx

14.13 Principal stresses: At any point in a stressed solid there are three mutually orthogonal planes where the shear stresses vanish and only the normal stresses remain. Those three normal stresses are called the principal stresses. The principal stresses are utilized to define the most common failure criteria theories for most materials. Therefore, their calculation is usually obtained at each quadrature point in a stress analysis. The principal stresses are found from a small 3 × 3 classic eigen-problem:

\[
[[\sigma] - \lambda_k[I]] = 0, k = 1, ..., n_s \quad (14.13-1)
\]

where \([I]\) is the identity matrix, \([\sigma]\) is the symmetric second order stress tensor, \(\lambda_k\), is a real principal stress value giving the algebraic maximum stress, \(\lambda_1 = \sigma_1\), the minimum stress \(\lambda_2 = \sigma_2\), and for three-dimensional solids \(\lambda_3 = \sigma_3\) the intermediate stress. (This process is valid for any other second order symmetric tensor, such as the moment of inertia tensor.) The principle stresses (in reverse order) are obtained from the Matlab function \textit{eig}, with only \([\sigma]\) as the single argument. Finite element formulations often use the Voigt stress notation as a condensed form of the stress tensor:

\[
\sigma^T = [\sigma_x \hspace{5pt} \sigma_y \hspace{5pt} \sigma_z \hspace{5pt} \tau_{xy} \hspace{5pt} \tau_{xz} \hspace{5pt} \tau_{yz}] \Leftrightarrow [\sigma] = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_{zz}
\end{bmatrix}
\]

The principal stresses define other physically important quantities, such the measure of material failure by the distortional energy criterion:

\[
\sigma_E = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \quad (14.14-2)
\]

\[
\ldots \sigma_E = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{yy} - \sigma_{xx})^2 + (\sigma_{zz} - \sigma_{xx})^2 + (\sigma_{zz} - \sigma_{yy})^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)} \quad (14.14-3)
\]

which has the units of stress, but is not a physical stress. That measure is called the Equivalent stress or the von Mises stress. It requires that the compressive yield stress and the tensile yield stress are equal. If they differ then another failure criterion must be used, like the Burzynski criterion.

Another terminology is the ‘stress intensity’ is defined as \(\sigma_I = \sigma_1 - \sigma_2 = 2\tau_{max}\) where \(\tau_{max}\) is the absolute maximum shear stress on any three-dimensional plane in the solid at that point. There is also a simple mathematical upper bound for the maximum shear stress: \(\tau_{limit} = \sigma_E/\sqrt{3}\). For plane-stress problems it is common to find the maximum in plane shear stress from

\[
\tau_{plane} = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2} \leq \tau_y = \frac{1}{2}\sigma_y \quad (14.14-4)
\]

But, if the thickness of the planar part is not very small the absolute maximum shear stress can lie in a different plane and has to be found from the above three-dimensional intensity.
For a ductile material, its tensile yield stress, $\sigma_Y$, is compared to the $\sigma_E$ value to see if it has failed based on distortional energy at the point. Material failure is declared when $\sigma_E \geq \sigma_Y$. That is checked because in a uniaxial tension test of material failure

$$\sigma_{test} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_Y - 0)^2 + (\sigma_Y - 0)^2 + (0 - 0)^2} = \frac{\sqrt{2}}{\sqrt{2}} \sigma_Y = \sigma_Y$$

The yield stress of a ductile material is also compared to the stress intensity because in a uniaxial tension test of material failure the maximum shear stress is $\tau_{max} = \sigma_Y / 2$ and the Intensity measure becomes

$$\sigma_{test} = \sigma_Y - 0 = 2\tau_{max}.$$

**Example 14.14-1 Given:** The stress at a point is

$$\sigma^T = [1 \ 1 \ 4 \ -3 \ \sqrt{2} \ -\sqrt{2}] \Leftrightarrow [\sigma] = \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix} \text{MPa}$$

Develop a Matlab script to determine the principal stresses, the intensity, and the equivalent stress. **Solution:** The script in Fig. 14.14-1 gives principal stresses of $\sigma(1:3) = [6, 2, -2]$ MPa, an intensity of 8 MPa (and thus a maximum shear stress of 4 MPa), and an equivalent stress of $\sigma_E = 6.93$ MPa and a shear stress upper bound of 4 MPa.

