Linear Spring Models, J.E. Akin, Rice University

Modern structural analysis relies almost extensively on the finite element method. The most popular integral formulation, based on the variational calculus of Euler, is the Principle of Minimum Total Potential Energy. Basically, it states that the displacement field that satisfies the displacement boundary conditions and minimizes the total potential energy is the unique one that corresponds to the state of static equilibrium. This implies that displacements are the primary unknowns. The total potential energy, *Π*, is the strain energy, *U*, of the structure minus the mechanical work, W, done by the external forces. From introductory mechanics, the mechanical work, *W*, done by a force is the scalar dot product of the force vector, ***F***, and the displacement vector, ***u***, at its point of application.

The well-known linear elastic spring will be reviewed to illustrate the concept of obtaining equilibrium equations from an energy formulation. Consider a linear spring, of stiffness *k*, and length *L*, that has an applied force, *F*, at the free (right) end, and is restrained from displacement at the other (left) end, as in Figure 1.

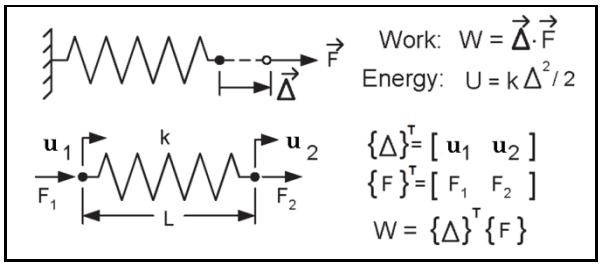

Figure 1 Classic (top) and general linear spring element

The free end of the spring undergoes a displacement of Δ. The work done by the single external force is $W = \vec{\Delta} \circ \vec{F} = \Delta_x F_x = u F$. The spring stores potential energy, or strain energy, due to its deformation (change in length). That stored energy is given by $U = \frac{1}{2} k \Delta_x^2$. Therefore, the Total Potential Energy for the loaded spring with one end fixed is

$$\Pi = \frac{1}{2} k \Delta_x^2 - \Delta_x F_x . \tag{1}$$

The equation of equilibrium is obtained by minimizing this total potential energy with respect to the unknown displacement, Δ_x. That is, the partial derivative of the total potential energy with respect to each displacement is set to zero. That yields one equilibrium equation per unknown displacement.

$$\frac{\partial \Pi}{\partial \Delta_x} = 0 = \frac{2}{2} k \Delta_x - F_x . \tag{2}$$

This simplifies to the common single scalar equation

$$k \Delta_x = F, \text{ or } \Delta_x = F/k \tag{3}$$

which is the well-known equilibrium equation for a linear spring.

In most applications it is necessary to obtain the gradient of the solution in each element. For the simple linear spring the displacement gradient is just the change in length divided by the original length:

$$\varepsilon \equiv \partial \Delta / \partial x = (\Delta_x - 0)/L.$$

This example was slightly simplified, since we started with the condition that the left end of the spring had no displacement (a Dirichlet boundary condition). Next we will consider a spring where either end can be fixed or free to move. To obtain the equilibrium condition now one more step is required. In addition to minimizing the total potential energy it is also necessary impose all of the given displacement restraints. Now the spring model has two end displacements, say u1 and u2, and two associated axial external forces, say F1 and F2. The net deformation of the bar is $\delta = u_1 - u_2$. The total vector of displacement components and the associated vector of external forces are denoted as

$$\vec{\Delta} = \{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{ and } \vec{F} = \{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}, \tag{4}$$

respectively. The mechanical work done on the spring is $W = \{u\}^T \{F\} = u_1F_1 + u_2F_2$. Then the spring's strain energy is now

$$U = \frac{1}{2}k(u_2 - u_1)^2 = \frac{1}{2}(u_2 - u_1)k(u_2 - u_1) = \frac{1}{2}k(u_1u_1 - u_1u_2 - u_2u_1 + u_2u_2).$$

