5. Terminology from differential equations

5.1 Definitions*: An ordinary differential equation (ODE) has a dependent variable, say $u$, and an independent variable (here usually the spatial coordinate $x$, or time $t$). The dependent variable is to be found for all values of the independent variable within some specified length $L$. The boundary conditions are applied at the two bounding end points of that region.

Let $x$ be the independent variable and let $u(x)$ be the dependent variable to be determined inside a specific range of $x$. The differential equation is said to be of order $n$ if the $n$-th derivative of $u$ with respect to (wrt) $x$ is the highest derivative in that equation. An ODE of order $n$ can be written as a set of $n$ first order differential equations. Some authors utilize that approach when applying the least squares finite element analysis (FEA). That change allows the calculus required inter-element continuity to be reduced to simply having the function continuous at the interfaces between elements.

Here, we will consider mainly even order equations, $n = 2m$, in space and first or second order in time. A first-order ODE is said to be linear it can be written in the form

$$\frac{du}{dx} + f(x)u = p(x)$$

or $u' + f(x)u = p(x)$ where $(\cdot)' = du/dx$, and $f(x)$ and $p(x)$ are any given functions of $x$. When $p(x) = 0$, the equation is said to be homogeneous; otherwise it is call non-homogeneous. Likewise, a second-order ODE is said to be linear if it can be written as:

$$\frac{d^2u}{dx^2} + f(x)\frac{du}{dx} + g(x)u = p(x)$$

or $u'' + f(x)u' + g(x)u = p(x)$. For example, $u'' + 2u + \sin(x) = 0$ is linear while $u''u + u' = x$ is non-linear since it contains a product of terms involving the solution and or its derivatives. The
previous equations would be non-linear if their coefficients depended on the solution, for example, if \( f(x) = f(u(x)) \).

Again, if \( p(x) = 0 \) such an equation is said to be homogeneous. The solution of a homogeneous differential equation is called the homogeneous solution or complementary solution. When \( p(x) \neq 0 \), the resulting solution is called the particular solution. The general solution of a differential equation is the sum of the homogeneous solution and the particular solution. A solution is not unique until the boundary conditions have been applied (or enforced) to evaluate the unknown constants in the solution. This requirement carries forward to when we replace the differential (strong) form with an integral (weak) form. The integral form solution is not unique until it satisfies the essential boundary conditions (EBC). Likewise, when finite elements are employed to convert an integral form into a matrix form the matrix system is non-unique (singular) until the matrix system is modified to enforce the essential boundary conditions. Then the matrix solution can be solved and will yield the unique solution.

It can be proved that a finite element approximation of a differential equation will be only exact at the nodes of the mesh if the piecewise spatial approximation (the element interpolation function) contains at least the homogeneous solution. The solution at other points within the element usually will not be exact, but can be exact in some cases. If the element interpolation functions include at least the general solution then the results are exact everywhere in the element.

5.2 Boundary conditions: Boundary conditions are extremely important since they must be enforced to describe and obtain a unique solution. There are two classes of boundary conditions; the essential (or Dirichlet) conditions and the secondary boundary conditions. The essential boundary conditions (EBC) assign known values to the solution, \( u(x, y, z) \), at specified regions on the boundary of the geometric shape, inside which the differential defines the solution. The secondary boundary conditions (NBC) have two versions. The first is the Neumann condition which assigns known values to the derivative of the solution normal (perpendicular) to the boundary like:

\[
a_n \frac{\partial u}{\partial n} = b
\]

(5.2-1)

where \( b \) is the known boundary condition value, \( \vec{n} \) is the direction normal to the boundary, and \( a_n \) frequently denotes a known material constitutive term (property) component evaluated in that direction. In the common case where \( b = 0 \) this becomes a “natural boundary condition” (NatBC) because it is the default on all surfaces in a finite element analysis. It requires no action at all, since the integral of zero is zero.
In one-dimensional problems, keep in mind that at $x_{\text{max}}$ the normal derivative of the solution is $\frac{\partial u}{\partial n} = \frac{du}{dx}$, but at $x_{\text{min}}$ it changes sign to $\frac{\partial u}{\partial n} = -\frac{du}{dx}$ because the normal to that end is in the negative x-direction. The Neumann condition becomes a \textit{natural boundary condition} when $b=0$, because it is the default (finite element) boundary condition on all boundary regions that do not have any other type of boundary condition specified.