**Figure 14.14-1 Computing ductile material failure criteria**

**Example 14.14-2 Given:** The stress at a point in a plane-stress model is

$$\sigma^T \Leftrightarrow [\sigma] = \begin{bmatrix} 20 & 10 & 0 \\ 10 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ksi}$$

Determine the in-plane maximum shear stress, the absolute maximum shear stress, the shear stress limit, and compare the first two. **Solution:** Modifying the script in Fig. 14.14-1 gives principal stresses of $\sigma(1:3) = [26.2, 3.82, 0]$ ksi, and the intensity gives the absolute maximum shear stress as $\tau_{max} = 13.1$ ksi, and the upper bound on the shear stress is 14.1 ksi. The maximum in-plane shear stress equation gives $\tau_{plane} = 11.2$ ksi which is about 18% low.
14.15 Mohr’s Circle for principal stresses *: The graphical form of the principal stress calculations is called Mohr’s circle. For more than one-hundred years engineers have used it as a practical way to graphically represent the principal normal stresses and to interpret the general three-dimensional stress state at a point. Figure 14.15-2 (top) shows a Mohr’s circle of three-dimensional stress with the above stress terminology denoted with additional lines denoting the scalar upper bound on the shear stress (actually a point on the shear stress axis) and the scalar value of the Von Mises effective stress (not an actual normal stress, but is compared to them). In the top of that figure the data ($\sigma^T = [40 \ 20 \ -15 \ 10 \ 20 \ -5]$ ksi) gives a Von Mises stress greater than the maximum principal stress and the bottom shows where a change in the sign of one stress value ($\sigma^T = [40 \ 20 \ 15 \ 10 \ 20 \ -5]$ ksi) gives an Effective stress that is less than the maximum principal stress. Today, it is still a useful visualization tool, but it is now constructed from the computed eigenvalues, as shown in Fig. 14.15-3.
Figure 14.15-2 Mohr’s circles where $\sigma_E > \sigma_1$ (top) and $\sigma_E < \sigma_1$

Figure 14.15-3 Mohr’s circles of stress from stress tensor eigenvalues
14.16 Time Independent Schroedinger Equation* (TISE): The TISE is very important in quantum mechanics. It is a wave equation in terms of a waveform, $\Psi$, and the system energy, $E$, which predicts the probability of the distribution of an event or an outcome. The complex time dependent Schroedinger equation is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + V(r) \psi(r, t) = i\hbar \frac{\partial}{\partial t} \psi(r, t)$$

For a harmonic oscillator, $\psi(r, t) = \Psi(r) e^{-i\omega t}$. Substituting that yields the time independent form

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi(r) + V(r) \Psi(r) = \hbar \omega \psi(r, t) = E \Psi(r)$$

Here $\hbar$ is the reduced Planck constant, $m$ is the particle mass, $\omega$ is the frequency of vibration, and $V$ is the Potential Energy. For a one-dimensional quantum harmonic oscillator the form of the Hamiltonian operator is

$$H(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} k x^2.$$ 

The operator on the left is the Hamiltonian, $H(r)$, and the TISE is often written as

$$H(r) \Psi(r) = E \Psi(r)$$

The TISE is an eigenvalue problem with eigenvalues $E_k$ and the corresponding eigenvectors $\Psi_k$. For bound states the eigenvalues are discrete and easily calculated by the finite element method. In quantum mechanics the eigenvalues are desired to be accurate to at least one part in $10^{12}$. The waveform (eigenvector) is an infinitely differentiable ($C^\infty$) function which rapidly oscillates with a decreasing period.

