Chapter 5

ONE-DIMENSIONAL INTEGRATION

5.1 Introduction

Since the finite element method is based on integral relations it is logical to expect that one should strive to carry out the integrations as efficiently as possible. In some cases we will employ exact integration. In others we may find that the integrals can become too complicated to integrate exactly. In such cases the use of numerical integration will prove useful or essential. The important topics of local coordinate integration and Gaussian quadratures will be introduced here. They will prove useful when dealing with higher order interpolation functions in complicated element integrals.

5.2 Local Coordinate Jacobian

We have previously seen that the utilization of local element coordinates can greatly reduce the algebra required to establish a set of interpolation functions. Later we will see that some 2-D elements must be formulated in local coordinates in order to meet the interelement continuity requirements. Thus, we should expect to often encounter local coordinate interpolation. However, the governing integral expressions must be evaluated with respect to a unique global or physical coordinate system. Clearly, these two coordinate systems must be related. The relationship for integration with a change of variable (change of coordinate) was defined in elementary concepts from calculus. At this point it would be useful to review these concepts from calculus. Consider a definite integral

$$I = \int_a^b f(x)\, dx\,, \qquad a < x < b \tag{5.1}$$

where a new variable of integration, r, is to be introduced such that $x = x(r)$. Here it is required that the function $x(r)$ be continuous and have a continuous derivative in the interval $\alpha \le r \le \beta$. The region of r directly corresponds to the region of x such that when r varies between α and β, then x varies between $a = x(\alpha)$ and $b = x(\beta)$. In that case

 © 2003 J.E. Akin

$$I = \int_a^b f(x)\, dx = \int_\alpha^\beta f(x(r))\, \frac{dx}{dr}\, dr \tag{5.2}$$

or

$$I = \int_\alpha^\beta f(r)\, J\, dr \tag{5.3}$$

where $J = dx/dr$ is called the *Jacobian* of the coordinate transformation.

5.3 Exact Polynomial Integration *

If we utilize the unit coordinates, then $\alpha = 0$ and $\beta = 1$. Then from Sec. (4.2), the Jacobian is $J = L^e$ in an element domain defined by linear interpolation. By way of comparison, if one employs natural coordinates, then $\alpha = -1$, $\beta = +1$, and from Eq. (4.16) $J = L^e/2$. Generally, we will use interpolation functions that are polynomials. Thus, the element integrals of them and/or their derivatives will also contain polynomial terms. Therefore, it will be useful to consider expressions related to typical polynomial terms. A typical polynomial term is r^m where m is an integer. Thus, from the above

$$I = \int_{x_1^e}^{x_2^e} r^m\, dx = \int_0^1 r^m\, L^e\, dr = L^e\, \frac{r^{(1+m)}}{1+m}\,\Bigg|_0^1 = \frac{L^e}{(1+m)}\,. \tag{5.4}$$

A similar expression can be developed for the natural coordinates. It gives

$$I = \int_{L^e} n^m\, dx = \frac{L^e}{m+1} \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\,. \end{cases} \tag{5.5}$$

Later we will tabulate the extension of these concepts to two- and three-dimensional integrals. As an example of the use of Eq. (5.5), consider the integration

$$I = \int_{L^e} \mathbf{H}^{e^T}\, \mathbf{H}^e\, dx\,.$$

Recall that the integral of a matrix is the matrix resulting from the integration of each of the elements of the original matrix. If linear interpolation is selected for $\mathbf{H}^e$ on a line element then typical terms will include H_1^2, $H_1\, H_2$, etc. Thus, one obtains:

$$I_{11} = \int_{L^e} H_1^2(r)\, dx = \int_{L^e} (1-r)^2\, dx = L^e(1 - 2/2 + 1/3) = \frac{L^e}{3}$$

$$I_{12} = \int_{L^e} H_1\, H_2\, dx = \int_{L^e} (1-r)\, r dx = L^e\,(1/2 - 1/3) = \frac{L^e}{6}$$

$$I_{22} = \int_{L^e} H_2^2\, dx = \int_{L^e} r^2\, dx = \frac{L^e}{3}$$

so that

$$I = \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{5.6}$$

Similarly, if one employs the Lagrangian quadratic $\mathbf{H}$ in Eq. (4.17) one obtains:

 © 2003 J.E. Akin. All rights reserved.

$$I = \frac{L^e}{30}\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix}. \tag{5.7}$$

By way of comparison, if one selects the hierarchical quadratic polynomial in Eq. (4.22) the above integral becomes

$$I = \frac{L^e}{6}\left[\begin{array}{ccc|c} 2 & 1 & & -1/4 \\ 1 & 2 & & -1/4 \\ - & - & & - \\ \hline -1/4 & -1/4 & & 1/10 \end{array}\right].$$

Note that the top left portion of this equation is the same as Eq. (5.7) which was obtained from the linear polynomial. This desirable feature of hierarchical elements was mentioned in Sec. 4.6.

Before leaving the subject of simplex integrations one should give consideration to the common special case of axisymmetric geometries, with coordinates (ρ, z). Recall from calculus that the Theorem of Pappus relates a differential volume and surface area to a differential area and length in the (ρ, z) plane of symmetry, respectively. That is, $dv = 2\pi\rho\, dA$ and $dS = 2\pi\rho dl$, where ρ denotes the radial distance to the differential element. Thus, typical axisymmetric surface integrals reduce to

$$\mathbf{I}_S = \int_S \mathbf{H}^T\, dS = 2\pi \int_{L^e} \mathbf{H}^T \rho\, dl = 2\pi\left(\int_{L^e} \mathbf{H}^T\mathbf{H}\, dl\right)\boldsymbol{\rho}^e = \frac{2\pi L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{\rho}^e,$$

since $\rho = \mathbf{H}\boldsymbol{\rho}^e$. Many workers like to omit the 2π term and work on a per-unit-radian basis so that they can more easily do both two-dimensional and axisymmetric calculations with a single program.

5.4 Numerical Integration

Numerical integration is simply a procedure that approximates (usually) an integral by a summation. To review this subject we refer to Fig. 5.4.1. Recall that the integral

$$I = \int_a^b f(x)\, dx \tag{5.8}$$

can be viewed graphically as the area between the x-axis and the curve $y = f(x)$ in the region of the limits of integration. Thus, we can interpret numerical integration as an approximation of that area. The *trapezoidal rule* of numerical integration simply approximates the area by the sum of several equally spaced trapezoids under the curve between the limits of a and b. The height of a trapezoid is found from the integrand, $y_j = y(x_j)$, evaluated at equally spaced points, x_j and x_{j+1} . Thus, a typical contribution is $A = h(y_j + y_{j+1})/2$, where $h = x_{j+1} - x_j$ is the spacing. Thus, for n points (and $n - 1$ spaces), the well-known approximation is

 © 2003 J.E. Akin. All rights reserved.