The other secondary boundary condition type is the mixed boundary condition (or \textit{Robins} or \textit{Cauchy condition}). In addition to the normal derivative, a mixed condition couples the scaled normal derivative to the unknown value of the solution at the boundary:

$$a \frac{\partial u}{\partial n} + cu = b$$ \hspace{1cm} (5.2-2)

where $c$ is a known boundary property (like a surface-fluid convection coefficient).

Many engineering differential equations are even order (second, fourth or sixth). Consider an even order PDE of order $2m$ in a domain with essential boundary conditions (EBC) and secondary boundary conditions (EBC):

$$\frac{\partial^{2m} u}{\partial x^{2m}} + \ldots + \frac{\partial u}{\partial x} + g(x)u + q(x) = 0$$

\textbf{Table 5.2-1 Boundary condition classes for even order partial differential equations}

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Essential Boundary Conditions</th>
<th>Non-essential Boundary Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial^{2m} u}{\partial x^{2m}} + \ldots + \frac{\partial u}{\partial x} + g(x)u + q(x) = 0$</td>
<td>$u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{(m-1)} u}{\partial x^{(m-1)}}$</td>
<td>$\frac{\partial^{(m)} u}{\partial x^{(m)}} + \ldots + \frac{\partial^{(m+1)} u}{\partial x^{(m+1)}} = 0$</td>
</tr>
<tr>
<td>Note: $\frac{\partial^0 u}{\partial x^0} \equiv u$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let the even order be $n = 2m$ for $m \geq 1$, then the essential boundary conditions involve specifying the value of $u$ and its $(m - 1)$ derivatives. The zero-th derivative is the function value. For even order differential equations the secondary boundary conditions involve specifying the $(m$-th) to $(2m - 1)$ order derivatives. Two common applications to be studied later are listed here:

<table>
<thead>
<tr>
<th>Application</th>
<th>ODE</th>
<th>n</th>
<th>m</th>
<th>EBC [0 to m-1]</th>
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3
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<td>$EI u'''' = p$</td>
<td>4</td>
<td>2</td>
<td>$u$ and $u_{,n}$</td>
<td>$EI u_{,nn} = M$ and $EI u_{,nnn} = V$</td>
</tr>
</tbody>
</table>

where $(,)_{,n} = \partial (,) / \partial n$. In the case of heat conduction, an example of a mixed boundary condition (surface convection) is: $k_n u_{,n} + c u = b$. An ODE of order $n$ must have $n$ boundary conditions applied. In theory, they can be any mixture of EBCs and NBCs. Applying only NBCs defines the solution only to within an arbitrary constant. In that case, using a finite word length computer, the resulting equations are usually singular and it is necessary to specify an arbitrary value of $u$ on the interior of the region. Then, the true solution is the computed one, relative to that given point, plus an arbitrary constant.

5.3 Adjoint operator*: If a PDE is not even order, then the boundary conditions can be established by formulating the adjoint of the PDE. For example, let a homogeneous PDE operator be represented as $L (u) = 0$, and form its inner product with another function, say $v$. That is, integrate the product of the PDE and the second function over the region where the PDE acts:

$$\langle L(u), v \rangle \equiv \int_{\Omega} v L(u) d\Omega.$$ Integrating by parts (sometimes repeatedly) gives an alternate weak form

$$\langle L(u), v \rangle = \langle u, L^* \rangle + \int_{\Gamma} [F(v) G(u) - F(u) G^*(v)] d\Gamma. \tag{5.3-1}$$

Here, $F$ and $G$ are differential operators that appear naturally from the integration(s) by parts processes. The operator $L^*$ is called the adjoint of $L$ and $L$ is called self-adjoint if $L^* = L$. Then, $G^* = G$ also. The $F(u)$ terms are the essential boundary conditions on $\Gamma_1$ and the $G(u)$ terms are the secondary boundary conditions on $\Gamma_2$. Self-adjoint operators always lead to a symmetric algebraic system of equations in FEA.

