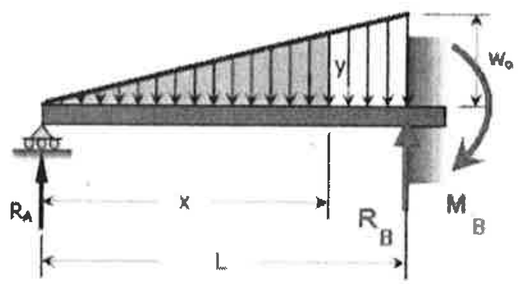
MECH 517 Homework 9

Problem:
a) Determine the angle at point 1 and the force and moment reactions based on the matrix equation below. Check the reactions with Newton's equilibrium equations.
b) Also graph the moment and shear along with their exact values obtained from the below deflection equation.



$$R_A = \frac{w_o L}{10}$$

Thus, the deflection equation is

$$EIy = \frac{R_A x^3}{6} - \frac{w_o x^5}{120L} + \left(\frac{w_o L^3}{24} - \frac{R_A L^2}{2}\right)x$$

Exact

Figure 1: Cantilever Beam with Triangular Line Loading

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{W_0 L}{2}\begin{Bmatrix} 3/10 \\ L/15 \\ 7/10 \\ -L/10 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ 0 \\ F_2 \\ M_2 \end{Bmatrix}$$

FEA

Solution:
a) The first step is to solve for θ_1 based on the above matrix equation.

$$\frac{EI}{L^3}(4L^2)\theta_1 = \frac{W_0L^2}{30}$$

$$\theta_1 = \frac{W_0L^3}{120EI}$$

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{W_0L^3}{EI}\begin{Bmatrix} 0 \\ 1/120 \\ 0 \\ 0 \end{Bmatrix}$$

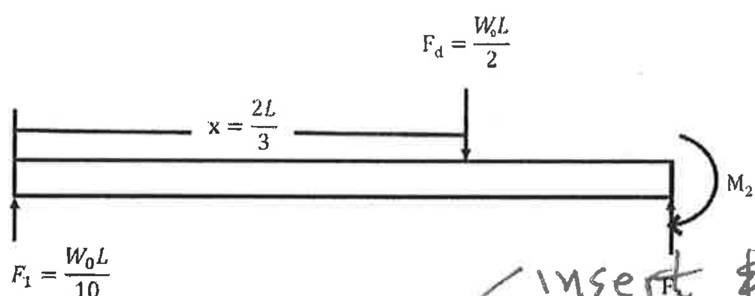
Now that these terms are known, we can substitute them back into the original matrix and solve for F_1, F_2, and M_2 with simple algebra.

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \frac{W_0L}{2}\begin{Bmatrix} 3/10 \\ L/15 \\ 7/10 \\ -L/10 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} - \frac{W_0L}{2}\begin{Bmatrix} 3/10 \\ L/15 \\ 7/10 \\ -L/10 \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \mathbf{LW_0}\begin{Bmatrix} -1/10 \\ 0 \\ -2/5 \\ L/15 \end{Bmatrix}}$$

These results can be compared to the results obtained from a simple free-body diagram of the system and the reactions resulting from the force & moment balance equations as shown below:



insert & verify = 0

Figure 2: Free Body Diagram of Beam with Triangle Loading

$$\sum F_y = 0: \quad -\frac{W_0L}{2} + \frac{W_0L}{10} + F_2 = 0 \Rightarrow \boxed{\mathbf{F_2} = \frac{\mathbf{4W_0L}}{\mathbf{10}}}$$

$$\sum M = 0: \quad -M_2 + \left(\frac{W_0L}{2}\right)\left(\frac{L}{3}\right) - \frac{W_0L^2}{10} = 0 \Rightarrow \boxed{\mathbf{M_2} = \frac{\mathbf{W_0L^2}}{\mathbf{15}}}$$

The above results indicate that the free body diagram has been drawn correctly and the magnitudes equal that of the FEA solution presented above.

b) The exact solutions for the deflection, shear, and moment are shown below.

$$
\begin{aligned}
y(x) &= \frac{1}{EI}\left(\frac{R_A x^3}{6} - \frac{W_0 x^5}{120L} + \frac{W_0 L^3 x}{24} - \frac{R_A L^2 x}{2}\right)\\
M(x) = y''(x) &= \frac{1}{EI}\left(R_A x - \frac{W_0 x^3}{6}\right)\\
V(x) = y'''(x) &= \frac{1}{EI}\left(R_A - \frac{W_0 x^2}{2}\right)
\end{aligned}
$$

I set the values of W, E, I, and L arbitrarily to:

$$
\begin{aligned}
W &= -1 \ N\\
E &= 1 \ Pa\\
I &= 1 \ m^4\\
L &= 1 \ m
\end{aligned}
$$