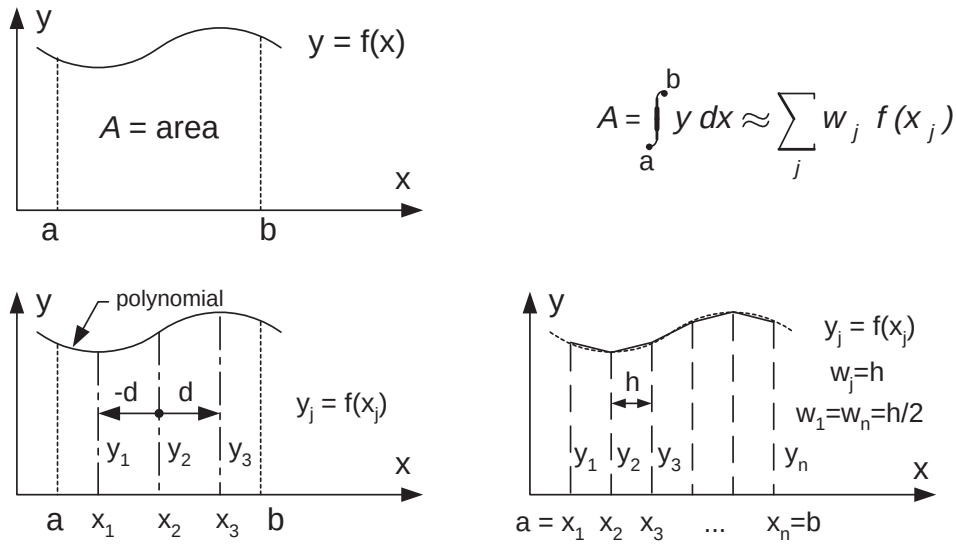


Figure 5.4.1 One-dimensional numerical integration

$$I \approx h\left(\frac{1}{2}\,y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{1}{2}\,y_n\right), \quad I \approx \sum_{j=1}^{n} w_j\, f(x_j) \tag{5.9}$$

where $w_j = h$, except $w_1 = w_n = h/2$. A geometrical interpretation of this is that the area under curve, I, is the sum of the products of certain heights, $f(x_j)$ times some corresponding widths, w_j. In the terminology of numerical integration, the locations of the points, x_j, where the heights are computed are called *abscissae* and the widths, w_j, are called *weights*. Another well-known approximation is the *Simpson rule*, which uses parabolic segments in the area approximation. For most functions the above rules may require 20 to 40 terms in the summation to yield acceptable accuracy. We want to carry out the summation with the minimum number of terms, n, in order to reduce the computational cost. What is the minimum number of terms? The answer depends on the form of the integrand $f(x)$. Since the parametric geometry usually involves polynomials we will consider that common special case for $f(x)$.

The famous mathematician Gauss posed this question: What is the minimum number of points, n, required to exactly integrate a polynomial, and what are the corresponding abscissae and weights? If we require the summation to be exact when $f(x)$ is any one of the power functions $1, x, x^2, \ldots, x^{2n-1}$, we obtain a set of $2n$ conditions that allow us to determine the n abscissae, x_i, and their corresponding n weights, w_j. The *n Gaussian quadrature* points are symmetrically placed with respect to the center of the interval, and will exactly integrate a polynomial of order $(2n + 1)$. The center point location is included in the abscissae list when n is odd, but the end points are never utilized in a Gauss rule. The Gauss rule data are usually tabulated for a non-dimensional *unit coordinate* range of $0 \le t \le 1$, or for a *natural coordinate* range of $-1 \le t \le +1$. Table 5.1 presents the low-order Gauss rule data in natural coordinates, and the alternate unit coordinate data are in Table 5.2. A two-point Gauss rule can often

 © 2003 J.E. Akin. All rights reserved.

Table 5.1 Abscissas and weights for Gaussian quadrature

$$\int_{-1}^{+1} f(x)\,dx = \sum_{i=1}^{n} w_i\ f(x_i)$$

$\pm x_i$		w_i
0.00000 00000 00000 00000 0000	$n = 1$	2.00000 00000 00000 00000 000
0.57735 02691 89625 76450 9149	$n = 2$	1.00000 00000 00000 00000 000
0.77459 66692 41483 37703 5835	$n = 3$	0.55555 55555 55555 55555 556
0.00000 00000 00000 00000 0000		0.88888 88888 88888 88888 889
0.86113 63115 94052 57522 3946	$n = 4$	0.34785 48451 37453 85737 306
0.33998 10435 84856 26480 2666		0.65214 51548 62546 14262 694
0.90617 98459 38663 99279 7627	$n = 5$	0.23692 68850 56189 08751 426
0.53846 93101 05683 09103 6314		0.47862 86704 99366 46804 129
0.00000 00000 00000 00000 0000		0.56888 88888 88888 88888 889
0.93246 95142 03152 02781 2302	$n = 6$	0.17132 44923 79170 34504 030
0.66120 93864 66264 51366 1400		0.36076 15730 48138 60756 983
0.23861 91860 83196 90863 0502		0.46791 39345 72691 04738 987
0.94910 79123 42758 52452 6190	$n = 7$	0.12948 49661 68869 69327 061
0.74153 11855 99394 43986 3865		0.27970 53914 89276 66790 147
0.40584 51513 77397 16690 6607		0.38183 00505 05118 94495 037
0.00000 00000 00000 00000 0000		0.41795 91836 73469 38775 510