For an even order PDE, the highest derivative of $u$ occurring in the variational or Galerkin integral form will also be of order $m$. If the highest derivative of $u$ in any integral, over some region, is of order $m$ then that integral can alternately be evaluated as the sum of integrals over the same region if, and only if, $u$ and its $(m-1)$ normal derivatives are continuous across the interface of the adjacent sub-regions.

In the integral form, the NBCs are carried into the integrals and the conversion to a matrix system places them in the column matrix with known values. If mixed NBC are present, then they also contribute known values to the square matrix of the algebraic system.

5.4 Three classes of PDEs*: The above general comments on the EBC and NBC also apply to partial differential equations (PDEs). A typical second order two-dimensional partial differential equation, for any two independent variables, and with $(,)_{,x} = \partial (,) / \partial x$ and etc. is:

$$A(x, y) u_{,xx} + 2B(x, y) u_{,xy} + C(x, y) u_{,yy} + D(x, y) u_{,x} + E(x, y) u_{,y} + F(x, y) u + G(x, y) = 0. \tag{5.4-1}$$

The solutions, $u(x, y)$, fall into three classes:
$B^2 - AC < 0$ for elliptic PDEs.

The solutions are as smooth as the coefficients interior to the solution region. Singular points may occur on the boundary. This book will mainly address even order elliptic PDEs like the Poisson equation:

$$\nabla^2 V + G(x,y) = 0$$

$B^2 - AC = 0$ for parabolic PDEs:

Parabolic PDEs are typically associated with problems where the quantity of interest varies slowly compared to the changes in the input sources. Only parabolic problems advancing in time are covered in this book, and mainly involve transient Fourier heat transfer. Then, the EBC and NBC vary with time, and the value of the solution everywhere at the beginning of the time is given and is referred to as initial conditions (IC).

$B^2 - AC > 0$ for hyperbolic PDEs:

The solution and/or its derivatives can have discontinuities. Hyperbolic PDEs will not be considered in this book. A hyperbolic PDE typically addresses a wave propagation problem, like

$$\nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

and structural dynamics problems like

$$m \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + k u(t) = f(t)$$

and in hyperbolic (non-Fourier) heat transfer.

### 5.5 Eigen-problems

Another type of problem is where the solution coefficient in (5.4-1) is an unknown global constant to be determined, $F(x,y) \equiv \lambda$. This is called an eigen-problem (from the German word *eigen* meaning belonging distinctly to a group). A common PDE containing eigen-values is $\nabla^2 V + \lambda V = 0$ which is solved for multiple eigen-values, $\lambda_n$, and a corresponding set of solution eigen-vectors, $V_n$. Eigen-problems are addressed in detail in Chapter 14.

### 5.6 Model Elliptic PDE

The PDE in (5.4-1) often takes the form of a two-dimensional boundary value problem (BVP) in the region $\Omega$ is:

$$\frac{\partial}{\partial x} \left( k_{xx} \frac{\partial u}{\partial x} \right) + 2 \frac{\partial}{\partial x} \left( k_{xy} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( k_{yy} \frac{\partial u}{\partial y} \right) + g(x,y) u + q(x,y) = 0. \quad (5.6-1)$$

Here, the coefficients $k_{xx}$ etc. usually represent directionally dependent material properties of the domain in which the equation is to be solved. The essential boundary condition, on boundary region, $\Gamma_1$, of the solution region, $\Omega$, is: $u = u_{known}$. On the interior of the region the quantity

$$k_{xx} \frac{\partial u}{\partial x} = \mp q_x$$

represents the $x$-component of a vector quantity, $q_x$, that corresponds to the flux of some quantity per unit area in the $x$-direction. The $y$-component is defined in a similar fashion. In heat transfer,
Fourier’s Law requires the negative sign since the heat, per unit area, flows from higher temperatures to lower temperatures. In ideal fluid flow, using a velocity potential, the plus sign is needed to define the mass flow rate components (or the velocity vector components since the mass density is taken as constant).