The scalar energy can be written as a triple matrix product

$$U = \frac{1}{2}[u_1 \quad u_2] \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{2}\{\mathbf{u}\}^T [\mathbf{k}] \{\mathbf{u}\} \tag{5}$$

where the "spring stiffness matrix" is found to be

$$[\mathbf{k}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{6}$$

The total potential energy, Π, becomes $\Pi = \frac{1}{2}\{\mathbf{u}\}^T [\mathbf{k}] \{\mathbf{u}\} - \{\mathbf{u}\}^T\{\mathbf{F}\}$ or

$$\Pi = \frac{1}{2}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}. \tag{7}$$

Note that each term has the units of energy, i.e. force times length. The matrix equations of equilibrium will come from the minimization of the above total potential energy with respect to each and every displacement component, as well as from satisfying all displacement restraints. The minimization requires that the partial derivative of all the displacements vanish:

$$\frac{\partial \Pi}{\partial \{u\}} = \{0\}, \text{ or } \frac{\partial \Pi}{\partial u_j} = 0_j, \; 1 \le j \le n. \tag{8}$$

That represents only the first stage system of algebraic equations of equilibrium for the elastic system:

$$k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}. \tag{9}$$

However, the square stiffness matrix has a zero determinant and therefore cannot be inverted. These two symmetric equations do not yet reflect the presence of any boundary condition on the displacements which are required to define a unique solution and/or to eliminate the axial rigid body motion (RBM). In other words, the full system must be modified to impose the known displacement boundary condition(s) before the unknown displacements can be computed.

In order to enforce the displacement boundary conditions (Dirichlet conditions) on this small matrix system note that the matrix can be partitioned into even smaller matrices associated with the known (k) and unknown (u) displacements as:

$$\begin{bmatrix} \mathbf{S_{kk}} & \mathbf{S_{ku}} \\ \mathbf{S_{uk}} & \mathbf{S_{uu}} \end{bmatrix} \begin{Bmatrix} \mathbf{u_k} \\ \mathbf{u_u} \end{Bmatrix} = \begin{Bmatrix} \mathbf{r_k} \\ \mathbf{c_u} \end{Bmatrix} \tag{10}$$

where $\mathbf{S_{kk}} = [k]$, $\mathbf{S_{ku}} = [-k]$, $\mathbf{u_k} = \{u_1\} = \{u_{given}\}$, $\mathbf{r_k} = \{R\}$
$\mathbf{S_{uk}} = [-k]$, $\mathbf{S_{uu}} = [k]$, $\mathbf{u_u} = \{u_2\}$, $\mathbf{c_u} = \{F\}$,

and both u_2 and R are the unknowns. Only the lower row(s) are independent equations for the displacements, $\mathbf{u_u}$. Once they are computed, then the top row(s) are the independent equations to compute the reactions, $\mathbf{r_k}$, at the displacement boundary conditions. The displacements, $\mathbf{u_u}$, which satisfy both equilibrium and the imposed displacements, $\mathbf{u_k}$, are computed from the bottom partition:

$$\mathbf{u_u} = \mathbf{S_{uu}}^{-1}(\mathbf{c_u} - \mathbf{S_{uk}}\mathbf{u_k}) \tag{11}$$

Next the reactions can be found, if desired:

$$\mathbf{S_{kk}}\mathbf{u_k} + \mathbf{S_{ku}}\mathbf{u_u} = \mathbf{r_k}. \tag{12}$$

This process works on arrays of any size.

For example, consider the classic spring with which this study began and assume that the left node has a known displacement ($u_1 = u_{given}$) and the right end has the known force, $f_2 = F$. The unknowns are the right displacement, u_2, and the left end reaction force, say $f_1 = R$. The now unique analytic equilibrium relation is partitioned between the known displacements (with unknown reactions) and the independent unknown displacements being subjected to known forces:

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{given} \\ u_2 \end{Bmatrix} = \begin{Bmatrix} R \\ F \end{Bmatrix}. \tag{13}$$

There are still two unknowns, R and u_2, related to two known quantities, F and u_{given}. This matrix form has the very desirable property of being symmetric. In theory, the matrices could be rearranged to solve for R and u_2 simultaneously but that would destroy the important symmetry property. Instead, a two-step process is used: first solve the symmetric sub-set of equations involving the independent unknown displacements, and then after all displacements are known the equations for the reactions can be solved (but they don't have to be solved).