Table 5.2 Unit abscissas and weights for Gaussian quadrature

$$\int_{0}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i\ f(x_i)$$

x_i		w_i
0.50000 00000 00000 00000 000	$n = 1$	1.00000 00000 00000 00000 000
0.21132 48654 05187 11774 543	$n = 2$	0.50000 00000 00000 00000 000
0.78867 51345 94812 88225 457		0.50000 00000 00000 00000 000
0.11270 16653 79258 31148 208	$n = 3$	0.27777 77777 77777 77777 778
0.50000 00000 00000 00000 000		0.44444 44444 44444 44444 444
0.88729 83346 20741 68851 792		0.27777 77777 77777 77777 778
0.06943 18442 02973 71238 803	$n = 4$	0.17392 74225 68726 92868 653
0.33000 94782 07571 86759 867		0.32607 25774 31273 07131 347
0.66999 05217 92428 13240 133		0.32607 25774 31273 07131 347
0.93056 81557 97026 28761 197		0.17392 74225 68726 92868 653
0.04691 00770 30668 00360 119	$n = 5$	0.11846 34425 28094 54375 713
0.02307 65344 94715 84544 818		0.23931 43352 49683 23402 065
0.50000 00000 00000 00000 000		0.28444 44444 44444 44444 444
0.76923 46550 52841 54551 816		0.23931 43352 49683 23402 065
0.95308 99229 69331 99639 881		0.11846 34425 28094 54375 713

© 2003 J.E. Akin. All rights reserved.

exceed the accuracy of a 20-point trapezoidal rule. When computing norms of an exact solution, to be compared to the finite element solution, we often use the trapezodial rule. That is because the exact solution is usually not a polynomial and the Gauss rule may not be accurate.

Sometimes it is desirable to have a numerical integration rule that specifically includes the two end points in the abscissae list when ($n \geq 2$). The *Lobatto rule* is such an alternate choice. Its n points will exactly integrate a polynomial of order $(2n - 3)$ for $n > 2$. Its data are included in Table 5.3. It is usually less accurate than the Gauss rule but it can be useful. Mathematical handbooks give tables of Gauss or Lobatto data for much higher values of n. Some results of Gauss's work are outlined below. Let y denote $f(x)$ in the integral to be computed. Define a change of variable

$$x(r) = 1/2\,(b - a)\,r + 1/2\,(b + a) \tag{5.10}$$

so that the non-dimensional limits of integration of r become -1 and $+1$. The new value of $y(r)$ is

$$y = f(x) = f\,[\,1/2\,(b - a)\,r + 1/2\,(b + a)\,] = \Phi(r)\,. \tag{5.11}$$

Noting from Eq. (5.10) that $dx = 1/2\,(b - a)\,dr$, the original integral becomes

$$I = \frac{1}{2}\,(b - a)\int_{-1}^{1} \Phi(r)\,dr\,. \tag{5.12}$$

Gauss showed that the integral in Eq. (5.12) is given by

Table 5.3. Abscissas and weight factors for Lobatto integration

$$\int_{-1}^{+1} f(x)\,dx \approx \sum_{i=1}^{n} w_i\ f(x_i)$$

$\pm\, x_i$		w_i
0.00000 00000 00000	$n = 1$	2.00000 00000 00000
1.00000 00000 00000	$n = 2$	1.00000 00000 00000
1.00000 00000 00000	$n = 3$	0.33333 33333 33333
0.00000 00000 00000		1.33333 33333 33333
1.00000 00000 00000	$n = 4$	0.16666 66666 66667
0.44721 35954 99958		0.83333 33333 33333
1.00000 00000 00000	$n = 5$	0.10000 00000 00000
0.65465 36707 07977		0.54444 44444 44444
0.00000 00000 00000		0.71111 11111 11111
1.00000 00000 00000	$n = 6$	0.06666 66666 66667
0.76505 53239 29465		0.37847 49562 97847
0.28523 15164 80645		0.55485 83770 35486