On a boundary surface, having a unit normal vector of \( \vec{n} = n_x \hat{i} + n_y \hat{j} \), the dot product of that normal vector with the above flux vector, \( \vec{q} \), gives the scalar flow rate per unit area, say \( q_s \), entering or exiting the domain:

\[
(k_{xx} \frac{\partial u}{\partial x}) n_x + (k_{yy} \frac{\partial u}{\partial y}) n_y = k_n \frac{\partial u}{\partial n} = \mp q_s.
\]

A secondary boundary condition, on boundary region \( \Gamma_2 \) often involves specifying that entering or exiting flux:

\[
(k_{xx} \frac{\partial u}{\partial x}) n_x + (k_{yy} \frac{\partial u}{\partial y}) n_y = b_{\text{known}} \text{, or } (k_n \frac{\partial u}{\partial n}) = b_{\text{known}}
\]

where the unit normal vector on the boundary is \( \vec{n} = n_x \hat{i} + n_y \hat{j} \). This is often given as a natural boundary condition, on \( \Gamma_3 \), of:

\[
(k_n \frac{\partial u}{\partial n}) = 0
\]

or a mixed boundary condition, on \( \Gamma_3 \):

\[
(k_n \frac{\partial u}{\partial n}) + cu = b_{\text{known}}
\]

where the union (\( \cup \)) of all the boundary regions forms the total boundary of the solution domain \( \Gamma = \bigcup_k \Gamma_k \). An example image of a solution with an internal point source with an essential or a natural boundary condition on one boundary is sketched in Fig. 5.2-1.

### 5.7 Directionally dependent data

Often the coefficients of the PDF, \( k_{xx} \), \( k_{xy} \), and \( k_{yy} \) vary with direction and location. Any data that are direction dependent are called \textit{anisotropic}. Any data that vary with location are called \textit{non-homogeneous}. The finite element method automatically includes the ability to treat such real world materials. Usually and anisotropic material is assumed to be constant in a single element. Fig. 5.7-1 shows such an element and the angle, \( \theta \), to the principal material axes that were used to experimentally establish its directionally dependent properties.

![Figure 5.7-1 Constant principle material directions in an element](image)

The user must define the ‘direction angles’ to the material axes along with the principle material properties. An orthotropic material has a diagonal material matrix when viewed in the
principle material axes. Since the PDE is written in the global axes the material data must be transformed from the local element material axes to the global coordinates. Typically, that is done via a triple matrix product:

\[ k^e(\theta^e)_{\text{Global}} = T(\theta^e)^T \ k^e_{\text{Local}} \ T(\theta^e) \]  

(5.7-1)

where the matrix \( T \) depends on the direction cosines to the local axes. When the property matrix is diagonal the material is called orthotropic. In two-dimensional heat transfer the data represents the thermal conductivity in the local axes:

\[ k_{\text{Global}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} = k_{\text{Global}} \ T \ T(\theta^e) \]  

(5.7-2)

In stress analysis, the material properties are the elastic modulus and Poisson ratios in the local directions and the transformation matrix is the Mohr’s circle for stress written in matrix form.

\[ \begin{align*}
\text{Example 5.7-1 Given:} & \quad \text{An orthotropic thermal conductivity occurs at an angle of } \theta^e = 65 \text{ degrees in an element with the properties:} \\
& \quad k_{\text{Local}} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 0.007 & 0 \\ 0 & 0.7 \end{bmatrix} \text{ BTU/in} \cdot \text{sec} \cdot \text{F} \\
& \quad \text{Convert those data to the global values for the two-dimensional Laplace equation. Solution:} \quad \text{At 65 degrees the transformation is} \\
& \quad T(\theta^e) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.423 & 0.906 \\ -0.906 & 0.423 \end{bmatrix}. \\
& \quad \text{Evaluating the triple matrix product:} \\
& \quad k_{\text{Global}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} = \begin{bmatrix} 0.423 & -0.906 \\ 0.906 & 0.423 \end{bmatrix} \begin{bmatrix} 0.007 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 0.423 & 0.906 \\ -0.906 & 0.423 \end{bmatrix} \\
& \quad k_{\text{Global}} = \begin{bmatrix} 0.576 & -0.266 \\ -0.266 & 0.131 \end{bmatrix} \text{ BTU/in} \cdot \text{sec} \cdot \text{F} \\
\end{align*} \]