Here, the independent displacement sub-set is found from the second row:

$$[k]\{u_2\} = \{F\} - [-k]\{u_{given}\}$$

where the known displacement effects have been moved to the right hand side. Multiplying both sides by the inverse matrix $[1/k]$ gives the solution

$$\{u_2\} = [k]^{-1}\big(\{F\} + [k]\{u_{given}\}\big) = \{F/k\} + \{u_{given}\}. \tag{14}$$

This is the same solution as the common form when $\{u_{given}\} = 0$:

$$\{u_2\} = \{F/k\}.$$

Now the system reaction force necessary to maintain $\{u_{given}\}$ can be obtained from the first row of the matrix system:

$$k[1 \quad -1\ ]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \{R\}$$

$$k[\ 1 * u_{given} \quad -1 * (u_{given} + F/k)\ ] = \{-F\} = \{R\} \qquad (15)$$

Thus, the reaction force is equal and opposite to the applied load: R = −F, as expected. For this form of the linear spring the displacement gradient again is just the change in length divided by the original length:

$$\varepsilon \equiv \partial u/\partial x = \Delta u/\Delta L\ = (u_2 - u_1)/L.$$

Next, a system of linear springs, shown in Figure 2, will be analyzed. Clearly there are a total of five displacements, or degrees of freedom, of which only three are independent. The five system displacements and the five external forces are

$$\Delta = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5] \text{ and } \boldsymbol{F^T} = [R_1 \quad R_2 \quad 0 \quad 0 \quad P],$$

respectively, where P is a known external load and $R_1$and R_2 are unknown external reactions, and nodes 3 and 4 have no external loads. The system stiffness matrix, K, will be 5 x 5 in size and is initially zero.

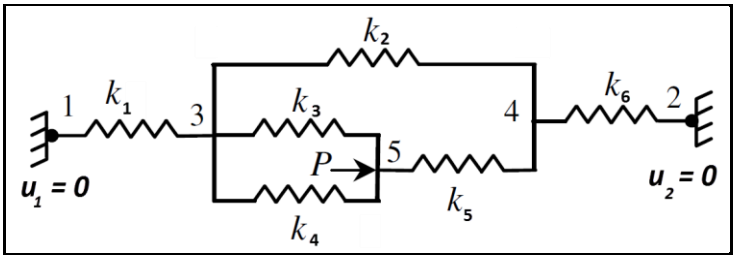

Figure 2 A mesh of six springs with rigid links

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A system of springs is described by a 'connection list' which gives the first and second node connected to the spring:

Spring	Stiffness	Length	Node 1	Node 2
1	k_1	L_1	1	3
2	k_2	L_2	3	4
3	k_3	L_3	3	5
4	k_4	L_4	3	5
5	k_5	L_5	5	4
6	k_6	L_6	4	2

The stiffness matrix of the j-th spring is

$$\boldsymbol{k^e} = \begin{bmatrix} k_j & -k_j \\ -k_j & k_j \end{bmatrix}$$

and its four components will be directly scattered (added) to the rows and columns of the 5 by 5 system stiffness matrix, ***K*,** to which the j-th spring is connected. Begin forming the system stiffness matrix **K** by allocating it and setting the matrix to zero before looping over all of the springs in the system. Scatter in the first element having node connections 1 and 3 by adding its four terms only into those rows and columns. For example, $k^e(1,1)$ is added to $K(1,1)$, and $k^e(1,2)$ is added to $K(1,3)$:

$$\boldsymbol{K}=\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$, add spring 2 at 3 and 4: $\boldsymbol{K}=\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & (k_1+k_2) & -k_2 & 0 \\ 0 & 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

add spring 3 at nodes 3 and 5: $\boldsymbol{K}=\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & (k_1+k_2+k_3) & -k_2 & -k_3 \\ 0 & 0 & -k_2 & k_2 & 0 \\ 0 & 0 & -k_3 & 0 & k_3 \end{bmatrix}$,

add spring 4 at nodes 3 and 4: $\boldsymbol{K}=\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & (k_1+k_2+k_3+k_4) & -k_2 & (-k_3-k_4) \\ 0 & 0 & -k_2 & k_2 & 0 \\ 0 & 0 & (-k_3-k_4) & 0 & (k_3+k_4) \end{bmatrix}$,

add spring 5 at nodes 5 and 4 so $k^e(1,1)$ is added to $K(5,5)$, and $k^e(2,2)$ is added to $K(4,4)$:

