MECH 517 Homework 12

Problem:
Refer to the text example 14.4-3. Derive the average mass matrix that is given in that example for a quadratic, three-node, line element.

Solution:
The averaged mass matrix for the quadratic line element is given in Example 14.4-3 as follows:

$$\mathbf{M}^{\mathbf{e}}_{\mathbf{avg}} = \frac{m^e}{60}\begin{bmatrix} 9 & 2 & -1 \\ 2 & 36 & 2 \\ -1 & 2 & 9 \end{bmatrix}$$

To derive this averaged mass matrix, the first step is to derive the consistent mass matrix based on the quadratic interpolation functions.

$$\mathbf{M}^{\mathbf{e}} = \int_{L^e} \mathbf{H}(r)^T \rho^e \mathbf{H}(r) dx$$

$$\mathbf{M}^{\mathbf{e}} = \int_0^1 \mathbf{H}(r)^T \rho^e \mathbf{H}(r) L^e dr$$

$$\mathbf{M}^{\mathbf{e}} = \rho^e L^e \int_0^1 \mathbf{H}(r)^T \mathbf{H}(r) dr$$

$$\mathbf{M}^{\mathbf{e}} = \rho^e L^e \int_0^1 \begin{bmatrix} 1 - 3r + 2r^2 \\ 4r - 4r^2 \\ 2r^2 - r \end{bmatrix} \begin{bmatrix} 1 - 3r + 2r^2 & 4r - 4r^2 & 2r^2 - r \end{bmatrix} dr$$

$$\mathbf{M}^{\mathbf{e}} = \rho^e L^e \int_0^1 \begin{bmatrix} (1 - 3r + 2r^2)^2 & (-4r^2 + 4r)(2r^2 - 3r + 1) & -(-2r^2 + r)(2r^2 - 3r + 1) \\ (-4r^2 + 4r)(2r^2 - 3r + 1) & (4r - 4r^2)^2 & -(-2r^2 + r)(-4r^2 + 4r) \\ -(-2r^2 + r)(2r^2 - 3r + 1) & -(-2r^2 + r)(-4r^2 + 4r) & (2r^2 - r)^2 \end{bmatrix} dr$$

$$\mathbf{M}^{\mathbf{e}} = \frac{\rho^e L^e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Here, $\rho^e L^e$ can be equated to m^e. Now that the consistent mass matrix has been computed, the

averaged mass matrix can be computed as the average of the consistent mass matrix and its diagonalized form. This calculation can be completed by first taking the ratio of the total sum of the elements in the consistent mass matrix to the sum of the elements in the diagonal mass matrix.

$$\sum \mathbf{M^e} = 1$$
$$\sum \mathbf{M^e}_{diagterms} = \frac{4}{5}$$
$$ratio = \frac{1}{0.8} = \frac{5}{4}$$

The next step is to multiply only the diagonal terms in the consistent mass matrix by this ratio.

$$\mathbf{M^e}_{diag} = \frac{5m^e}{120}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Finally, we can take the average of this diagonalized mass matrix with the consistent mass matrix to find the averaged mass matrix.

$$\mathbf{M^e}_{avg} = \frac{\mathbf{M^e} + \mathbf{M^e_{diag}}}{2}$$
$$\boxed{\mathbf{M^e}_{avg} = \frac{m^e}{60}\begin{bmatrix} 9 & 2 & -1 \\ 2 & 36 & 2 \\ -1 & 2 & 9 \end{bmatrix}}$$

The MATLAB code used to derive these values has been appended on the next page.