 © 2003 J.E. Akin. All rights reserved.

$$\int_{-1}^{1} \Phi(r)\, dr \;=\; \sum_{i=1}^{n} W_i\, \Phi(r_i)\,,$$

where W_i and r_i represent tabulated values of the *weight functions* and *abscissae* associated with the n points in the non-dimensional interval (−1, 1). The final result is

$$I \;=\; \frac{1}{2}(b-a)\sum_{i=1}^{n} W_i\, \Phi(r_i) \;=\; \sum_{i=1}^{n} f(\,x(r_i)\,)\, W_i\,. \tag{5.13}$$

Gauss also showed that this equation will exactly integrate a polynomial of degree $(2n-1)$. For a higher number of space dimensions (which range from -1 to $+1$), one obtains a multiple summation. Since Gaussian quadrature data are often tabulated in references for the range $-1 \le r \le +1$, it is popular to use the natural coordinates in defining element integrals. However, one can convert the tabulated data to any convenient system such as the unit coordinate system where $0 \le r \le 1$. The latter may be more useful on triangular regions. As an example of Gaussian quadratures, consider the following one-dimensional integral:

$$I \;=\; \int_{1}^{2} \begin{bmatrix} 2 & 2x \\ 2x & (1+2x^2) \end{bmatrix} dx \;=\; \int_{1}^{2} \mathbf{F}(x)\;\; dx\,.$$

If two Gauss points are selected, then the tabulated values from Table 5.1 give $W_1 = W_2 = 1$ and $r_1 = 0.57735 = -r_2$. The change of variable gives $x(r) = (r+3)/2$, so that $x(r_1) = 1.788675$ and $x(r_2) = 1.211325$. Therefore, from Eq. (5.13)

$$I \;=\; \tfrac{1}{2}(2-1)\Big[\, W_1\, \mathbf{F}(\,x(r_1)\,) + W_2\, \mathbf{F}(\,x(r_2)\,)\,\Big]$$

$$= \tfrac{1}{2}(1)\left((1)\begin{bmatrix} 2 & 2(1.788675) \\ sym. & 1+2(1.788675)^2 \end{bmatrix} + (1)\begin{bmatrix} 2 & 2(1.211325) \\ sym. & 1+2(1.211325)^2 \end{bmatrix}\right)$$

$$I \;=\; \begin{bmatrix} 2.00000 & 3.00000 \\ 3.00000 & 5.66667 \end{bmatrix},$$

which is easily shown to be in good agreement with the exact solution. As another example consider a typical term in Eq. (5.8). Specifically, from Eqs. (4.16) and (4.18)

$$I_{33} \;=\; \int_{L^e} H_3^2\; dx \;=\; \frac{L^e}{2}\int_{-1}^{+1} (1-n^2)^2\; dn\,.$$

Since the polynomial terms to be integrated are fourth order, we should select $(2m-1) = 4$, or $m = 3$ Gaussian points. Then,

$$I_{33} \;=\; \int_{L^e} H_3^2\; dx \;=\; \frac{L^e}{2}\int_{-1}^{+1} H_3^2\,(n)\; dn$$

 © 2003 J.E. Akin. All rights reserved.

$$I_{33} = \frac{L^e}{2}\left(0.55556\left[1.00000 - (0.77459)^2 \right]^2 \right.$$

$$\left. + 0.88889\left[1.00000 - (0.0)^2 \right]^2 + 0.55556\left[1.00000 - (+0.77459)^2 \right]^2 \right)$$

$$I_{33} = \frac{L^e}{2}(0.08889 + 0.88889 + 0.08889) = 0.5333\, L^e ,$$

which agrees well with the exact value of $16\, L^e/30$ given in Eq. (5.7).