$$\boldsymbol{K}=\begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & (k_1+k_2+k_3+k_4) & -k_2 & (-k_3-k_4) \\ 0 & 0 & -k_2 & (k_2+k_5) & -k_5 \\ 0 & 0 & (-k_3-k_4) & -k_5 & (k_3+k_4+k_5) \end{bmatrix},$$

and add spring 6 at nodes 4 and 2 to give the final assembled partitioned system stiffness matrix:

$$\boldsymbol{K}=\left[\begin{array}{ccc|ccc} k_1 & 0 & \vdots & -k_1 & 0 & 0 \\ 0 & k_6 & \vdots & 0 & -k_6 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_1 & 0 & \vdots & (k_1+k_2+k_3+k_4) & -k_2 & (-k_3-k_4) \\ 0 & -k_6 & \vdots & -k_2 & (k_2+k_5+k_6) & -k_5 \\ 0 & 0 & \vdots & (-k_3-k_4) & -k_5 & (k_3+k_4+k_5) \end{array}\right]. \quad (16)$$

Note the general rules that the system stiffness matrix is symmetric and its diagonal element on each row (corresponding to a node) has as many sums as elements connected to that node. Also, the diagonal terms are always positive and the off diagonal terms are often negative. This means that the assembled system has another important mathematical property: it is 'diagonally dominant'. The combinations of these matrix properties yield algorithms that can efficiently solve for hundreds of thousands of displacements.

The final (singular) system matrix equilibrium equations are partitioned as $\boldsymbol{Ku}=\boldsymbol{F}$

$$\left[\begin{array}{ccc|ccc} k_1 & 0 & \vdots & -k_1 & 0 & 0 \\ 0 & k_6 & \vdots & 0 & -k_6 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_1 & 0 & \vdots & (k_1+k_2+k_3+k_4) & -k_2 & (-k_3-k_4) \\ 0 & -k_6 & \vdots & -k_2 & (k_2+k_5+k_6) & -k_5 \\ 0 & 0 & \vdots & (-k_3-k_4) & -k_5 & (k_3+k_4+k_5) \end{array}\right]\begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}=\begin{Bmatrix} R_1 \\ R_2 \\ \cdots \\ 0 \\ 0 \\ P \end{Bmatrix} \quad (17)$$

Here rows 1 and 2 are not independent equations for displacements since $u_1$and u_2 are to be specified as boundary conditions. Those equations are just optional equations for finding the reaction forces needed to

enforce the specified displacements at the two ends of the spring assembly. Only the last three rows are independent equations for the system displacements. Also, the first two columns of those rows are stiffnesses multiplied by known displacements so those products must be carried to the right hand side (RHS) of known values. The system started with five degrees of freedom, but applying two displacement boundary conditions left only three independent degrees of freedom. Applying sufficient boundary conditions always results in a non-singular stiffness matrix partition to find the independent displacements. The active 3 by 3 independent equations of equilibrium, $\boldsymbol{K}^*\boldsymbol{u}^* = \boldsymbol{F}^*$, are

$$\begin{bmatrix} (k_1+k_2+k_3+k_4) & -k_2 & (-k_3-k_4) \\ -k_2 & (k_2+k_5+k_6) & -k_5 \\ (-k_3-k_4) & -k_5 & (k_3+k_4+k_5) \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} - u_1 \begin{Bmatrix} -k_1 \\ 0 \\ 0 \end{Bmatrix} - u_2 \begin{Bmatrix} 0 \\ -k_6 \\ 0 \end{Bmatrix} \quad (18)$$

and they can be solved once the known end displacements, $u_1$and u_2, are specified to make the results unique. Note that this general approach allows for non-zero displacements as the end boundary conditions.

Using the vector subscript notations of Matlab and Fortran the independent matrices are

$$\boldsymbol{K}^* = \boldsymbol{K}(3{:}5, 3{:}5) = \mathbf{S_{uu}},\ u^* = \boldsymbol{u}(3{:}5) = \mathbf{u_u},\ \boldsymbol{F}^* = \boldsymbol{F}(3{:}5) = \mathbf{c_u}$$

or

$$\boldsymbol{K}^* = \boldsymbol{K}(\boldsymbol{free}, \boldsymbol{free}),\ u^* = \boldsymbol{u}(\boldsymbol{free}),\ \boldsymbol{F}^* = \boldsymbol{F}(\boldsymbol{free})$$

where the vector subscript array is $\boldsymbol{free} = [3\ \ 4\ \ 5]$**.** The latter form is used later since it is a more powerful programming approach because it does not require the independent displacement numbers to be sequential.

