MECH 517 – Finite Element Analysis

Homework 5

1) Resolve Example 8.1-2 using a linear element with $\boldsymbol{H}(r) = [(1-r) \quad r]$

$$Q(x) = c$$

$$u^*(r) \cong \boldsymbol{H}(r)u^e$$

$$\boldsymbol{H}(r) = [(1-r) \quad r]$$

$$\int_{x_1}^{x_3} u(x)Q(x)dx = u^{eT}\left\{\int_{x_1}^{x_3} \boldsymbol{H}(r)^T Q(x)^e\, dx\right\} \equiv \boldsymbol{u}^{eT}\boldsymbol{c}^e$$

$$c^e = \int_{x_1}^{x_3} \boldsymbol{H}(r)^T Q(x)^e\, dx = \int_0^1 \boldsymbol{H}(r)^T c\frac{dx}{dr}dr = c\int_0^1 \boldsymbol{H}(r)^T(L^e dr) = cL^e\int_0^1 \begin{Bmatrix}(1-r)\\ r\end{Bmatrix}dr$$

$$= cL^e\begin{Bmatrix} r-\dfrac{r^2}{2}\\ \dfrac{r^2}{2}\end{Bmatrix}_0^1 = cL^e\begin{Bmatrix}\dfrac{1}{2}\\ \dfrac{1}{2}\end{Bmatrix}$$

2) Resolve Example 8.1-3 using a linear element with $\boldsymbol{H}(r) = [(1-r) \quad r]$

$$I = \left[Ku(x_2)\frac{du(x_2)}{dx} - Ku(x_1)\frac{du(x_1)}{dx}\right] - \int_{x_1}^{x_2}\frac{du}{dx}K\frac{du}{dx}dx + \int_{x_1}^{x_2} u(x)Q(x)dx = 0$$

$$\left[Ku(x_2)\frac{du(x_2)}{dx} - Ku(x_1)\frac{du(x_1)}{dx}\right] \equiv \boldsymbol{u}^T\boldsymbol{c}_{NBC}$$

Stiffness matrix formed as follows:

$$-\boldsymbol{u}^{eT}\boldsymbol{S}^e\boldsymbol{u}^e = -\boldsymbol{u}^{eT}\int_0^1\left(\frac{d\boldsymbol{H}(r)^T}{dr}\frac{1}{L^e}\right)K\frac{dH(r)}{dr}\frac{1}{L^e}(L^e dr)\boldsymbol{u}^e =$$

$$\rightarrow -\boldsymbol{u}^{eT}\int_0^1\frac{1}{L^e}\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{K}{L^e}[-1 \quad 1](L^e dr)\boldsymbol{u}^e = -\boldsymbol{u}^{eT}\frac{K}{L^e}\int_0^1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dr\,\boldsymbol{u}^e =$$

$$\rightarrow -\boldsymbol{u}^{eT}\frac{K}{L^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\boldsymbol{u}^e = -\boldsymbol{u}^{eT}\boldsymbol{S}^e\boldsymbol{u}^e$$

The governing equation is then

$$\boldsymbol{S}^e\boldsymbol{u}^e = \boldsymbol{c}_{NBC} + \boldsymbol{c}^e \rightarrow \frac{K}{L^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-K\dfrac{du(0)}{dx}\\ K\dfrac{du(L)}{dx}\end{Bmatrix} + cL^e\begin{Bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{Bmatrix}$$

✓

3) (MECH 517) In addition to all of the above assignments, use numerical integration to form the matrix

$$\boldsymbol{m} = \int_0^1 \boldsymbol{H}(r)^T\boldsymbol{H}(r)dr$$

For a cubic line element.

$$\boldsymbol{H}(r) = \frac{1}{2}\begin{bmatrix}(2-11r+18r^2-9r^3)\\ (18r-45r^2+27r^3)\\ (-9r+36r^2-27r^3)\\ (2r-9r^2+9r^3)\end{bmatrix}^T = [H_1 \quad H_2 \quad H_3 \quad H_4]$$

✓

This corresponds with a Lagrange L4 element of polynomial degree 3. Their product is degree 3+3 ~~7~~

$$Degree \leq (2n_q-1) \rightarrow 3+3 \leq (2n_q-1) \rightarrow \cancel{7} \leq 2n_q \rightarrow 3.5 \leq n_q \rightarrow n_q = 4$$

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$$\boldsymbol{m} = \sum_{q=1}^{4}\boldsymbol{H}(r_q)^T\boldsymbol{H}(r_q)\,\boldsymbol{w}_q$$

✓

$$r_q = \begin{Bmatrix}0.06943\\ 0.33001\\ 0.66999\\ 0.93057\end{Bmatrix}$$

✓

$$w_q = \begin{Bmatrix}0.17393\\ 0.32607\\ 0.32607\\ 0.17393\end{Bmatrix}$$

✓

Subsequent analysis performed using MathCAD. Numerical integration using Gaussian quadrature results in the same answer vs. using standard integration.

vvv ✓

Problem 3:

$$s = \sum_k \boldsymbol{H}_k(r) = \tfrac{1}{2}(2 - 11r + 18r^2 - 9r^3 + 18r - 45r^2 + 27r^3 - 9r + 36r^2 - 27r^3 + 2r - 9r^2 + 9r^3)$$
$$s = \tfrac{1}{2}((2) + (-11 + 18 - 9 + 2)r + (18 - 45 + 36 - 9)r^2 + (-9 + 27 - 27 + 9)r^3$$
$$s = 1$$

HW5

Problem 1:

Using $u(x) = (u^e)^T \boldsymbol{H}(x)^T$, then

$$\int_{\Omega^e} u(x)Q(x)dx = \int_{\Omega^e} (u^e)^T \boldsymbol{H}(x)^T Q(x)dx = (u^e)^T \int_{\Omega^e} \boldsymbol{H}(x)^T Q(x)dx = (u^e)^T \int_{\Omega^e} \boldsymbol{H}(r)^T c\frac{dx}{dr}dr$$

$$= (u^e)^T c \int_0^1 \boldsymbol{H}(r)^T L^e dr = (u^e)^T cL^e \int_0^1 \begin{bmatrix} 1-r \\ r \end{bmatrix} dr = (u^e)^T cL^e \begin{bmatrix} r - r^2/2 \\ r^2/2 \end{bmatrix}_0^1 = (u^e)^T cL^e \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Therefore

$$\int_{\Omega^e} u(x)Q(x)dx = (u^e)^T \frac{cL^e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 2:

Using $u(x) = (u^e)^T \boldsymbol{H}(x)^T$, then

$$-\int_{\Omega^e} \frac{du}{dx} K \frac{du}{dx} dx = -(u^e)^T \int_0^1 \left(\frac{d\boldsymbol{H}(r)^T}{dr} \frac{1}{L^e} \right) K \left(\frac{d\boldsymbol{H}(r)}{dr} \frac{1}{L^e} \right) (L^e dr)\ u^e$$

$$= -(u^e)^T \int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{K}{L^e} \begin{bmatrix} -1 & 1 \end{bmatrix} dr\ u^e = -(u^e)^T \frac{K}{L^e} \int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dr\ u^e$$

$$= -(u^e)^T \frac{K}{L^e} \begin{bmatrix} -r & r \\ -r & r \end{bmatrix}_0^1 u^e = -(u^e)^T \frac{K}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u^e.$$

Therefore

$$-\int_{\Omega^e} \frac{du}{dx} K \frac{du}{dx} dx = -(u^e)^T \frac{K}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u^e.$$