5.5 Variable Jacobians

When the parametric space and physical space have the same number of dimensions then the Jacobian is a square matrix. Otherwise, we need to use more calculus to evaluate the integrals. For example, we often find the need to execute integrations along a two-dimensional curve defined by our one-dimensional parametric representation. Consider a planar curve in the xy-plane such as that shown in Fig. 5.4.2. We may need to know its length and first moments (centroid), which are defined as

$$L = \int_L ds \ , \qquad \bar{x}\, L = \int_L x(r)\, ds \ , \qquad \bar{y}\, L = \int_L y(r)\, ds \ ,$$

respectively, where ds denotes the physical length of a segment, dr, of the parametric length. To evaluate these quantities we need to convert to an integral in the parametric space. For example,

$$L = \int_L ds = \int_0^1 \frac{ds}{dr}\, dr \ .$$

To relate the physical and parametric length scales, we must first recall the planar relation that $ds^2 = dx^2 + dy^2$. Since both x and y are defined in terms of r, we can extend this identity to the needed quantity

$$\left(\frac{ds}{dr} \right)^2 = \left(\frac{dx}{dr} \right)^2 + \left(\frac{dy}{dr} \right)^2$$

where dx/dr can be found from the spatial interpolation functions, etc. for dy/dr. Thus, our physical length is defined in terms of the parametric coordinate, r, and the spatial data for the nodes defining the curve location in the xy-plane (i.e., x^e and y^e) :

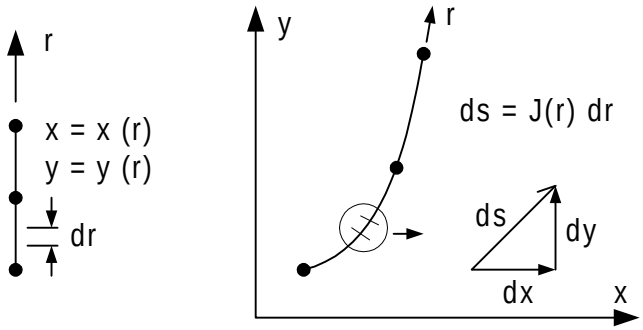
$$L = \int_0^1 \sqrt{\left(\frac{dx}{dr} \right)^2 + \left(\frac{dy}{dr} \right)^2}\ dr$$

where

$$\frac{dx(r)}{dr} = \sum_{i=1}^{N} \frac{d\, H_i(r)}{dr}\, x_i^e \ , \qquad dy\, \frac{(r)}{dr} = \sum_{i=1}^{N} \frac{d\, H_i(r)}{dr}\, y_i^e \ .$$

Note that this does preserve the proper units for L , and it has what we could refer to as a variable Jacobian. The preceding integral is trivial only when the planar curve is a straight line ($N = 2$). Then, from linear geometric interpolation $dx/dr = x_2^e - x_1^e$ and $dy/dr = y_2^e - y_1^e$ are both constant, and the result simplifies to

 © 2003 J.E. Akin. All rights reserved.



$$J \equiv \sqrt{(dx/dr)^2 + (dy/dr)^2}$$

Figure 5.4.2 A variable curve metric or Jacobian

$$L^e = \sqrt{\left(x_2^e - x_1^e\right)^2 + \left(y_2^e - y_1^e\right)^2} \int_0^1 dr = L$$

which, by inspection, is exact. For any other curve shape, these integrals become unpleasant to evaluate, and we would consider their automation by means of numerical integration.

To complete this section we outline an algorithm to automate the calculation of the y-centroid:

1. Recover the N points describing the curve, $\mathbf{x}^e$, $\mathbf{y}^e$.
2. Recover the M-point quadrature data, r_j and w_j.
3. Zero the integrals: $L = 0$, $\bar{y}\, L = 0$.
4. Loop over all the quadrature points: $1 \le j \le M$. At each local quadrature point, t_j:

A. Find the length scales (i.e., Jacobian):

(i) Compute local derivatives of the N interpolation functions

$$\mathbf{DLH}_j \equiv \left.\frac{\partial \mathbf{H}}{\partial r}\right|_{r=r_j}$$

(ii) Get x- and y-derivatives from curve data

$$\frac{dx}{dr_j} \equiv \mathbf{DLH}_j\, \mathbf{x}^e = \sum_{i=1}^{N} \frac{\partial H_i(r_j)}{\partial r}\, x_i^e, \qquad \frac{dy}{dr_j} \equiv \mathbf{DLH}_j\, \mathbf{y}^e$$

 © 2003 J.E. Akin. All rights reserved.