To simplify this system assume that all six springs have the same stiffness, k. Then the above system becomes

$$k \begin{bmatrix} (4) & -1 & (-2) \\ -1 & (3) & -1 \\ (-2) & -1 & (3) \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} - u_1 \begin{Bmatrix} -k \\ 0 \\ 0 \end{Bmatrix} - u_2 \begin{Bmatrix} 0 \\ -k \\ 0 \end{Bmatrix} \quad (19)$$

Now assign numerical values of $k = 120\ kN/m$, $P = 20\ kN$, and $u_1 = u_2 = 0\ m$. Then

$$\begin{bmatrix} 480 & -120 & -240 \\ -120 & 360 & -120 \\ -240 & -120 & 360 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20 \end{Bmatrix}$$

and solving for the independent displacements using the Matlab command $\boldsymbol{u}^* = \boldsymbol{K}^* \backslash \boldsymbol{F}^*$ gives

$$u^* = \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0.08974 \\ 0.07692 \\ 0.14103 \end{Bmatrix} m$$

Now that all of the displacements, $\boldsymbol{u}$, are known the optional first two rows of the original 5 by 5 system can be used to recover the two system end reaction forces. For the current numerical values the reactions are

$$\begin{bmatrix} 120 & 0 & -120 & 0 & 0 \\ 0 & 120 & 0 & -120 & 0 \end{bmatrix} kN/m \begin{Bmatrix} 0 \\ 0 \\ 0.08974 \\ 0.07692 \\ 0.14103 \end{Bmatrix} m = \begin{Bmatrix} -10.7692 \\ -9.2307 \end{Bmatrix} kN = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}.$$

Those two system reaction forces are shown in Figure 3 along with the external applied force. Checking for equilibrium using Newton's third law shows that the sum of the external axial forces is indeed zero.

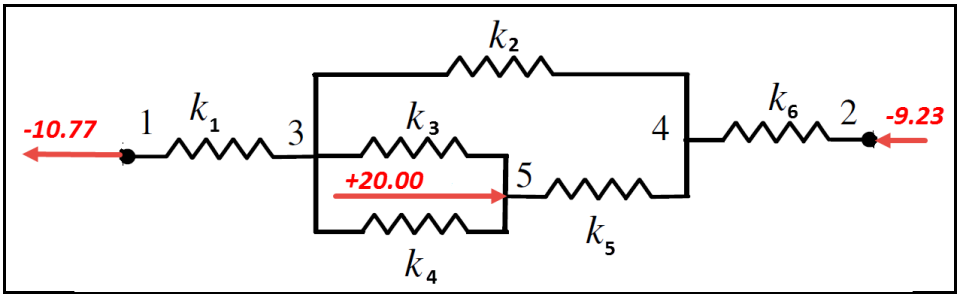

Figure 3 System external load and end reaction forces (kN)

At this point there is usually a loop over all of the elements where their end displacements are gathered to find the solution gradient and optionally to find the element nodal forces. For each spring the displacement gradient again is just the change in length divided by the original length: $\varepsilon \equiv \partial u/\partial x$. From the element connection list the first spring is connected to nodes 1 and 3. Those connections can be represented as the entries into a vector subscript array $\boldsymbol{list} = [1 \quad 3]$. The first spring end displacements are a sub-set of the system displacements, $\boldsymbol{\delta} \subset \boldsymbol{u}$. Here the two 'gathered' spring end displacements on the first spring are

$$\{\boldsymbol{\delta}\} = \{\boldsymbol{u}(\boldsymbol{list})\} = \begin{Bmatrix} \boldsymbol{u}(1) \\ \boldsymbol{u}(3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.08974 \end{Bmatrix} m$$

From other sources, like Figure 7, the length of the first spring is known to be $L_1 = 2\ m$. Thus, the solution gradient for the first element is

$$\varepsilon \equiv \frac{\partial u}{\partial x} = \frac{u(3) - u(1)}{L_1} = \frac{0.08974\ m - 0\ m}{2\ m} = 0.04487\ m/m.$$

A spring network is one of the rare cases where the solution gradient is not an important item, so the remaining gradients will not be given in detail. In the vast majority of applications the solution gradient in each element is of vital importance and is always calculated. Conversely, since springs are structural members it is often important to determine the individual spring end forces, typically called the element reactions. Most non-structural applications do not bother to recover the individual element reactions.

