MECH 517 Homework 6

Problem 1:
Beginning with the assembled matrix equilibrium equations for the three bar problem, find the displacements, reaction, and element stresses in each bar for the boundary conditions below:

(a) $u_1 = 1$ in at node 1
(a) $u_3 = 0$ in at node 3

Solution:
(a) Setting $u_1 = 1$m at node 1, we obtain the following displacement values:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 1.0080 \\ 1.0164 \\ 1.0308 \end{bmatrix} in$$

the following reactions at node 1:

$$R_1 = 0lbf$$

and the following stresses in each element:

$$\sigma = \begin{bmatrix} \sigma_x^1 \\ \sigma_x^2 \\ \sigma_x^3 \end{bmatrix} = \begin{bmatrix} 10000 \\ 7000 \\ 9000 \end{bmatrix} psi$$

(b) Setting $u_3 = 0$m at node 3, we obtain the following displacement values:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.0164 \\ -0.0084 \\ 0.0000 \\ 0.0144 \end{bmatrix} in$$

The reaction force at node 3 remains as above, since no change was made to the input forces:

$$R_3 = 0lbf$$

and the stresses remain the same in each element:

$$\sigma = \begin{bmatrix} \sigma_x^1 \\ \sigma_x^2 \\ \sigma_x^3 \end{bmatrix} = \begin{bmatrix} 10000 \\ 7000 \\ 9000 \end{bmatrix} psi$$

Problem 2:

The Gaussian one-dimensional quadrature points are symmetrically placed with respect to the center of the interval, and n_q points in 1D will EXACTLY integrate a 1D integrand that is a polynomial of degree $2n_q - 1$. In other words,

$$(\text{1D Integrand Polynomial Degree}) \leq 2n_q - 1$$

The mass matrix for any element is defined as the integral of the tranpose of the interpolation funtions times the interpolation functions times the mass density times the differential volume. In 1D:

$$m = \rho A \int_0^r H(r)^T H(r) dr$$

For a four noded cubic line element the interpolation functions are

$$\begin{aligned} H(r) &= [H_1, H_2, H_3, H_4] \\ H(r) &= [(1 - \frac{11r}{2} + 9r^2 - \frac{9r^3}{2}), (9r - \frac{45r^2}{2} + \frac{27r^3}{2}), \ldots \\ &\quad (\frac{-9r}{2} + 18r^2 - \frac{27r^3}{2}), (r - \frac{9r^2}{2} + \frac{9r^2}{2})] \end{aligned}$$

(a) What is the polynomial degree of the interpolation functions?
(b) What is the polynomial degree of the element mass matrix?
(c) How many integration (quadrature) points, n_q, are needed to exactly integrate the element mass matrix?
(d) Compute the integral of the cubic interpolation functions over the domain 0 to 1.

Solution:

(a) The polynomial degree of each cubic interpolation function is 3. The polynomial degree of the integrand, $H(r)^T H(r)$, is 6. = 3+3
(b) The polynomial degree of the element mass matrix is 7. after integration
(c) Since the polynomial degree of the element ~~mass matrix is 7,~~ we use the following relationship: Degree $\leq 2n_q - 1$. This yields:

integrand is 6

$$6 \; \cancel{7} = 2n_q - 1$$
$$n_q = 4$$

(d) The integral of the cubic interpolation functions over the domain 0 to 1 yields the following matrix:

$$K = \int_0^1 H(r)dr$$

$$K = \left[\begin{array}{cccc} -\frac{9r^4}{8} + 3r^3 - \frac{11r^2}{4} + r, & \frac{3r^2(9r^2 - 20r + 12)}{8}, & -\frac{3r^2(9r^2 - 16r + 6)}{8}, & \frac{r^2(3r-2)^2}{8} \end{array} \right]$$

Evaluated from r = 0 to r = 1 gives:

$$K = \left[\begin{array}{cccc} 0.125 & 0.375 & 0.375 & 0.125 \end{array} \right]$$

$= \frac{1}{8}[1 \quad 3 \quad 3 \quad 1]$ ✓