Using typical element interpolation functions gives the matrix form $[S - E_k M] \{ \Psi_k \} = \{ 0 \}$ which becomes the general matrix eigen-problem

$$[S - E_k M] = 0$$

Applying $C^0$ Lagrangian interpolation methods has required as many as 40,000 DOF to reach an acceptable accuracy. Since the waveform is $C^\infty$, it makes sense to employ elements with at least slope and curvature continuity. For one-dimensional elements the Hermite family of interpolations easily give $C^n$ interpolations when $(n + 1)$ DOF per node are used. Also, $C^n$ rectangular elements can be formed from tensor products of the one-dimensional Hermite elements. The domains are typically semi-infinite so geometric errors at the boundary are less important. Triangular and solid elements are desirable for the TISE but it is very difficult to obtain even $C^1$ elements in those forms. There the isogeometric elements seem like a natural choice since they can easily yield at least $C^2$ interpolations.
14.17 Summary

PDE Eigenvalue Form: \[ \nabla^2 u(x, y, t) + \lambda u(x, y, t) = 0 \]

Interpolation: \[ u(x) = H(r) u^e \]

Matrix form: \[ \det [K + \lambda M] = |K + \lambda M| = 0 \]

Eigen-group: \[ [K + \lambda_j M] \delta_j = 0, \quad j = 1, 2, \ldots \]

Iron’s Bound Theorem: \[ \lambda_{min}^e \leq \lambda_{min} \leq \lambda_{max} \leq \lambda_{max}^e \]

Vibration: \[ \lambda = \omega^2, \quad \omega \text{ real} \]

Rigid body motion: \[ \omega = 0 \]

Spring-mass: \[ \omega = \sqrt{k/m} \]

Stiffnesses: \( K_E = \) elastic, \( K_k = \) foundation stiffness, \( K_G = \) geometric stiffness

Buckling: \[ [K_E + K_k + B L F K_G] \delta = 0 \]

Beam-Column: \[ |(K_E + N_B K_n) - \omega^2 M| = 0, \quad K_G = N_B K_n \]

Principle stresses: \[ \sigma_1 > \sigma_2 > \sigma_3 \]

Hydrostatic pressure: \[ p = (\sigma_1 + \sigma_2 + \sigma_3)/3 \]

Experimental yield stresses: \( \sigma_t \) tensile, \( \sigma_c \) compressive, \( \kappa \equiv \sigma_c/\sigma_t \) ratio

Von Mises criterion: \[ \sigma_E = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \]

Burzynski criterion: \[ \sigma_B = \left[ 3p(\kappa - 1) + \sqrt{9p^2(\kappa - 1)^2 + 4\kappa \sigma_E^2} \right]/2\kappa, \quad \sigma_B(\kappa = 1) \equiv \sigma_E \]

Ductile material failure criterion: \[ \sigma_E \geq \sigma_t \text{ or } \sigma_B \geq \sigma_t \]

Lagrange linear line element stiffness matrix:
\[ K^e_{bar} = \frac{E^e A^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, K^e_{shaft} = \frac{G^e f^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, K^e_{spring} = k^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Lagrange quadratic line element stiffness matrix:
\[ K^e_{bar} = \frac{E^e A^e}{3 L^e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \]
Lagrange cubic line element stiffness matrix:

\[
K^e_{\text{bar}} = \frac{E^e A^e}{40L^e} \begin{bmatrix}
148 & -189 & 54 & -13 \\
-189 & 432 & -297 & 54 \\
54 & -297 & 432 & -189 \\
-13 & 54 & -189 & 148
\end{bmatrix}
\]

Lagrange linear line element consistent mass matrix:

\[
M^e = \frac{m^e}{6} \begin{bmatrix} 2 & 1 \\
1 & 2 \end{bmatrix}
\]

Lagrange quadratic line element consistent mass:

\[
M^e = \frac{m^e}{30} \begin{bmatrix} 4 & 2 & -1 \\
2 & 16 & 2 \\
-1 & 2 & 4 \end{bmatrix}
\]

Lagrange cubic line element consistent mass matrix:

\[
M^e = \frac{m^e}{1680} \begin{bmatrix} 128 & 99 & -36 & 19 \\
99 & 648 & -81 & -36 \\
-36 & -81 & 648 & 99 \\
19 & -36 & 99 & 128 \end{bmatrix}
\]

Lagrange linear triangle consistent mass matrix:

\[
M^e_x = \frac{m^e}{12} \begin{bmatrix} 2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2 \end{bmatrix} = M^e_y
\]

\[
M^e(\text{Odd, Odd}) = M^e_x, \quad M^e(\text{Even, Even}) = M^e_y
\]