From Figure 3 the forces in springs 1 and 6 are obvious from Newton's third law, but the finite element analysis gives them in a systematic way by requiring that each element be in equilibrium: $[\boldsymbol{k}]\{\boldsymbol{\delta}\} = \{\boldsymbol{r}\}$. As shown above, the first spring is connected to nodes 1 and 3 and its end displacements were gathered above. Substituting the data for the first element into the single spring matrix equilibrium equations $[\boldsymbol{k}]_1\{\boldsymbol{\delta}\}_1 = \{\boldsymbol{r}\}_1$ gives the reaction set as

$$120\ kN/m \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.08974 \end{Bmatrix} m = \begin{Bmatrix} -10.769 \\ 10.769 \end{Bmatrix} kN = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

for the first spring. Note that the sign of the second reaction determines if the spring is in tension (+) or in compression (-). For spring 2 the node connections are $\boldsymbol{list} = [3 \quad 4]$ and its reactions are

$$120\ kN/m \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.08974 \\ 0.07692 \end{Bmatrix} m = \begin{Bmatrix} 1.5384 \\ -1.5384 \end{Bmatrix} kN = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

which shows that it is compression. Likewise, for springs 3 through 6 their second node reactions are 6.1538, 6.1538, -7.6923, and -9.2308 kN, respectively. Figure 4 shows the free body diagrams of internal nodes 3 and 5 to verify that they are also in equilibrium via Newton's law. The reader should try drawing the equilibrium forces at internal node 4.

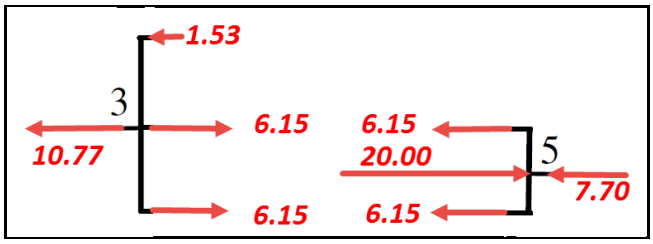

Figure 4 Equilibrium forces (kN) at internal nodes 3 and 5

The nodes in this network mesh would have axial coordinates even if they were not required. They could be used to establish the length of each spring for plotting the mesh. In most applications the length of each element is an important piece of data that appears in the 'stiffness'.

It so happens that there are several field of engineering where this same mesh and matrix equations represent other types of equilibrium, and the terms have different units. Such applications include elastic bars, steady DC electrical circuits, heat conduction, torsion of circular shafts, laminar pipe flow through a network, etc. Below the interpretations of some of these alternate forms are listed.

Study	**K (units)**	**u (units)**	**F (units)**	**R (units)**
Linear Spring	Stiffness per unit Length (N/m)	Displacement (m)	External force (N)	Reaction force (N)
Axial Bar	Axial stiffness k=$E\ A/L$ (N/m)	Displacement (m)	External force (N)	Reaction force (N)
Heat Conduction	Axial conductivity k=$\kappa\ A/L$ $(W/°C)$	Temperature (°C)	External heat flow (W)	Reaction heat flow (W)
Torsional Shaft	Torsion stiffness k=$G\ J/L$ (N-m)	Twist angle (radians)	External torque (N-m)	Reaction torque (N-m)
Electric DC Circuit	Inverse resistance k=1/R (amp/V)	Voltage (V)	External current source (Amp)	Reaction current (Amp)