(iii) Find length scale at point r_j

$$\frac{ds}{dr_j} = \sqrt{\left(\frac{dx}{dr}\right)_j^2 + \left(\frac{dy}{dr}\right)_j^2}$$

B. Evaluate the integrand at point r_j :

(i) Evaluate the N interpolation functions :

$$\mathbf{H}_j \equiv \mathbf{H}(r_j)$$

(ii) Evaluate y from curve data

$$y_j = \mathbf{H}_j \, \mathbf{y}^e = \sum_{i=1}^{N} H_i(r_j) \, y_i^e$$

C. Form products and add to previous values

$$L = L + \frac{ds}{dr_j} w_j \ , \qquad \bar{y} \, L = \bar{y} \, L + y_j \, \frac{ds}{dr_j} w_j \ .$$

5. Evaluate items computed from the completed integrals : $\bar{y} = \bar{y} \, L / L$, etc. for $\bar{x}$.

Note that to automate the integration we simply need: (1) storage of the quadrature data, r_j and w_j, (2) access to the curve data, $\mathbf{x}^e$, $\mathbf{y}^e$, (3) a subroutine to find the parametric derivative of the interpolation functions at any point, r , and (4) a function program to evaluate the integrand(s) at any point, r . This usually requires (see Step **4B**) a subroutine to evaluate the interpolation functions at any point, r . The evaluations of the interpolation products, at a point r_j, can be thought of as a dot product of the data array $\mathbf{x}^e$ and the evaluated interpolation quantities $\mathbf{H}_j$ and $\mathbf{DLH}_j$.

5.6 Exercises

1. In Tables 5.1 and 5.3 the sum of the weights is exactly 2, but in Table 5.2 the sum is exactly 1. Explain why.

2. Note that the Gauss abscissas are always interior to the domain. The Lobatto rules for $n \geq 2$ always includes the two end points. Discuss an advantage or disadvantage of including the end points.

3. For a one-dimensional quadratic element use Gaussian quadratures to numerically evaluate the matrices:

$$a) \quad \mathbf{C}^e = \int_{L^e} \mathbf{H}^T \, dx, \; b) \quad \mathbf{M}^e = \int_{L^e} \mathbf{H}^T \, \mathbf{H} \, dx \, ,$$

$$c) \quad \mathbf{S}^e = \int_{L^e} \frac{d\mathbf{H}^T}{dx} \, \frac{d\mathbf{H}}{dx} \, dx \, , \; d) \quad \mathbf{U}^e = \int_{L^e} \mathbf{H}^T \, \frac{d\mathbf{H}}{dx} \, dx \, .$$

4. Use Lobatto quadratue to evaluate the matrices in problem 3.

 © 2003 J.E. Akin. All rights reserved.

5.7 References

[1] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, National Bureau of Standards (1964).

[2] Carey, G.F. and Oden, J.T., *Finite Elements − Computational Aspects*, Englewood Cliffs: Prentice-Hall (1984).

[3] Hinton, E. and Owen, D.R.J., *Finite Element Programming*, London: Academic Press (1977).

[4] Hughes, T.J.R., *The Finite Element Method*, Englewood Cliffs: Prentice-Hall (1987).

[5] Stroud, A.H. and Secrest, D., *Gaussian Quadrature Formulas*, Englewood Cliffs: Prentice-Hall (1966).

[6] Zienkiewicz, O.C. and Taylor, R.L., *The Finite Element Method*, 5th Edition, London: Arnold (2000).

 © 2003 J.E. Akin. All rights reserved.