Lagrange linear line element averaged mass matrix:

\[
M^e = \frac{m^e}{12} \begin{bmatrix} 5 & 1 \\
1 & 5 \end{bmatrix}
\]

Lagrange quadratic line averaged mass:

\[
M^e = \frac{m^e}{60} \begin{bmatrix} 9 & 2 & -1 \\
2 & 36 & 2 \\
-1 & 2 & 9 \end{bmatrix}
\]

Lagrange cubic line averaged mass matrix:

\[
M^e = \frac{m^e}{325920} \begin{bmatrix} 25856 & 9603 & -3492 & 1843 \\
9603 & 130896 & -7857 & -3492 \\
-3492 & -7857 & 130896 & 9603 \\
1843 & -3492 & 9603 & 25856 \end{bmatrix}
\]

Lagrange linear triangle averaged mass matrix:

\[
M^e_x = \frac{m^e}{24} \begin{bmatrix} 6 & 1 & 1 \\
1 & 6 & 1 \\
1 & 1 & 6 \end{bmatrix} = M^e_y
\]

\[
M^e(\text{Odd, Odd}) = M^e_x, \quad M^e(\text{Even, Even}) = M^e_y
\]

Lagrange linear line element diagonalized mass matrix:

\[
M^e = \frac{m^e}{2} \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}
\]
Lagrange quadratic line diagonalized mass:  
$$M^e = \frac{m^e}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lagrange cubic line element diagonalized mass matrix:  
$$M^e = \frac{m^e}{194} \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 81 & 0 & 0 \\ 0 & 0 & 81 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

Lagrange linear triangle diagonalized mass matrix:  
$$M^e_x = \frac{m^e}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M^e_y$$

$$M^e(\text{Odd, Odd}) = M^e_x, \quad M^e(\text{Even, Even}) = M^e_y$$

### 14.18 Exercises

1. Repeat the prior guitar string frequency study to obtain an estimate of the first two smallest frequencies and their relative mode shapes by using three two-node linear elements. (Ans: \(\omega_1 = 3.2863 \sqrt{T/\rho L^2}, \omega_2 = 7.3485 \sqrt{T/\rho L^2}\))

2. Given the stress tensor at a point in a ductile material of  
$$\sigma = \begin{bmatrix} 40 & 10 & 20 \\ 10 & 20 & -5 \\ 20 & -5 & -15 \end{bmatrix} \text{ MPa},$$

determine the maximum compression (-) stress and the limit on the maximum shear stress. (Ans: -23.0, 36.1 MPa)

3. Use a quadratic line element with the consistent mass matrix to find the first two frequency estimates for the axial vibration of a bar and give their percent error (see Ex. 14.4-3). (Ans: %, %)

4. Use a quadratic line element with its diagonalized mass matrix to find the first two frequency estimates for the axial vibration of a bar and give their percent error (see Ex. 14.4-3). (Ans: %, %)

5. For the simple truss in Fig. 14.8-2 determine the critical buckling value if the compressive force at node 1 is horizontal (see Ex. 14.8-2).

6. For the simple truss in Fig. 14.8-2 determine the critical buckling value if the compressive force at node 1 has a slope of 4 horizontal to 3 vertical (see Ex. 14.8-2).

### 14.19 List of examples

- Ex 14.3-1  Natural frequency of a bar with distributed and point masses, L2
- Ex 14.4-1  Effect of tension on string natural frequency, L3
- Ex 14.4-2  Matlab script for string vibration modes and frequencies, L3
- Ex 14.4-3  First two frequencies of fixed-free elastic bar, L3
- Ex 14.5-1  Matlab script for torsional vibrations of a fixed-free shaft, L3
- Ex 14.8-1  Buckling load for a two-bar truss, L2
- Ex 14.8-2  Buckling load for a fixed-pinned beam-column, L3C1
Ex 14.8-3 Buckling of a fixed-pinned beam-column with spring support, L3C1
Ex 14.8-4 Matlab script for buckling of fixed-pinned beam-column, L3C1
Ex 14.9-1 Matlab script for buckling of fixed-pinned tensioned beam-column, L3C1
Ex 14.14-1 Matlab script to find principal stresses for 3D stress tensor
Ex 14.14-2 Find maximum shear stress for 3D stress tensor

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