As an example of non-zero boundary conditions let the prior spring mesh represent a DC current electrical network. Assign node 1 to have a value of 100 Volts and node 2 to be ground at 0 Volts, and let $P = 20\ amp$be an external current source. Let each wire have a resistance of only 1/120 Ohm. Then k = 120 1/Ω and the only initial numerical change in the prior equations is the voltage at node 1, $V_1 = 100$. Then Eq. 19 becomes

$$k\begin{bmatrix}(4) & -1 & (-2)\\ -1 & (3) & -1\\ (-2) & -1 & (3)\end{bmatrix}\begin{Bmatrix}V_3\\V_4\\V_5\end{Bmatrix} = \begin{Bmatrix}0\\0\\P\end{Bmatrix} - V_1\begin{Bmatrix}-k\\0\\0\end{Bmatrix} - V_2\begin{Bmatrix}0\\-k\\0\end{Bmatrix} \qquad (19')$$

with the new numerical resultant source values of

$$\begin{bmatrix} 480 & -120 & -240 \\ -120 & 360 & -120 \\ -240 & -120 & 360 \end{bmatrix} \begin{Bmatrix} V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20 \end{Bmatrix} - 100 \begin{Bmatrix} -120 \\ 0 \\ 0 \end{Bmatrix} - 0 \begin{Bmatrix} 0 \\ -120 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.2e4 \\ 0 \\ 20 \end{Bmatrix}$$

and solving for the independent voltages using the Matlab command $\boldsymbol{u}^* = \boldsymbol{K}^* \backslash \boldsymbol{F}^*$ gives

$$\boldsymbol{u}^* = \begin{Bmatrix} V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 61.6282 \\ 38.5385 \\ 53.9872 \end{Bmatrix} Volts$$

Now that all of the voltages, $\boldsymbol{u}$, are known the optional first two rows of the original 5 by 5 system can be used to recover the two system end reaction currents. For the current numerical values the reaction currents are

$$\begin{bmatrix} 120 & 0 & -120 & 0 & 0 \\ 0 & 120 & 0 & -120 & 0 \end{bmatrix} 1/\Omega \begin{Bmatrix} 100 \\ 0 \\ 61.6282 \\ 38.5385 \\ 53.9872 \end{Bmatrix} Volts = 1e3 \begin{Bmatrix} 4.6046 \\ -4.60462 \end{Bmatrix} Amps = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Looping over all of the wires where their end voltages are gathered to find the element end currents. A current entering the first node of a wire is positive (+). The first spring is connected to nodes 1 and 3. Those connections are represented by the vector subscript array $\boldsymbol{list} = [1 \quad 3]$. The wire voltages are a sub-set of the system voltages, $\boldsymbol{\delta} \subset \Delta$. Here the two 'gathered' end voltages are

$$\{\boldsymbol{\delta}\} = \{\Delta(\boldsymbol{list})\} = \begin{Bmatrix} \Delta(1) \\ \Delta(3) \end{Bmatrix} = \begin{Bmatrix} 100 \\ 61.6282 \end{Bmatrix} Volts$$

and the element equilibrium requirement $[\boldsymbol{k}]\{\boldsymbol{\delta}\} = \{\boldsymbol{r}\}$ gives

$$120\ 1/\Omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 100 \\ 61.6282 \end{Bmatrix} Volts = 1e3 \begin{Bmatrix} 4.6046 \\ -4.6046 \end{Bmatrix} Amps = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

for the first wire. Note that the current enters that wire from the 100 V boundary condition. For wire 2 the node connections are $\boldsymbol{list} = [3 \quad 4]$ and its reactions are

$$120\ 1/\Omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 61.6282 \\ 38.5385 \end{Bmatrix} V = \begin{Bmatrix} 2.7708 \\ -2.7708 \end{Bmatrix} Amps = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

which shows its current enters at node 3 and exits at node 4. Likewise, for wires 3 through 6 their first node reactions are 916.9231, 916.9231, 1.8538e3, and 4.6246e3 Amp, respectively. Figure 5 shows Kirchhoff's nodal current law for internal nodes 3 and 5 to verify that they are also in equilibrium. The reader should try drawing Kirchhoff's law at internal node 4. The network nodal voltages can be sketched using the nodal coordinates that did not enter into the solution process, in this example, as shown in Figure 6.