5.8 Point singularities*: It is a property of all elliptic PDEs that the boundary of the solution domain often has points where the solution is singular. Such points occur where there is a discontinuity in assigned EBC values, where an EBC jumps to an adjacent NBC and where the solution domain contains a sharp reentrant corner. As such points are approached the solution has an infinite derivative in the radial direction outward from the point \( \partial u/\partial r \to \infty \text{ as } r \to 0 \) and tends to change rapidly in the transverse direction, \( \theta \).

For a re-entrant corner, the ‘strength’ of the singularity depends on the value of the radial exponent, \( \alpha = \pi/\beta < 1 \), where the local singular solution in cylindrical coordinates varies as \( u = \sum k c k r^\alpha f_k(\theta) \). The first and most important term simply depends on the angle, \( \beta \), interior to the domain at the tip of a re-entrant sharp corner. The strongest case is a re-entrant slit or crack \( \beta = 2\pi \text{ so } \alpha = 1/2 \). The most common case of a singular point is a right angle corner \( \beta = 3\pi/2 \text{ so } \alpha = 2/3 \) which is much less severe that a crack geometry.
Figure 5.7-1 Typical singularity point locations
5.8 Summary

Boundary condition classes for even order partial differential equations:

<table>
<thead>
<tr>
<th>Differential Equation</th>
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</tr>
</thead>
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<tr>
<td>Essential Boundary Conditions</td>
<td>$u, \ \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{(m-1)}u}{\partial x^{(m-1)}}$</td>
</tr>
<tr>
<td>Non-essential Boundary Conditions</td>
<td>$\frac{\partial^{(m)}u}{\partial x^{(m)}}, \ \frac{\partial^{(m+1)}u}{\partial x^{(m+1)}}, \ldots, \frac{\partial^{(2m-1)}u}{\partial x^{(2m-1)}}$</td>
</tr>
</tbody>
</table>

Note: $\frac{\partial^0 u}{\partial x^0} \equiv u$

Boundary conditions for PDE $L(u) = 0$ via the Adjoint:

$\int_{\Omega} L(u(x,y)) \ v(x,y) \ d\Omega \equiv \langle L(u), v \rangle = \langle u, L^* \rangle + \int_{\Gamma} [F(v)G(u) - F(u)G^*(v)] \ d\Gamma$

Essential boundary condition: $F(u)$ on $\Gamma_1$.
Non-essential boundary condition: $G(u)$ on $\Gamma_2$

Common one-dimensional boundary conditions:

<table>
<thead>
<tr>
<th>Application</th>
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<th>n</th>
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<td>$EI \ u_{nn} = M$ and $EI \ u_{nnn} = V$</td>
</tr>
</tbody>
</table>

Anisotropic material data transformations:

$k^e(\theta^e)_{Global} = T(\theta^e)^T \ k^e_{Local} \ T(\theta^e)$

Typical elliptical two-dimensional PDE:

$\frac{\partial}{\partial x} (k_{xx} \frac{\partial u}{\partial x}) + 2 \frac{\partial}{\partial x} (k_{xy} \frac{\partial u}{\partial y}) + \frac{\partial}{\partial y} (k_{yy} \frac{\partial u}{\partial y}) + g(x,y)\ u + q(x,y) = 0$

A mixed boundary condition: ($b$ and/or $c$ can be zero)

$\left(k_n \frac{\partial u}{\partial n}\right) + cu = b$
5.9 Exercises

510 List of examples
Ex 5.7-1 Transform local orthotropic conduct to global anisotropic values

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