Global MWR in FEA notation

2.15 $L(u) \equiv \dfrac{d^2u}{dx^2} + u + x = 0$, $u(0) = 0 = u(1)$

exact $u(x) = \dfrac{\sin x}{\sin 1} - 1$.

satisfies BC

Guess a solution $u^*(x) = g(x)\, f(x, \Delta_i)$ including constants Δ_i for $1 \le i \le n$. Here, try a polynomial $f(x, \Delta_i) = \Delta_1 + \Delta_2 x + \dots + \Delta_n x^{(n-1)}$

2.17 Try $n = 2$ so $u^*(x) = x(1-x)(\Delta_1 + \Delta_2 x)$

$$u^*(x) = (x - x^2)\Delta_1 + (x^2 - x^3)\Delta_2$$
$$= \sum_{i=1}^{n=2} H_i(x)\, \Delta_i$$

with $H_1(x) = (x - x^2)$, $H_2(x) = (x^2 - x^3)$ or in matrix form

$$\underset{1\times1}{u^*(x)} = \underset{1\times2}{\begin{bmatrix} H_1(x) & H_2(x) \end{bmatrix}} \underset{2\times1}{\begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix}}$$

2.17' $u^*(x) = \underset{1\times2}{[H(x)]}\,\underset{2\times1}{\{\Delta\}}$

The derivatives of $u^*(x)$ are

$$\frac{du^*}{dx} = \sum_{i=1}^{n} \frac{dH_i(x)}{dx} \Delta_i$$

$$= (1-2x)\Delta_1 + (2x-3x^2)\Delta_2$$

$$= \begin{bmatrix} (1-2x) & (2x-3x^2) \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{dH_1(x)}{dx} & \frac{dH_2(x)}{dx} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix}$$

$$\frac{du^*}{dx} = \underbrace{\left[\frac{dH(x)}{dx}\right]}_{1\times 2} \underbrace{\{\Delta\}}_{2\times 1} \quad \text{or} \quad \frac{d}{dx}[H(x)]\{\Delta\}$$

and

$$\frac{d^2u^*}{dx^2} = \sum_{i=1}^{n} \frac{d^2H_i(x)}{dx^2} \Delta_i$$

$$= (-2)\Delta_1 + (2-6x)\Delta_2$$

$$= \begin{bmatrix} -2 & (2-6x) \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix}$$

$$\frac{d^2u^*}{dx^2} = \underbrace{\left[\frac{d^2H(x)}{dx^2}\right]}_{1\times 2} \underbrace{\{\Delta\}}_{2\times 1}$$

The residual error is

$$\left(\frac{d^2u^*}{dx^2} + u^*(x) + x\right) = R(x) \neq 0$$

2.18

The residual is

$$R(x) = \left((-2)\Delta_1 + (2-6x)\Delta_2\right)$$

$$+ \left((x-x^2)\Delta_1 + (x^2-x^3)\Delta_2\right) + x \neq 0$$

2.18 $$= x + (-2+x-x^2)\Delta_1 + (2-6x+x^2-x^3)\Delta_2 = R(x)$$

or
$$R(x) = \frac{d^2}{dx^2}[H(x)]\{\Delta\} + [H(x)]\{\Delta\} + x \neq 0$$

$$= \sum_{i=1}^{n} \frac{d^2 H_i(x)}{dx^2}\Delta_i + \sum_{i=1}^{n} H_i(x)\Delta_i + \underbrace{x}_{R_0} \neq 0$$

2.20 $$R(x) = \sum_{j=1}^{n} h_j(x)\Delta_j + R_0$$

with
$$h_j = \frac{d^2 H_j(x)}{dx^2} + H_j(x)$$

2.19 Require $$\int_0^1 W_i(x)\, R(x)\, dx = 0 \qquad 1 \le i \le n$$

to give n equations for finding the n Δ_i values.

2.21 $$\int_0^1 W_i(x) R_0\, dx + \int_0^1 W_i(x) \sum_{j=1}^{n} h_j(x)\,\Delta_j = 0_i \qquad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$$

or

2.22 $$\underset{n\times 1}{\{C\}} + \underset{n\times n}{[S]}\underset{n\times 1}{\{\Delta\}} = \underset{n\times 1}{\{0\}}$$

Galerkin Method; Recall

$$u^*(x) = \sum_{i=1}^{n} H_i(x)\,\Delta_i$$

$$= (x-x^2)\Delta_1 + (x^2-x^3)\Delta_2$$

and

$$R(x) = R_0 + \sum_{j=1}^{n} h_j(x)\,\Delta_j \neq 0$$

$$= x + (-2+x-x^2)\Delta_1 + (2-6x+x^2-x^3)\Delta_2$$

The Galerkin method chooses the spatial part of the solution

2.27 $$W(x)_i = \frac{\partial u^*(x)}{\partial \Delta_i} = H_i(x)$$

for use in

$$\int_0^1 W_i(x)\,R(x)\,dx = 0 \qquad 1 \le i \le n$$

$$i=1 \quad \int_0^1 (x-x^2)\left[x + \sum_{j=1}^{n} h_j(x)\,\Delta_j\right] dx = 0$$

$$i=2 \quad \int_0^1 (x^2-x^3)\left[\qquad\qquad\qquad\right] dx = 0$$

or

$$-\frac{1}{12} + \frac{3}{10}\Delta_1 + \frac{3}{20}\Delta_2 = 0$$

$$-\frac{1}{20} + \frac{3}{20}\Delta_1 + \frac{13}{105}\Delta_2 = 0$$

or

$$\begin{bmatrix} 3/10 & 3/20 \\ 3/20 & 13/105 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/20 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 71/369 \\ 7/41 \end{Bmatrix}$$