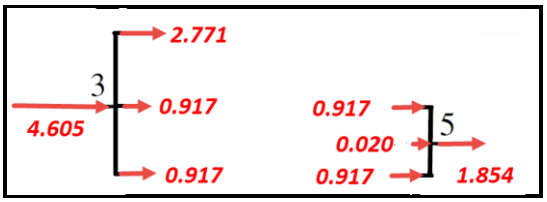

Figure 5 Kirchhoff's nodal current law ($1e3$ Amp) at internal nodes 3 and 5

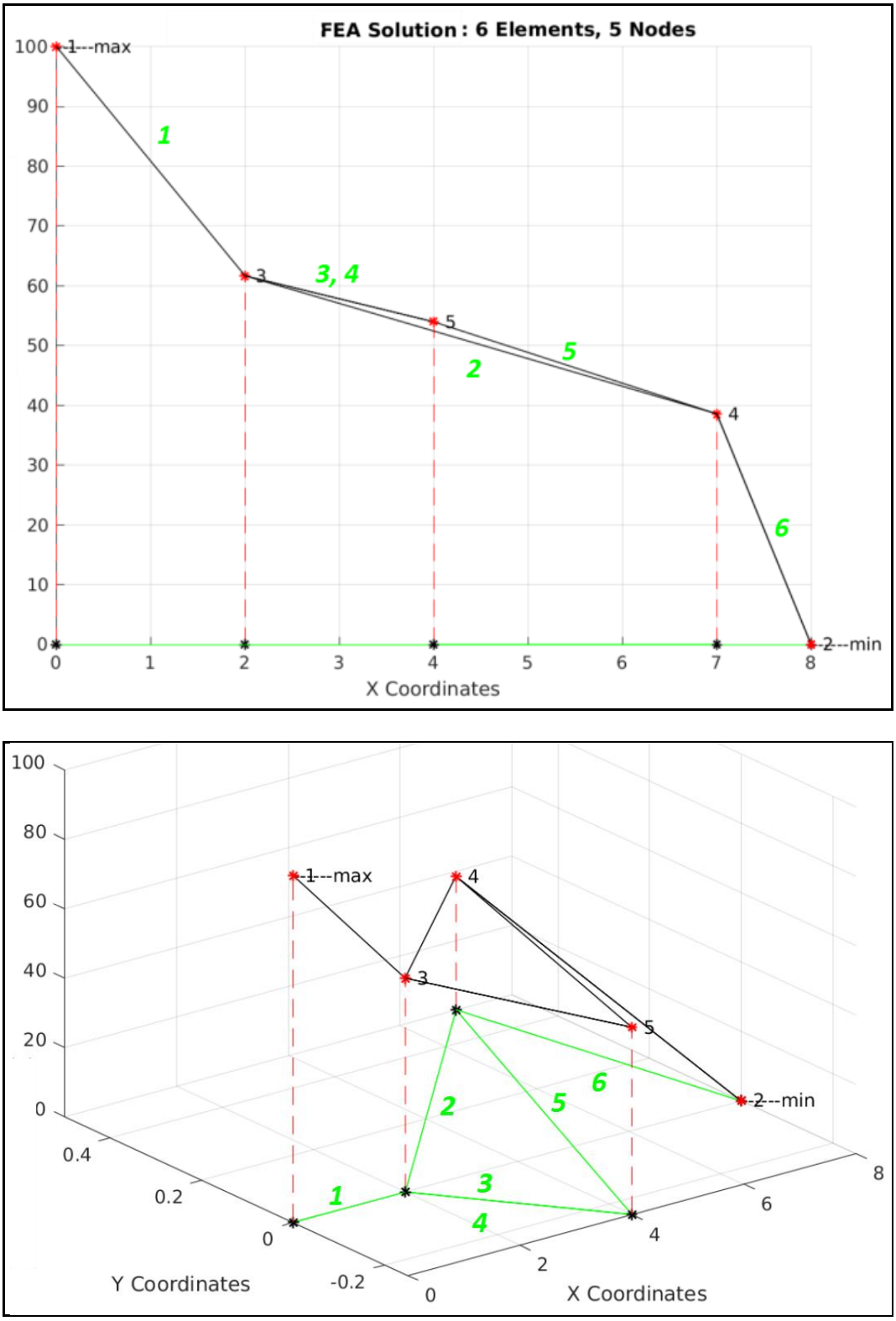

Figure 7 Voltages in the DC circuit