Numerical Integration

$$\int_{\boxdot} \boldsymbol{F}(r)\ d\boxdot \approx \sum_{q=1}^{n_q} \boldsymbol{F}(r_q)\, w_q$$

$$\int_{\Omega} \boldsymbol{F}(r)\, d\Omega = \int_{\boxdot} \boldsymbol{F}(r)\, |\boldsymbol{J}^e|\, d\boxdot \approx \sum_{q=1}^{n_q} \boldsymbol{F}(r_q)\, |\boldsymbol{J}^e(r_q)| w_q$$

Number of quadrature points for exact 1-D polynomial integration: $Degree \le (2n_q - 1)$

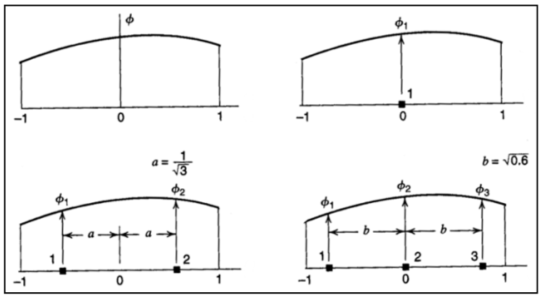

Figure 3.1-1 Natural 1, 2, and 3 point quadrature points, ξ, on a line

Example 3.1-1 Given: Using numerical integration determine the physical length of the cubic line element in Ex. 2.2-1 with straight line coordinates of $\boldsymbol{x}^{e^T} = [2 \quad 4 \quad 6 \quad 8]\ cm$. **Solution**: The length is

$$L^e = \int_{x_1}^{x_2} dx = \int_0^1 \frac{dx(r)}{dr}\, dr \equiv \int_0^1 |J^e(r)|\, dr.$$

Recall that the interpolation function and its derivative are

$$\boldsymbol{H}(r) = [(2 - 11r + 18r^2 - 9r^3) \quad (18r - 45r^2 + 27r^3) \quad \ldots$$
$$(-9r + 36r^2 - 27r^3) \quad (2r - 9r^2 + 9r^3)]/2$$

$$\partial\boldsymbol{H}(r)/\partial r = [(-11 + 36r - 27r^2) \quad (18 - 90r + 81r^2) \ldots$$
$$(-9 + 72r - 81r^2) \quad (2 - 18r + 27r^2)]/2$$

Since the coordinate is interpolated by the above cubic polynomial, $x(r) = \boldsymbol{H}(r)\, \boldsymbol{x}^e$, the degree of the integrand is 3 and the number of Gauss points is only $n_q = 2$ to get the exact answer. The numerical integration is

$$L^e = \int_0^1 \frac{dx(r)}{dr}\, dr = \sum_{q=1}^{n_q} \frac{dx(r_q)}{dr}\, w_q = \sum_{q=1}^{n_q} \frac{d\boldsymbol{H}\, \boldsymbol{x}^e}{dr}(r_q)\, w_q = \left[\sum_{q=1}^{n_q} \frac{d\boldsymbol{H}}{dr}(r_q)\, w_q\right] \boldsymbol{x}^e$$

where the determinant of the Jacobian at each point in the summation is

$$|J^e(r_q)| = \frac{dx(r_q)}{dr} = \frac{d\boldsymbol{H}}{dr}(r_q)\, \boldsymbol{x}^e$$

Here, that product will be evaluated at each quadrature point. The two tabulated quadrature locations in unit coordinates are $r_1 = 0.21132$, and $r_2 = 0.78868$, and the two tabulated weights are the same $w_1 = w_2 = 0.50000$. Set the sum total initially to zero, *L=0*, and begin the summation loop: set *q =1*, substituting $r = 0.21132$ into the derivative the Jacobian is

$$\boldsymbol{J}(r_1)^e = [-2.2990 \quad 1.2990 \quad 1.2990 \quad -0.2990]\begin{Bmatrix}2\\4\\6\\8\end{Bmatrix} cm = 6.0000\ cm$$

Multiply by the tabulated weight and add to the sum:

$$L^e = 0 + \boldsymbol{J}(r_1)^e w_1 = 0 + (6.0000\ cm)0.5000 = 3.0000\ cm.$$

At the second point the Jacobian is

$$\boldsymbol{J}(r_2)^e = [0.2990 \quad -1.2990 \quad -1.2990 \quad 2.2990]\begin{Bmatrix}2\\4\\6\\8\end{Bmatrix} cm = 6.0000\ cm$$

Multiply this by the tabulated weight and add to the sum

$$L^e = 3.0000\ cm + \boldsymbol{J}(r_2)^e w_2 = 3.0000 + (6.0000\ cm)0.5000 = 6.0000\ cm$$

That yields the exact physical length of the cubic line element.

However, since the physical nodes are equally spaced on a straight line the physical location only depends on the first and last node. In other words, the geometry mapping degenerates to a linear interpolation $x(r) = (1 - r)x_1 + rx_4$. For a linear polynomial or a constant the exact integration requires only one quadrature point. For a one-point rule the tabulated data are $r_1 = 0.5000$, $w_1 = 1.0000$. Because the physical coordinates were equally spaced the element will have a constant Jacobian:

$$J^e(r) = d\boldsymbol{H}(r)/dr\ \boldsymbol{x^e} = (-1)x_1 + 1x_4 = (x_4 - x_1) = L.$$

Continuing with the numerical evaluation of the length: $L^e = 0 + [-1 \quad 1]\begin{Bmatrix}2.\\8.\end{Bmatrix} = 6.\,cm.$

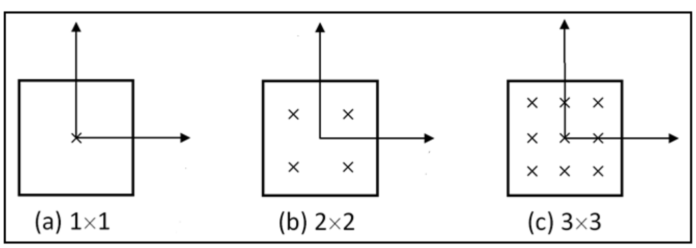

Figure 3.2-1 Product of 1, 2 and 3 quadrature point rules in a quadrilateral

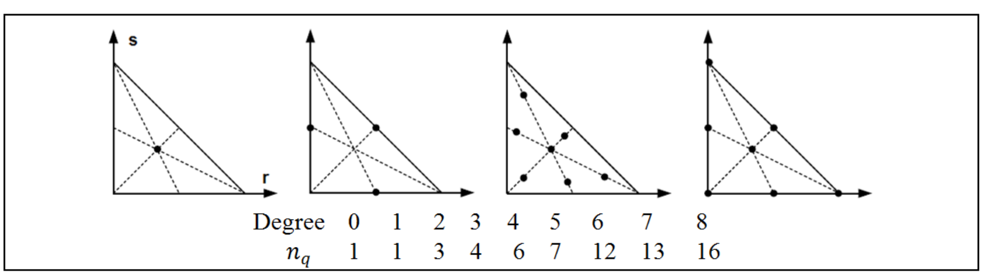

Figure 3.2-4 Some symmetric quadrature locations for unit triangle