For the finite element solutions, I used the following equations:

$$
\begin{aligned}
v(x) &= v_1(1 - 3r^2 + 2r^3) + \theta_1(r - 2r^2 + r^3)L + v_2(3r^2 - 2r^3) + \theta_2(r^3 - r^2)L\\
M(x) &= [v_1(12r - 6) + \theta_1(6r - 4)L + v_2(6 - 12r) + \theta_2(6r - 2)L]/L^2\\
V(x) &= [v_1(12) + \theta_1(6)L + v_2(-12) + \theta_2(6)L]/L^3
\end{aligned}
$$

The plots for the displacement, shear, and moment for both the exact solution and the cubic element approximation have been included below:

sign error

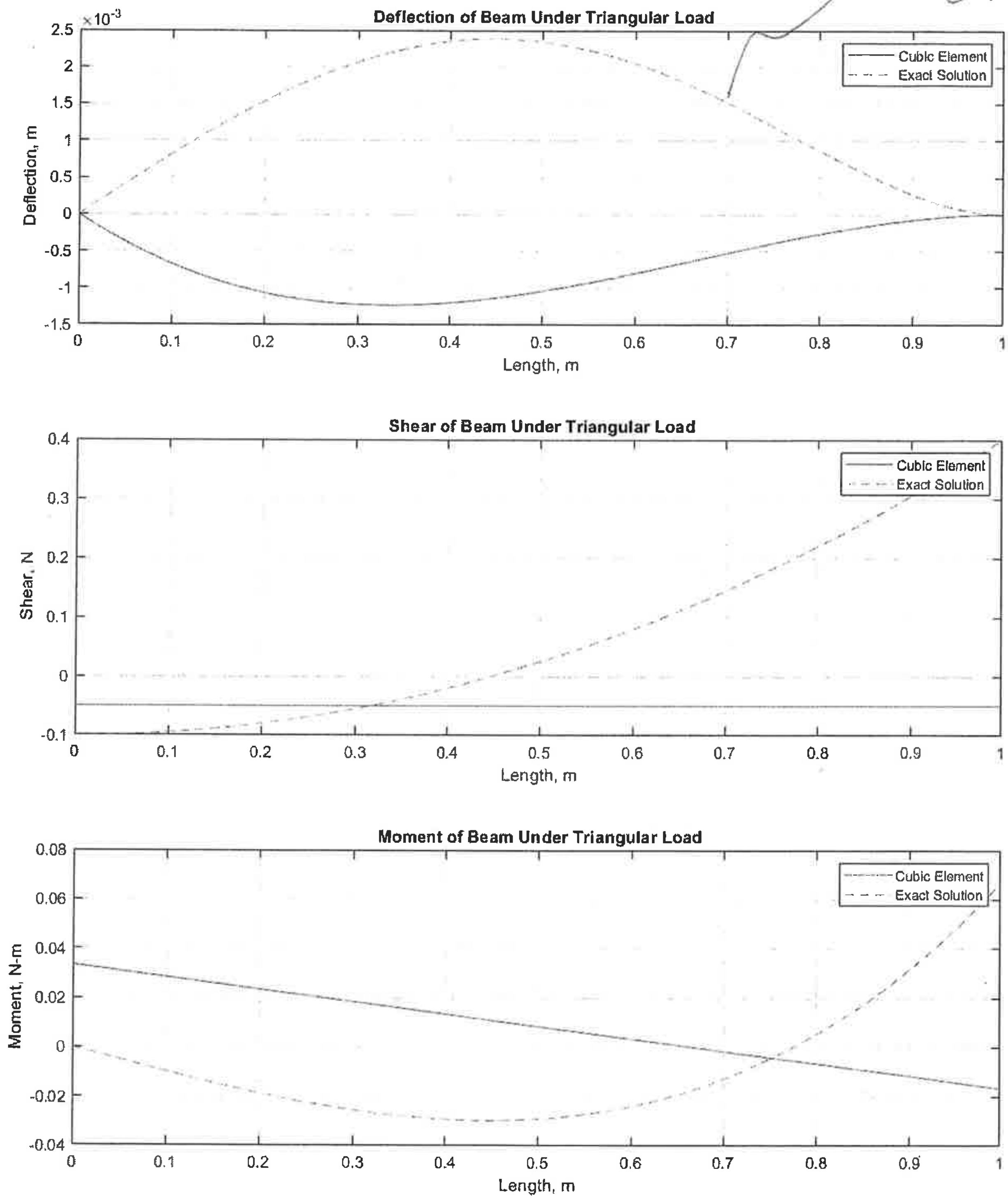


Figure 3: Deflection, Shear, and Moment Diagrams